Stochastic processes in continuous time

Motivating example: The Nelson-Aalen estimator

We have right-censored survival data $(\tilde{T}_i, D_i)$; $i = 1, 2, \ldots, n$; where the *uncensored* survival times T_i are iid with hazard rate $\alpha(t)$; cf. pages 30-31

The aggregated counting process

$$N(t) = \sum_{i=1}^{n} I\{\tilde{T}_i \le t, D_i = 1\}$$

has intensity process

$$\lambda(t) = \alpha(t) Y(t)$$

where $Y(t) = \sum_{i=1}^{n} I\{\tilde{T}_i \ge t\}$ is the number of individuals at risk "just before" time t

We make no assumptions on the form of the hazard $\alpha(t)$ and we want to estimate the cumulative hazard

$$A(t) = \int_0^t \alpha(s) ds$$

We have the decomposition

$$dN(t) = \alpha(t) Y(t) dt + dM(t)$$

Assuming for now that $Y(t) > 0$, this gives

$$\frac{dN(t)}{Y(t)} = \alpha(t) dt + \frac{dM(t)}{Y(t)}$$

By integration we obtain

$$\int_0^t \frac{dN(s)}{Y(s)} = \int_0^t \alpha(s) ds + \int_0^t \frac{dM(s)}{Y(s)}$$

The Nelson-Aalen estimator is given by

$$\hat{A}(t) = \int_0^t \frac{dN(s)}{Y(s)} \qquad \text{(a sum over the jump times of } N\text{)}$$

We have that

$$\hat{A}(t) = A(t) + \int_0^t \frac{dM(s)}{Y(s)}$$

To derive the statistical properties of the Nelson-Aalen estimator (and other estimators and test statistics), we need study stochastic integrals (and other stochastic processes in continuous time)

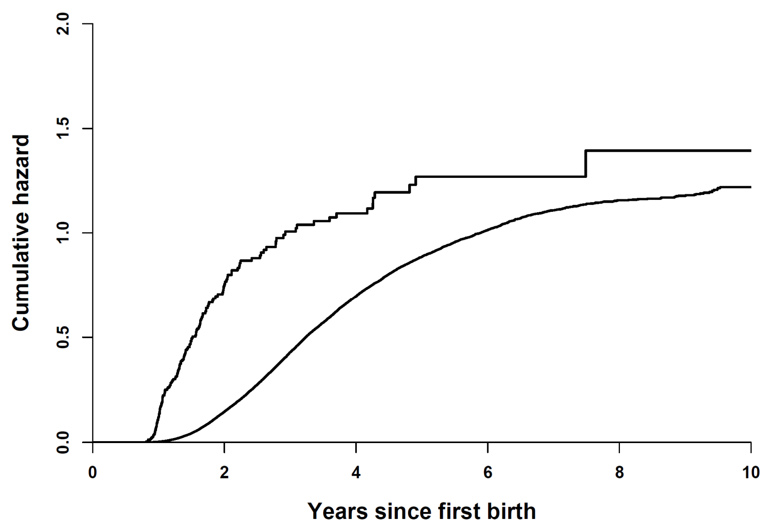


Fig. 3.1 *Nelson-Aalen estimates for the time between first and second births. Lower curve: first child survived one year; upper curve: first child died within one year.*

Some general properties of stochastic processes

Unless otherwise stated, we will consider time-continuous stochastic processes defined on a finite time interval $[0,\tau]$

A stochastic process $X = \{X(t); t \in [0,\tau]\}$ is adapted to a history $\{\mathscr{F}_t\}$ if at time t we know the value of $X(s)$ for all $s \le t$ (possibly apart from unknown parameters)

A realization of X is a function of t and is called a sample path

Unless otherwise stated, we will consider stochastic processes with sample paths that are right-continuous and have left-hand limits (cadlag)

A random variable $T > 0$ is a stopping time if we at time t know whether $T \le t$ or $T > t$

Martingales in continuous time

A stochastic process $M = \{M(t); t \in [0,\tau]\}$ is a martingale relative to the history $\{\mathscr{F}_t\}$ if it is adapted to the history and satisfies the martingale property:

$$\mathrm{E}(M(t) \mid \mathscr{F}_s) = M(s) \quad \text{for all } t > s$$

<u>Example</u>:

Let $N(t)$ be a Poisson process with intensity λ

Denote by $\mathscr{F}_t$ the information on all events that happen in $[0,t]$

Then $M(t) = N(t) - \lambda t$ is a martingale (cf. page 53)

Heuristically, M is a martingale provided that

$$\mathrm{E}(dM(t) \mid \mathscr{F}_{t-}) = 0$$

where $dM(t)$ is the increment of M over $[t, t + dt)$, and $\mathscr{F}_{t-}$ is the history "just before" time t

The properties of continuous time martingales parallel those of discrete time martingales

We will assume throughout that $M(0)=0$

Then $\mathrm{E}M(t) = 0$ and M is a mean zero martingale

A martingale has uncorrelated increments, i.e.

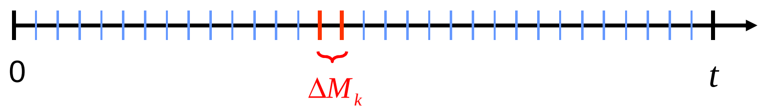
$$\mathrm{Cov}(M(t) - M(s), M(v) - M(u)) = 0$$

for all $0 \le s < t < u < v \le \tau$

Variation processes

The predictable variation process $\langle M \rangle$ and the optional variation process $[M]$ of a time-continuous martingale M are obtained as limits of the discrete time variation processes

Partition $[0,t]$ into n time intervals each of length t/n :



Then

$$[M](t) = \lim_{n\to\infty} \sum_{k=1}^{n} (\triangle M_k)^2$$

$$\langle M \rangle (t) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{Var}(\triangle M_k | \mathscr{F}_{(k-1)t/n})$$

Informally, the last expression gives:

$$d\langle M \rangle (t) = \mathrm{Var}(dM(t) \mid \mathscr{F}_{t-})$$

Example:

$N(t)$ is a Poisson process with intensity λ

$\mathscr{F}_t$ is the information on all events that happen in $[0,t]$

$M(t) = N(t) - \lambda t$ is a martingale

We have $dM(t) = dN(t) - \lambda\, dt$

Hence

$$d\langle M\rangle(t) = \text{Var}(dM(t) \mid \mathscr{F}_{t-}) = \text{Var}(dN(t) - \lambda\, dt \mid \mathscr{F}_{t-})$$
$$= \text{Var}(dN(t) \mid \mathscr{F}_{t-}) = \text{E}(dN(t) \mid \mathscr{F}_{t-}) = \lambda\, dt$$

By integration this gives

$$\langle M\rangle(t) = \lambda t$$

As for discrete time martingales we have that

$$M^2 - [M] \text{ is a mean zero martingale}$$

$$M^2 - \langle M\rangle \text{ is a mean zero martingale}$$

From this we obtain:

$$\text{Var}\,(M(t)) = \text{E}\left(M(t)^2\right) = \text{E}\langle M\rangle(t) = \text{E}[M](t)$$

For two martingales M_1 and M_2, we may define the predictable covariation process $\langle M_1, M_2\rangle$ and the optional covariation process $[M_1, M_2]$, see exercise 2.5 and page 50

Stochastic integrals

Stochastic integrals are the continuous time analogue to transformations for discrete time martingales

Let $M = \{M(t); t \in [0, \tau]\}$ be a martingale and $H = \{H(t); t \in [0, \tau]\}$ a predictable process, which intuitively means that for any time t we know the value of $H(t)$ "just before" t (a sufficient condition for predictability is that H is adapted and left-continuous)

We will define the stochastic integral

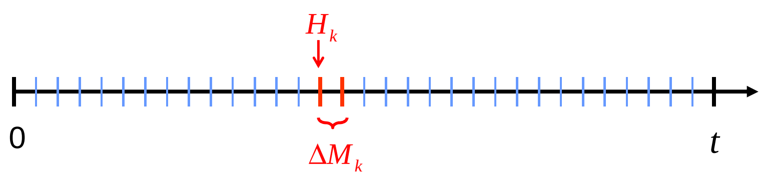
$$I(t) = \int_0^t H(s)\, dM(s)$$

In general it is quite intricate to define a stochastic integral (cf. the Itô integral), but for the situations we will consider it may be defined as a limit of transformations of discrete time martingales

We partition $[0,t]$ into n time intervals each of length t/n

Define $H_k = H((k-1)t/n)$

$\triangle M_k = M(kt/n) - M((k-1)t/n)$



Then

$$I(t) = \int_0^t H(s)\, dM(s) = \lim_{n\to\infty} \sum_{k=1}^{n} H_k \triangle M_k$$

A stochastic integral

$$I(t) = \int_0^t H(s)\,dM(s)$$

has similar properties as a transformation for discrete time martingales:

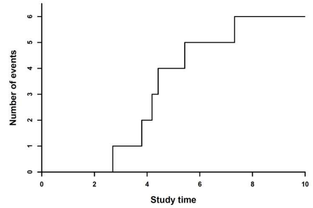
- $I(t) = \int_0^t H(s)\,dM(s)$ is a mean zero martingale
- $\langle I \rangle(t) = \left\langle \int HdM \right\rangle(t) = \int_0^t H^2(s)\,d\langle M \rangle(s)$
- $[I](t) = \left[\int HdM \right](t) = \int_0^t H^2(s)\,d[M](s)$

For results on covariance processes for two stochastic integrals, see exercise 2.8 and page 51

Counting processes

A counting process $N = \{N(t); t \in [0,\tau]\}$ is a stochastic process with sample paths that are right-continuous step functions with steps of size +1

We assume that the counting process is adapted to the history $\{\mathscr{F}_t\}$



The intensity process $\lambda(t)$ w.r.t the history $\{\mathscr{F}_t\}$ is given informally by

$$\lambda(t)dt = \mathrm{P}(dN(t) = 1 \mid \mathscr{F}_{t-}) = \mathrm{E}(dN(t) \mid \mathscr{F}_{t-})$$

Then, as seen earlier, the process

$$M(t) = N(t) - \int_0^t \lambda(s)ds$$

is a mean zero martingale

The optional variation process of $M(t)$ becomes

$$[M](t) = \lim_{n\to\infty} \sum_{k=1}^n (\triangle M_k)^2 = \lim_{n\to\infty} \sum_{k=1}^n (\Delta N_k)^2 = N(t)$$

For the predictable variation process, note that

$$d\langle M \rangle(t) = \mathrm{Var}(dM(t) \mid \mathscr{F}_{t-}) = \mathrm{Var}(dN(t) - \lambda(t)dt \mid \mathscr{F}_{t-})$$
$$= \mathrm{Var}(dN(t) \mid \mathscr{F}_{t-}) \approx \lambda(t)dt\{1 - \lambda(t)dt\} \approx \lambda(t)dt$$

This motivates the important relation

$$\langle M \rangle(t) = \int_0^t \lambda(s)ds$$

When considering two counting processes $N_1(t)$ and $N_2(t)$ adapted to the same history $\{\mathscr{F}_t\}$, we will assume that they do not jump at the same time

Then one may show that the corresponding martingales $M_1(t)$ and $M_2(t)$ are orthogonal, which means that

$$\langle M_1, M_2 \rangle(t) = 0 \quad \text{for all } t$$
$$[M_1, M_2](t) = 0 \quad \text{for all } t$$

(exercise 2.10)

Stochastic integrals for counting process martingales

Most stochastic integrals we will encounter are relative to counting process martingales

A stochastic integral relative to a counting process integral is simple to understand:

$$I(t) = \int_0^t H(s)\,dM(s) = \underbrace{\int_0^t H(s)\,dN(s)}_{= \sum_{T_j \le t} H(T_j)} - \underbrace{\int_0^t H(s)\,\lambda(s)\,ds}_{\text{an ordinary integral (pathwise)}}$$

a sum over the jump times $T_1 < T_2 < \ldots$

an ordinary integral (pathwise)

For a stochastic integral relative to a counting process martingale, the variance processes take the form

$$\left\langle \int H\,dM \right\rangle(t) = \int_0^t H^2(s)\lambda(s)ds$$

$$\left[\int H\,dM \right](t) = \int_0^t H^2(s)\,dN(s)$$

Example: For the Nelson-Aalen estimator, we have

$$\hat{A}(t) - A(t) = \int_0^t \frac{dM(s)}{Y(s)}$$

Thus $\hat{A} - A$ is a mean zero martingale with optional variation process

$$\left[\hat{A} - A\right](t) = \int_0^t \frac{dN(s)}{Y^2(s)}$$

Note that $\left[\hat{A} - A\right](t)$ is an unbiased estimator for the variance of the Nelson-Aalen estimator

The Doob-Meyer decomposition

An adapted process $X = \{X(t); t \in [0, \tau]\}$ is a submartingale if it satisfies

$$\mathrm{E}(X(t) \mid \mathscr{F}_s) \ge X(s) \quad \text{for all } t > s$$

The Doob-Meyer decomposition states that any submartingale can be decomposed *uniquely* as

$$X = X^* + M$$

where X^* is a nondecreasing predictable process, denoted the compensator of X, and M is a mean zero martingale

Heuristically we have

$$dX^*(t) = \mathrm{E}(dX(t) \mid \mathscr{F}_{t-})$$

$$dM(t) = dX(t) - \mathrm{E}(dX(t) \mid \mathscr{F}_{t-})$$

Examples:

A counting process $N(t)$ is a submartingale, and we have the decomposition

$$N(t) = \underbrace{\int_0^t \lambda(s)ds}_{\text{compensator}} + M(t)$$

$M^2(t)$ is a submartingale, and we have the decomposition

$$M^2(t) = \underbrace{\langle M \rangle(t)}_{\text{compensator}} + \text{martingale}$$