# UNIVERSITETET I OSLO

*Matematisk Institutt*

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| EXAM IN: | **STK 4060/9060 – Time Series** |
| WITH: | **Nils Lid Hjort** |
| AUXILIA: | **Calculator, plus one single sheet of paper with the candidate's own personal notes** |
| TIME FOR EXAM: | **Thursday 2/vi/2022, 15:00 – 19:00** |

This exam set contains four exercises and comprises four pages (including a simple appendix on the last page).

### Exercise 1: spectral densities

SUPPOSE THAT $x_t$ is a stationary time series process, for $t = 0, \pm 1, \pm 2, \ldots$, with finite covariances $\gamma(h) = \mathrm{cov}(x_t, x_{t+h})$ for $h = 0, \pm 1, \pm 2, \ldots$. Assume furthermore that the associated spectral domain distribution has a density $f(\omega)$ on $[-\frac{1}{2}, \frac{1}{2}]$. You may take for granted here that $\gamma(h)$ can be represented as $\int_{-1/2}^{1/2} \exp(2\pi i\omega h) f(\omega)\,\mathrm{d}\omega$, for each $h$, and that the spectral density can be expressed as

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) \exp(-2\pi i\omega h) \quad \text{for } -\tfrac{1}{2} \le \omega \le \tfrac{1}{2}.$$

For the following, you may find it useful to check with the few facts for complex numbers given in the Appendix.

(a) Show that $f(\omega) = \gamma(0) + 2\sum_{h=1}^{\infty} \gamma(h)\cos(2\pi h\omega)$. If the $x_t$ are actually independent, what is its $f(\omega)$?

(b) Now consider a transformation of the $x_t$ process to a new $y_t$ process, defined by

$$y_t = \sum_j c_j x_{t-j},$$

for certain coefficients $c_j$. Show that

$$\gamma^*(h) = \mathrm{cov}(y_t, y_{t+h}) = \sum_{r,s} c_r c_s \gamma(|h + r - s|).$$

(c) Use this to show that the $y_t$ process has spectral density

$$f^*(\omega) = |A(\omega)|^2 f(\omega),$$

in terms of the squared modulus of $A(\omega) = \sum c_j \exp(2\pi i j\omega)$.

(d) Then study the case of $y_t = x_t + b x_{t-1}$. Show that its spectral density can be written

$$f^*(\omega) = \{1 + b^2 + 2b\cos(2\pi\omega)\} f(\omega).$$

## Exercise 2: moving average of moving average

WE START OUT CONSIDERING a simple moving average process, of the type $x_t = w_t + a w_{t-1}$ for $t = 1, 2, \ldots$, in terms of i.i.d. white noise zero-mean variables $w_0, w_1, \ldots$, and where we for simplicity take these to have variance $\sigma_w^2 = 1$.

(a) Give formulae for the variance $\gamma(0)$ and for the covariances $\gamma(h) = \mathrm{cov}(x_t, x_{t+h})$ for $h = 1, 2, \ldots$. Show also that the one-step correlation, between $x_t$ and $x_{t+1}$, is $\rho(1) = a/(1 + a^2)$. What is the range of possible values, for this correlation?

(b) For an observed time series $x_1, \ldots, x_n$, give a formula for the empirical autocorrelation $\widehat{\rho}(1)$. Explain how you may estimate the parameter $a$ based on this.

(c) Show that the spectral density for the $x_t$ process becomes

$$f(\omega) = 1 + a^2 + 2a\cos(2\pi\omega).$$

(d) Then define the moving average of the initial moving average process, by $y_t = x_t + b x_{t-1}$. Find the spectral density $f^*(\omega)$ using results from Exercise 1.

(e) Show that the $y_t$ also can be expressed as a moving average process of order two. Find formulae for the covariances $\gamma(0), \gamma(1), \gamma(2)$ using this representation, and use this to find the spectral density once more. Verify that the two formulae you now have derived for the spectral density for the $y_t$ process are the same.

## Exercise 3: the periodogramme and the Whittle log-likelihood

THE DISCRETE FOURIER TRANSFORM, for an observed time series $x_1, \ldots, x_n$, is defined as

$$d(\omega) = (1/\sqrt{n}) \sum_{t=1}^{n} x_t \exp(-2\pi i \omega t).$$

(a) Give a formula for the inverse Fourier transform, where $x_1, \ldots, x_n$ can be retrieved from $d(\omega_j)$ computed at $\omega_j = j/n$ for $j = 0, 1, \ldots, n$. (You are not asked here to prove such a formula.)

(b) The periodogramme for the observed sequence is $I(\omega_j) = |d(\omega_j)|^2$, for $\omega_j = j/n$. Give a formula for this $I(\omega_j)$, in terms of $\sum_{t=1}^n x_t \cos(2\pi\omega_j t)$ and $\sum_{t=1}^n x_t \sin(2\pi\omega_j t)$.

(c) If the $x_t$ series is stationary, describe briefly how the periodogramme relates to its spectral density $f(\omega)$.

(d) Consider now the simple order-one zero-mean moving average process studied in Exercise 2, with $x_t = w_t + a w_{t-1}$ and with $\sigma_w = 1$, and where we have seen that the spectral density takes the form $f(\omega) = 1 + a^2 + 2a\cos(2\pi\omega)$. For an observed time series $x_1, \ldots, x_n$ from this model, define the Whittle log-likelihood function $\ell^w(a)$.

(e) I have data for two separate and independently observed time series, say $x_{A,1}, \ldots,$ $x_{A,100}$ and $x_{B,1}, \ldots, x_{B,100}$, both assumed to follow the simple MA(1) model above, but with parameters $a_A$ and $a_B$ that are not necessarily equal. In the figure below I've plotted the Whittle log-likelihood functions $\ell_A^w(a_A)$ and $\ell_B^w(a_B)$. For the A data, $\ell_A^w$ is maximised for 0.5622 with 2nd order derivative $-95.1402$; similarly, for the B data, $\ell_B^w$ is maximised for 0.3093 with 2nd order derivative $-105.6288$. (i) Give approximate 95 percent confidence intervals for $a_A$ and for $a_B$. (ii) Test the null hypothesis that the two parameters are equal.

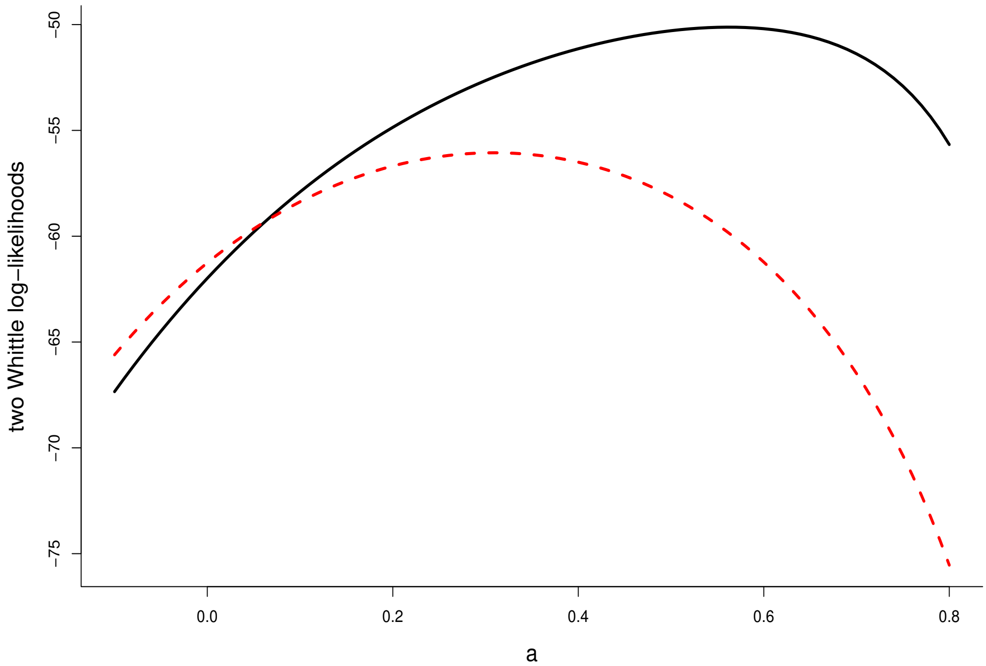


*Figure: The two Whittle log-likelihood functions, $\ell_A^w(a_1)$ (black, full) and $\ell_B^w(a_B)$ (red, dashed), for the two time series datasets.*

**Exercise 4: equations for the AR(2) model**

Consider the second-order autoregressive model, of the form

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t,$$

taken here for simplicity to have zero mean, and with the $w_t$ being i.i.d. with variance $\sigma_w^2$. Below, we let as usual $\gamma(h) = \mathrm{cov}(x_t, x_{t-h})$ for $h = 0, \pm 1, \pm 2, \ldots$.

(a) What is the requirement on $(\phi_1, \phi_2)$, to ensure that the $x_t$ process is stationary and does not explode (i.e. is causal, to use the book's term)? Assume in the following that this requirement is met.

(b) Multiply $x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} - w_t = 0$ with $x_t$, and show that

$$\gamma(0) - \phi_1\gamma(1) - \phi_2\gamma(2) = \sigma_w^2.$$

(c) Derive, in perhaps a similar manner, equations for $\gamma(1)$ and $\gamma(2)$. Show that these can be organised as

$$\begin{pmatrix} \gamma(0), & \gamma(1) \\ \gamma(1), & \gamma(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \gamma(1) \\ \gamma(2) \end{pmatrix}.$$

(d) Having observed a time series $x_1, \ldots, x_n$ from this zero-mean AR(2) model, explain how you can use the equations of (d) to estimate $\phi_1, \phi_2$. How can you then estimate $\sigma_w$?

**Appendix: just a few things for complex numbers**

For $z = a + ib$ a complex number, with $i$ the famous square root of $-1$, its conjugate number is $\bar{z} = a - ib$, and its squared length is $|z|^2 = z\bar{z} = a^2 + b^2$. Also, for $z = \sum_r c_r \exp(2\pi i r)$, it follows that

$$|z|^2 = \sum_{r,s} c_r c_s \exp(2\pi i(r - s)).$$

Here, as usual, $\exp(2\pi i u) = \cos(2\pi u) + i \sin(2\pi u)$; also, famously, $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$.