SOLUTION TO EXERCISES WEEK 35

Exercise 1.1

Assum $t > 0$

a) $S(t) = e^{-\gamma t}$

$$f(t) = -S'(t) = \gamma e^{-\gamma t}$$

$$\alpha(t) = -\frac{S'(t)}{S(t)} = \gamma$$

b) $\alpha(t) = b\, t^{k-1}$

$$A(t) = \int_0^t \alpha(u)\,du = \int_0^t b\, u^{k-1} = b\,(t/k)^k$$

$$S(t) = e^{-A(t)} = e^{-b(t/k)^k}$$

$$f(t) = -S'(t) = b\, t^{k-1} e^{-b(t/k)^k}$$

c) $f(t) = \frac{\gamma^k}{\Gamma(k)} t^{k-1} e^{-\gamma t}$

with $\Gamma(k) = \int_0^\infty u^{k-1} e^{-u} du$

$$S(t) = \int_t^\infty f(v)dv = \int_t^\infty \frac{\gamma^k}{\Gamma(k)} v^{k-1} e^{-\gamma v} dv$$

$$= \frac{\gamma^k}{\Gamma(k)} \int_{\gamma t}^\infty \left(\frac{u}{\gamma}\right)^{k-1} e^{-u} \frac{du}{\gamma} \qquad [u = \gamma v]$$

$$= \frac{1}{\Gamma(k)} \int_{\gamma t}^\infty u^{k-1} e^{-u} du = \frac{\Gamma(k, \gamma t)}{\Gamma(k)}$$

with $\Gamma(k, x) = \int_x^\infty u^{k-1} e^{-u} du$

$$\alpha(t) = -\frac{S'(t)}{S(t)} = \frac{f(t)}{S(t)} = \frac{\gamma^k}{\Gamma(k, \gamma t)} t^{k-1} e^{-\gamma t}$$

Exercise 1.2

ξ_p is defined by $F(\xi_p) = P(T \le \xi_p) = p$

Equivalently $1 - S(\xi_p) = p$, i.e.

$S(\xi_p) = 1 - p$

a) $$S(t) = e^{-\int_0^t \alpha(u)\,du} = e^{-A(t)}$$

$$1 - p = S(\xi_p) = e^{-A(\xi_p)}$$

$$\log(1-p) = -A(\xi_p)$$

$$A(\xi_p) = -\log(1-p)$$

b) Exponential distribution:

$$A(t) = \int_0^t \alpha(u)\,du = \int_0^t \gamma\,du = \gamma t$$

$$A(\xi_p) = \gamma \xi_p = -\log(1-p)$$

$$\xi_p = -\frac{1}{\gamma}\log(1-p)$$

Weibull distribution:

$$A(t) = b\,(t/k)^k$$

$$A(\xi_p) = b\,(\xi_p/k)^k = -\log(1-p)$$

$$\xi_p = k\left(-\tfrac{1}{b}\log(1-p)\right)^{1/k}$$

Exercise 1.3

Survival time T

$S(t) = P(T > t)$ with $S(\infty) = 0$

a) May write

$$T = \int_0^\infty I(T > u)\,du$$

Hence

$$ET = E\int_0^\infty I(T>u)\,du = \int_0^\infty E\{I(T>u)\}\,du$$

$$= \int_0^\infty P(T>u)\,du = \int_0^\infty S(u)\,du$$

b) Exponential distribution:

$$S(t) = e^{-\gamma t}$$

$$ET = \int_0^\infty S(u)\,du = \int_0^\infty e^{-\gamma u}\,du$$

$$= \left[-\frac{1}{\gamma} e^{-\gamma u}\right]_0^\infty = \frac{1}{\gamma}$$

Weibull distribution:

$$S(t) = e^{-b(t/k)^k}$$

$$ET = \int_0^\infty e^{-b(u/k)^k}\,du$$

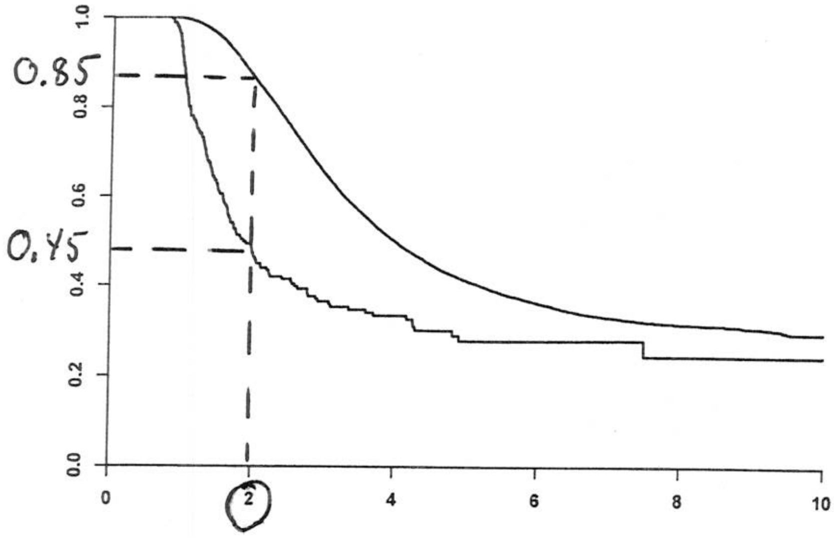
$$= \int_0^\infty e^{-v} \frac{1}{b}\left(\frac{v}{b}\right)^{1/k-1} dv \qquad \left[\begin{array}{l} v = b(u/k)^k \\ u = k(v/b)^{1/k} \end{array}\right]$$

$$= \frac{1}{b^{1/k}} \int_0^\infty v^{1/k-1} e^{-v}\,dv$$

$$= \frac{\Gamma(1/k)}{b^{1/k}}$$

Exercise 1.4

a)



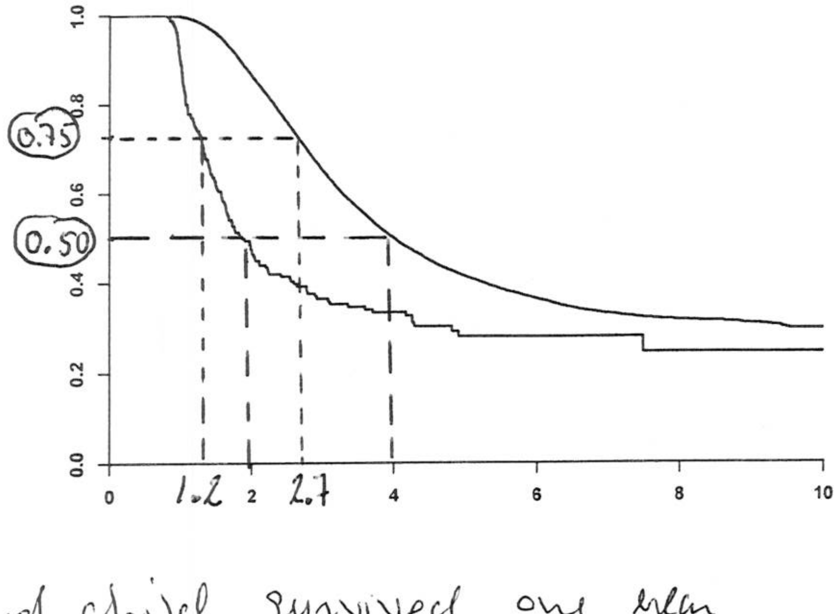
First child survived one year:

$\hat{P}$(second child within two years) $= 1 - \hat{S}(2) \approx 1 - 0.85 = 0.15$

First child died within one year:

$\hat{P}$(second child within 2 years) $\approx 1 - 0.45 = 0.55$

b)



First child survived one year

$\hat{\xi}_{0.25} \approx 2.7$ years $\qquad \hat{\xi}_{0.50} \approx 4.0$ years

First child died within one year

$\hat{\xi}_{0.25} \approx 1.2$ years $\qquad \hat{\xi}_{0.50} \approx 2.0$ years

Exercise 1.5

Covariates $x_{i1}, x_{i2}, \dots, x_{ip}$ for $i = 1, 2$

Hazard rates

$$\alpha_i(t) = \alpha_0(t) \exp\{\beta_1 x_{i1} + \dots + \beta_p x_{ip}\}$$

a) Hazard ratio

$$\frac{\alpha_2(t)}{\alpha_1(t)} = \frac{\exp\{\beta_1 x_{21} + \dots + \beta_p x_{2p}\}}{\exp\{\beta_1 x_{11} + \dots + \beta_p x_{1p}\}}$$

$$= \exp\{\beta_1 (x_{21} - x_{11}) + \dots + \beta_p (x_{2p} - x_{1p})\}$$

b) $x_{2j} = x_{1j} + 1$ and $x_{2l} = x_{1l}$ for $l \neq j$

$$\frac{\alpha_2(t)}{\alpha_1(t)} = e^{\beta_j}$$

Thus e^{β_j} is the hazard ratio for one unit's increase in the j-th covariate when all other covariates are the same.