Notes on the Nils exam set, STK 4060, Time Series, 2/vi/2022

Exercise 1: spectral densities

(a) This is standard, from $\gamma(h)=\gamma(-h)$, and $\exp(2\pi ih\omega)+\exp(-2\pi ih\omega)=2\cos(2\pi h\omega)$. Under independence, only $\gamma(0)$ remains, so $f(\omega)=\gamma(0)$ is constant.

(b) With $y_t=\sum_j c_j x_{t-j}$ we have

$$\begin{aligned}\gamma^*(h)=\mathrm{cov}(y_t,y_{t+h})&=\sum_{r,s} c_r c_s \mathrm{cov}(x_{t-r},x_{t+h-s})\\&=\sum_{r,s} c_r c_s \gamma(t-r-t-h+s)=\sum_{r,s} c_r c_s\gamma(h+r-s).\end{aligned}$$

(c) Further elaboration leads to

$$\begin{aligned}\gamma^*(h)&=\sum_{r,s} c_r c_s\int \exp(2\pi i(h+r-s)\omega)f(\omega)\,\mathrm{d}\omega\\&=\int \exp(2\pi ih)\sum_{r,s} c_r c_s\exp(2\pi i(r-s))f(\omega)\,\mathrm{d}\omega\\&=\int \exp(2\pi ih)|A(\omega)|^2 f(\omega)\,\mathrm{d}\omega,\end{aligned}$$

with $A(\omega)=\sum c_j\exp(2\pi ij\omega)$.

(d) Here $A(\omega)=1+b\exp(2\pi i\omega)$, with

$$|A(\omega)|^2=\{1+b\cos(2\pi\omega)\}^2+b^2\sin^2(2\pi\omega)=1+b^2+2b\cos(2\pi\omega).$$

Exercise 2: moving average of moving average

We study $x_t=w_t+aw_{t-1}$, with $\sigma_w^2=1$.

(a) We have $\gamma(0)=1+a^2$, $\gamma(1)=a$, and the other $\gamma(h)=0$. This gives $\rho(1)=a/(1+a^2)$, with range $[-\frac{1}{2},\frac{1}{2}]$.

(b) We may solve the equation $\widehat{\rho}=a/(1+a^2)$. There are a few issues here, since this function has limited range, and also has two solutions; we might pick the one closest to zero.

(c) That $f(\omega)=1+a^2+2a\cos(2\pi\omega)$ follows from the representation of Exercise 1.

(d) With $y_t=x_t+bx_{t-1}$ we may point to Exercise 1 and deduce

$$f^*(\omega)=(1+a^2+2a\cos(2\pi\omega))(1+b^2+2b\cos(2\pi\omega)).$$

(e) We have

$$y_t=w_t+aw_{t-1}+b(w_{t-1}+aw_{t-2})=w_t+(a+b)w_{t-1}+abw_{t-2},$$

which is an MA(2) with coefficients $1, a+b, ab$. This leads to

$$\widetilde{f}(\omega) = \gamma(0) + 2\gamma(1)\cos(2\pi\omega) + 2\gamma(2)\cos(4\pi\omega),$$

with $\gamma(0) = 1 + (a+b)^2 + (ab)^2$, $\gamma(1) = 1 \cdot (a+b) + (a+b)ab$, $\gamma(2) = ab$. – So these need to be equal. To verify this, use $x = \cos(2\pi\omega)$, so that $\cos(4\pi\omega) = 2x^2 - 1$. So

$$\begin{aligned} f^*(\omega) &= (1 + a^2 + 2ax)(1 + b^2 + 2bx), \\ \widetilde{f}(\omega) &= 1 + (a+b)^2 + a^2b^2 + 2(a+b)(1+ab)x + 2ab(2x^2 - 1). \end{aligned}$$

It's now algebra to verify that these are the same function. Or one may show that these, their 1st derivatives, and their 2nd derivatives, are equal at zero.

Exercise 3: the periodogramme and the Whittle log-likelihood

(a) We have

$$x_t = (1/\sqrt{n}) \sum_{j=0}^{n-1} d(\omega_j) \exp(2\pi i \omega_j t).$$

(b) So

$$|d(\omega)|^2 = \frac{1}{n} \Big\{ \sum_{t=1}^{n} x_t \cos(2\pi\omega t) \Big\}^2 + \frac{1}{n} \Big\{ \sum_{t=1}^{n} x_t \sin(2\pi\omega t) \Big\}^2,$$

giving also a formula for $I(\omega_j) = |d(\omega_j)|^2$.

(c) To the first order, $I(\omega)$ estimates $f(\omega)$. The variance isn't going to zero, though, as we have $I(\omega) \doteq f(\omega)U$, with U a standard exponential.

(d) The Whittle log-likelihood is

$$\ell^w(a) = -\sum_{0<\omega_j<1/2} \Big[\log(1 + a^2 + 2a\cos(2\pi\omega_j)) + \frac{I(\omega_j)}{1 + a^2 + 2a\cos(2\pi\omega_j)} \Big],$$

with $\omega_j = j/n$.

(e) The ML estimates are 0.5622 and 0.3093, for the A and B data. The variances are well approximated by 1/95.1402 and 1/105.6288, which means standard deviations 0.1025 and 0.0973. Confidence intervals for the two parameters are $0.5622 \pm 1.96 \cdot 0.1025$ and $0.3093 \pm 1.96 \cdot 0.0973$, which become $[0.3613, 0.7631]$ and $[0.1186, 0.5000]$. There is overlap, indicating that a_A and a_B are not far from each other. Also, with $d = a_A - a_B$, the Wald test is

$$W = \frac{0.5622 - 0.3093}{(1/95.1402 + 1/105.6288)^{1/2}} = \frac{0.2529}{0.1413} = 1.7898,$$

so d is not significantly different from zero.

Exercise 4: equations for the AR(2) model

(a) The characteristic polynomial is $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$, and its two roots must lie outside the unit circle in the complex plane. This is in turn due to the equation $\phi(z)\psi(z) = 1$ for $|z| < 1$, etc., where $\psi(z) = \sum_{j=0}^\infty \psi_j z^j$, for the representation $x_t = \sum_{j=0}^\infty \psi_j w_{t-j}$.

(b) From $x_t(x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} - w_t) = 0$ we find $\gamma(0) - \phi_1\gamma(1) - \phi_2\gamma(2) - \sigma_w^2 = 0$.

(c) Multiplying instead with x_{t-1}, and afterwards with x_{t-2}, leads to the two equations

$$\begin{aligned}\gamma(1) - \phi_1\gamma(0) - \phi_2\gamma(1) &= 0,\\ \gamma(2) - \phi_1\gamma(1) - \phi_2\gamma(0) &= 0.\end{aligned}$$

These may be rearranged to

$$\begin{aligned}\phi_1\gamma(0) + \phi_2\gamma(1) &= \gamma(1),\\ \phi_1\gamma(1) + \phi_2\gamma(0) &= \gamma(2),\end{aligned}$$

and then to

$$\begin{pmatrix}\gamma(0), & \gamma(1)\\ \gamma(1), & \gamma(0)\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\gamma(1)\\ \gamma(2)\end{pmatrix}.$$

These are the Yule–Walker equations of yore.

(d) Having observed a time series $x_1, \ldots, x_n$ from this zero-mean AR(2) model, we may compute the usual $\widehat\gamma(1), \widehat\gamma(2)$, and then

$$\begin{pmatrix}\widehat\phi_1\\ \widehat\phi_2\end{pmatrix} = \begin{pmatrix}\widehat\gamma(0), & \widehat\gamma(1)\\ \widehat\gamma(1), & \widehat\gamma(0)\end{pmatrix}^{-1}\begin{pmatrix}\widehat\gamma(1)\\ \widehat\gamma(2)\end{pmatrix}.$$

Of course we may derive even more explicit equations for $\widehat\phi_1, \widehat\phi_2$ by inverting the matrix. Also, we take $\widehat\sigma_w^2 = \widehat\gamma(0) - \widehat\phi_1\widehat\gamma(1) - \widehat\phi_2\widehat\gamma(2)$.