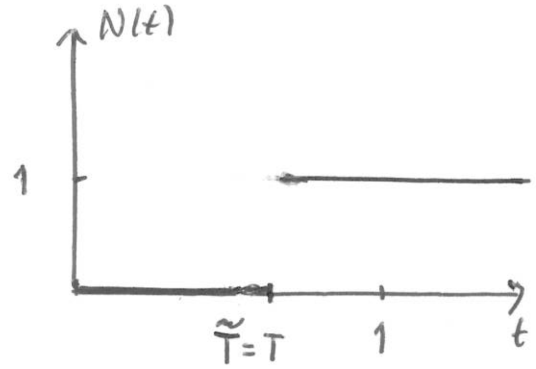
SOLUTION TO EXERCISES WEEK 36

Exercise 1.6

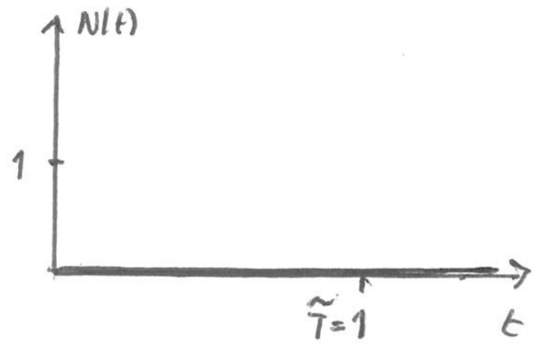
T exponentially distributed with hazard rate $\alpha(t) = 2$. We define $\tilde{T} = \min(T, 1)$ and $D = I(T \leq 1)$, and introduce the counting process $N(t) = I(\tilde{T} \leq t, D = 1)$.

a) Sketch of $N(t)$:

$T \leq 1, \; D = 1$



$T > 1, \; D = 0$



b) Intensity process $\lambda(t)$ is given by

$$\lambda(t)dt = P(dN(t) = 1 \mid \text{past}) = \begin{cases} \alpha(t)dt & \tilde{T} \geq t \\ 0 & \tilde{T} < t \end{cases}$$

Thus we have

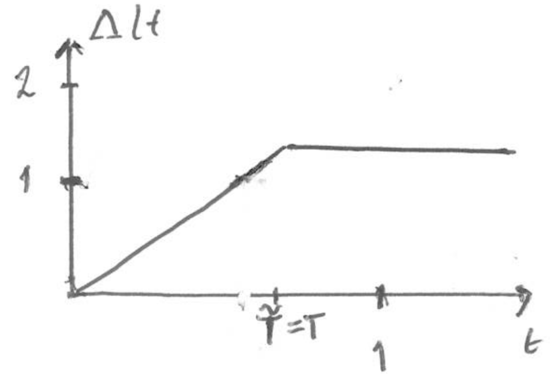
$$\lambda(t) = \alpha(t)\, I(\tilde{T} \geq t) = 2 \cdot I(\tilde{T} \geq t)$$

This gives

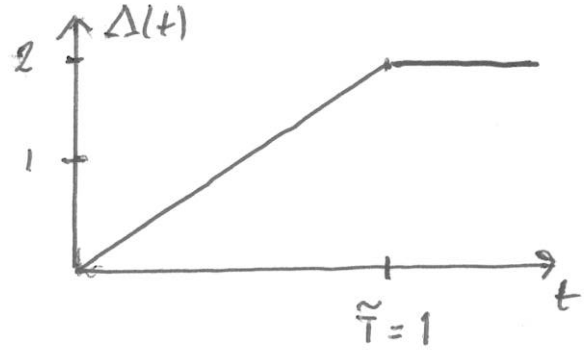
$$\Lambda(t) = \begin{cases} 2t & \text{for } t \leq \tilde{T} \\ 2\tilde{T} & \text{for } t > \tilde{T} \end{cases}$$

Sketch:

$T \leq 1,\ D = 1$

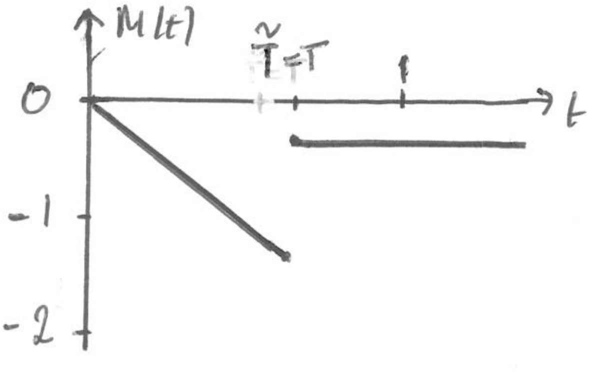


$T \geq 1,\ D = 0$

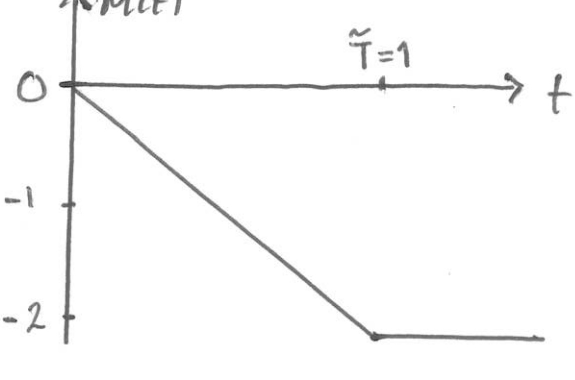


c) $M(t) = N(t) - \Lambda(t)$:

$T \leq 1,\ D = 1$



$T > 1,\ D = 0$



Exercise 1.9

T survival time with hazard rate $\alpha(t)$.
$v>0$ is a constant.
If we consider T conditional on $T>v$, we say that the survival time is <u>left-truncated</u>

a) The conditional survival function becomes

$$S(t|v) = P(T>t \mid T>v)$$

$$= \frac{P(T>t, T>v)}{P(T>v)}$$

$$= \frac{P(T>t)}{P(T>v)}$$

$$= \frac{e^{-\int_0^t \alpha(u)du}}{e^{-\int_0^v \alpha(u)du}}$$

$$= e^{-\int_v^t \alpha(u)du}$$

when $t>v$. If $t \leq v$, then $S(t|v) = 1$

Thus the (conditional) hazard rate becomes (cf. page 6 in ABG)

$$\alpha(t \mid v) = -\frac{d}{dt} \log S(t \mid v)$$

$$= \begin{cases} 0 & \text{for } t \leq v \\ \alpha(t) & \text{for } t > v \end{cases}$$

By the argument on page 28 in ABG, it then follows that $N(t) = I(v < T \leq t)$ has intensity process

$$\lambda(t) = \alpha(t \mid v)\, I(T \geq t)$$

$$= \alpha(t)\, I(v < t \leq T)$$

Note that the intensity process is derived from the conditional distribution of T given $T > v$

b) We then consider the left-truncated and right-censored survival time $\tilde{T} = T \wedge u$ obtained by censoring the left-truncated survival time at $u > v$ [$x \wedge y = \min(x, y)$] and let $D = I\{\tilde{T} = T\}$.

By the argument on page 31 in ABG, it follows that the counting process $N(t) = I(v < \tilde{T} \leq t, D = 1)$ has intensity process (derived from the conditional distribution of T given $T > v$):

$$\lambda(t) = \alpha(t \mid v)\, I\{\tilde{T} \geq t\}$$

$$= \alpha(t)\, I\{v < t \leq \tilde{T}\}$$

Note that this is of the same form as (1.22) in ABG, but with the at risk indicator given by $Y(t) = I\{v < t \leq \tilde{T}\}$, not by (1.23).

Exercise 2.2

$M = \{M_0, M_1, \dots\}$ is a zero mean martingale. Thus for all $n > m$:

$$\mathrm{Cov}(M_m, M_n - M_m) = E\{M_m(M_n - M_m)\}$$

$$= E\{E(M_m(M_n - M_m) \mid \mathcal{F}_m)\}$$

$$= E\{M_m E(M_n - M_m \mid \mathcal{F}_m)\}$$

$$= 0$$

since $E(M_n - M_m \mid \mathcal{F}_m) = E(M_n \mid \mathcal{F}_m) - M_m$

$$= M_m - M_m = 0$$

Exercise 2.4

We will show that $M^2 - \langle M \rangle$ is a zero mean martingale. We immediately have that $M_0^2 - \langle M \rangle_0^? = 0 - 0 = 0$

Further we have that (since $\langle M \rangle$ is predictable)

$$E\{M_n^2 - \langle M\rangle_n \mid \mathcal{F}_{n-1}\}$$

$$= E(M_n^2 \mid \mathcal{F}_{n-1}) - \langle M\rangle_n$$

$$= E((M_{n-1} + \Delta M_n)^2 \mid \mathcal{F}_{n-1}) - \langle M\rangle_n$$

$$= E(M_{n-1}^2 - 2M_{n-1}\Delta M_n + (\Delta M_n)^2 \mid \mathcal{F}_{n-1})$$

$$- \langle M\rangle_{n-1} - E((\Delta M_n)^2 \mid \mathcal{F}_{n-1})$$

$$= M_{n-1}^2 - 2M_{n-1}E(\Delta M_n \mid \mathcal{F}_{n-1})$$

$$+ E((\Delta M_n)^2 \mid \mathcal{F}_{n-1}) - \langle M\rangle_{n-1}$$

$$- E((\Delta M_n)^2 \mid \mathcal{F}_{n-1})$$

$$= M_{n-1}^2 - \langle M\rangle_{n-1}$$

Thus $M^2 - \langle M\rangle$ is a martingale

Exercise 2.6

If M is a mean zero martingale ($M_0 = 0$) and $H = \{H_0, H_1, \dots\}$ is a predictable process, we know that the trasformed process $Z = H \bullet M$ given by

$$Z_n = \sum_{s=1}^{n} H_s \, \Delta M_s$$

is a mean zero martingale

Let T be a stopping time. Then the process H given by $H_n = I(T \geq n)$ is predictable (since at time $n-1$ we know whether $T < n$ or $T \geq n$).

Therefore the stopped martingale

$$M_n^T = M_{n \wedge T} = \sum_{s=1}^{n} I(T \geq s) \, \Delta M_s$$

is a mean zero martingale

Exercise 2.7

We will prove that

$$[H \bullet M]_n = \sum_{s=1}^{n} H_s^2 \, \Delta [M]_s \qquad (2.15)$$

We have that

$$\Delta (H \bullet M)_s = H_s \, \Delta M_s$$

$$\Delta [M]_s = (\Delta M_s)^2$$

Therefore

$$[H \bullet M]_n = \sum_{s=1}^{n} (\Delta (H \bullet M)_s)^2$$

$$= \sum_{s=1}^{n} H_s^2 (\Delta M_s)^2$$

$$= \sum_{s=1}^{n} H_s^2 \, \Delta [M]_s$$