Additional exercise 3

In this problem we will study the gamma frailty model for clustered survival data when there are two units per cluster (e.g. twins). Consider one cluster, and denote the (uncensored) survival times for the two units by T_1 and T_2. We let Z be the frailty for the cluster, and assume that Z is gamma distributed with mean one and variance δ. Conditional on frailty, T_1 and T_2 are assumed to be independent and identically distributed with hazard rate $\alpha(t \mid Z) = Z\,\alpha(t)$. The joint survival function of T_1 and T_2 is given by $S(t_1, t_2) = P(T_1 > t_1, T_2 > t_2)$.

a) Show that

$$S(t_1, t_2) = \left(1 + \delta\{A(t_1) + A(t_2)\}\right)^{-1/\delta} \qquad (1)$$

where $A(t) = \int_0^t \alpha(u)\,du$.

We will find an expression for the conditional hazard rate of T_2 given $T_1 > t_1$, which is given by

$$\mu(t_2 \mid t_1) = \frac{-S'(t_2 \mid t_1)}{S(t_2 \mid t_1)}. \qquad (2)$$

Here $S(t_2 \mid t_1) = P(T_2 > t_2 \mid T_1 > t_1)$ is the conditional survival function of T_2 given $T_1 > t_1$ and the differentiation in the numerator is with respect to t_2.

b) Use (1) to find expressions for $S(t_2 \mid t_1)$ and $S'(t_2 \mid t_1)$.

c) Show that (2) may be given as

$$\mu(t_2 \mid t_1) = \frac{\alpha(t_2)}{1 + \delta\{A(t_1) + A(t_2)\}} \qquad (3)$$

Compare (3) with the marginal hazard for T_2 given by formula (6.8) in the ABG-book.