SOLUTIONS TO EXERCISES WEEK 37

Exercise 2.5

$M_1 = \{M_{10}, M_{20}, \dots\}$ and $M_2 = \{M_{20}, M_{21}, \dots\}$ are martingales with $M_{10} = M_{20} = 0$ w.r.t. the history $\{\mathcal{F}_n\}$.

<u>Predictable covariation process: $\langle M_1, M_2 \rangle$</u>

$$\langle M_1, M_2 \rangle_n = \sum_{i=1}^{n} E\{\triangle M_{1i} \triangle M_{2i} \mid \mathcal{F}_{i-1}\}$$

$$= \sum_{i=1}^{n} \mathrm{Cov}(\triangle M_{1i}, \triangle M_{2i} \mid \mathcal{F}_{i-1})$$

<u>Optional covariation process: $[M_1, M_2]$</u>

$$[M_1, M_2] = \sum_{i=1}^{n} \triangle M_{1i} \triangle M_{2i}$$

a) We imediately have that

$M_{10} M_{20} - \langle M_1, M_2 \rangle_0 = 0$ and

$M_{10} M_{20} - [M_1, M_2]_0 = 0$.

Further we have that (cf. exercise 2.4):

$$E(M_{1n} M_{2n} - \langle M_1, M_2 \rangle_n \mid F_{n-1})$$

$$= E\{(M_{1,n-1} + \Delta M_{1n})(M_{2,n-1} + \Delta M_{2n}) \mid F_{n-1}\} - \langle M_1, M_2 \rangle_n$$

$$= E\{M_{1,n-1} M_{2,n-1} + M_{1,n-1} \Delta M_{2n} + M_{2,n-1} \Delta M_{1n} + \Delta M_{1n} \Delta M_{2n} \mid F_{n-1}\}$$

$$- \langle M_1, M_2 \rangle_{n-1} - E(\Delta M_{1n} \Delta M_{2n} \mid F_{n-1})$$

$$= M_{1,n-1} M_{2,n-1} + 0 + 0 + E(\Delta M_{1n} \Delta M_{2n} \mid F_{n-1})$$

$$- \langle M_1, M_2 \rangle_{n-1} - E(\Delta M_{1n} \Delta M_{2n} \mid F_{n-1})$$

$$= M_{1,n-1} M_{2,n-1} - \langle M_1, M_2 \rangle_{n-1}$$

Thus $M_1 M_2 - \langle M_1, M_2 \rangle$ is a mean zero martingale.

Further we have (cf. page 45 in ABG)

$$E(M_{1n} M_{2n} - [M_1, M_2]_n \mid F_{n-1})$$

$$= E(M_{1,n-1} M_{2,n-1} + M_{1,n-1} \Delta M_{2n} + M_{2,n-1} \Delta M_{1n} - [M_1, M_2]_{n-1} \mid F_{n-1})$$

$$= M_{1,n-1} M_{2,n-1} + 0 + 0 - [M_1, M_2]_{n-1}$$

$$= M_{1,n-1} M_{2,n-1} - [M_1, M_2]_{n-1}$$

Thus $M_1 M_2 - [M_1, M_2]$ is a mean zero martingale.

b) Since M_1 and M_2 are mean zero martingales, we have

$$\mathrm{Cov}(M_{1n}, M_{2n}) = E(M_{1n} M_{2n})$$

Further by the results in a) we have that

$$E(M_{1n} M_{2n} - \langle M_1, M_2 \rangle_n) = 0$$

$$E(M_1 M_2 - [M_1, M_2]_n) = 0$$

This gives

$$\mathrm{Cov}(M_{1n}, M_{2n}) = E \langle M_1, M_2 \rangle_n = E [M_1, M_2]_n$$

Exercise 2.8

Let M_1 and M_2 be mean zero martingales as in exercise 2.5 and let $H_1 = \{H_{10}, H_{20}, \dots\}$ and $H_2 = \{H_{20}, H_{21}, \dots\}$ be predictable processes. Then we have

$$\langle H_1 \bullet M_1, H_2 \bullet M_2 \rangle_n$$

$$= \sum_{s=1}^{n} E\{ \Delta(H_1 \bullet M_1)_s \cdot \Delta(H_2 \bullet M_2)_s \mid \mathcal{F}_{s-1} \}$$

$$= \sum_{s=1}^{n} E\{ H_{1s} \Delta M_{1s} \cdot H_{2s} \Delta M_{2s} \mid \mathcal{F}_{s-1} \}$$

$$= \sum_{s=1}^{n} H_{1s} H_{2s} \, E\{ \Delta M_{1s} \Delta M_{2s} \mid \mathcal{F}_{s-1} \}$$

$$= \sum_{s=1}^{n} H_{1s} H_{2s} \, \Delta \langle M_1, M_2 \rangle_s$$

In a similar manner we have that

$$[H_1 \bullet M_1, H_2 \bullet M_2]_n$$

$$= \sum_{s=1}^{n} \Delta(H_1 \bullet M_1)_s \cdot \Delta(H_2 \bullet M_2)_s$$

$$= \sum_{s=1}^{n} H_{1s} \Delta M_{1s} \cdot H_{2s} \cdot \Delta M_{2s}$$

$$= \sum_{s=1}^{n} H_{1s} H_{2s} \Delta[M_1, M_2]_s$$

Exercise 2.10

For two martingales M_1 and M_2 we have

$$(*) \qquad \langle M_1 + M_2 \rangle = \langle M_1 \rangle + \langle M_2 \rangle + 2\langle M_1, M_2 \rangle$$

and

$$(**) \qquad [M_1 + M_2] = [M_1] + [M_2] + 2[M_1, M_2]$$

These results may be proved from

the definition of predictable and optional variation processes, cf (2.19) and (2.20) and the discussion on the top of page 50.

We then consider two counting processes N_1 and N_2 that have no simultaneous jump. The corresponding martingales are M_1 and M_2. Then $N_1 + N_2$ is a counting process (since all jumps have size $+1$) and it follows from (2.42) that

$$[M_1 + M_2](t) = N_1(t) + N_2(t)$$
$$= [M_1](t) + [M_2](t).$$

Using (**) we then obtain

$$[M_1, M_2](t) = 0$$

Next the counting process $N_1 + N_2$ has intensity process $\lambda_1(t) + \lambda_2(t)$, where $\lambda_j(t)$ is the intensity process of $N_j(t)$; $j=1,2$.

Hence

$$\langle M_1 + M_2 \rangle(t) = \int_0^t (\lambda_1(u) + \lambda_2(u))\,du$$

$$= \int_0^t \lambda_1(u)\,du + \int_0^t \lambda_2(u)\,du$$

$$= \langle M_1 \rangle(t) + \langle M_2 \rangle(t)$$

Using (*) we then obtain

$$\langle M_1, M_2 \rangle(t) = 0$$

Exercise 2.11

This exercise generalizes the results (2.36) and (2.37) to inhomogeneous Poisson processes.

$N(t)$ is an inhomogeneous Poisson process with intensity $\lambda(t)$ (a deterministic function). Then $N(t)-N(s)$ is Poisson distributed with parameter $\int_s^t \lambda(u)du$, and $N(t)-N(s)$ is independent of $N(s)$. We introduce

$M(t) = N(t) - \int_0^t \lambda(u)du$

a) We have that $(t>s)$

$$E(M(t) \mid \mathcal{F}_s) = E(N(t) - \int_0^t \lambda(u)du \mid N(s))$$

$$= E(N(s) + N(t) - N(s) \mid N(s)) - \int_0^t \lambda(u)du$$

$$= N(s) + E(N(t) - N(s)) - \int_0^t \lambda(u)du$$

$$= N(s) + \int_s^t \lambda(u)du - \int_0^t \lambda(u)du$$

$$= N(s) - \int_0^s \lambda(u)du = M(s)$$

b) $E\{M^2(t) - \int_0^t \lambda(u)du \mid F_s\}$

$= E\{(N(t) - \int_0^t \lambda(u)du)^2 / N(s)\} - \int_0^t \lambda(u)du$

$= E\{N^2(t) - 2N(t)\int_0^t \lambda(u)du + (\int_0^t \lambda(u)du)^2 / N(s)\}$

$- \int_0^t \lambda(u)du$

$= E\{(N(s) + N(t) - N(s))^2 / N(s)\}$

$- 2\int_0^t \lambda(u)du \; E\{N(s) + N(t) - N(s) \mid N(s)\}$

$+ (\int_0^t \lambda(u)du)^2 - \int_0^t \lambda(u)du$

$= N^2(s) + 2\, N(s)\, E\{N(t) - N(s)\} + E\{(N(t) - N(s))^2\}$

$- 2\int_0^t \lambda(u)du \cdot N(s) - 2\int_0^t \lambda(u)du \; E\{N(t) - N(s)\}$

$+ (\int_0^t \lambda(u)du)^2 - \int_0^t \lambda(u)du$

$= N^2(s) + 2N(s)\int_s^t \lambda(u)du + \mathrm{Var}\,(N(t) - N(s)) + (E(N(t) - N(s)))^2$

$- 2N(s)\int_0^t \lambda(u)du - 2\int_0^t \lambda(u)du \cdot \int_s^t \lambda(u)du$

$+ (\int_0^t \lambda(u)du)^2 - \int_0^t \lambda(u)du$

$$= N^2(s) + 2N(s)\int_s^t \lambda(u)du + \int_s^t \lambda(u)du + \left(\int_s^t \lambda(u)du\right)^2$$

$$- 2N(s)\int_0^t \lambda(u)du - 2\int_0^t \lambda(u)du \int_s^t \lambda(u)du + \left(\int_0^t \lambda(u)du\right)^2 - \int_0^t \lambda(u)du$$

$$= N^2(s) - 2N(s)\int_0^s \lambda(u)du + \left(\int_0^t \lambda(u)du - \int_s^t \lambda(u)du\right)^2 - \int_0^s \lambda(u)du$$

$$= N^2(s) - 2N(s)\int_0^s \lambda(u)du + \left(\int_0^s \lambda(u)du\right)^2 - \int_0^s \lambda(u)du$$

$$= M^2(s) - \int_0^s \lambda(u)du$$

Thus $M^2(t) - \int_0^t \lambda(u)du$ is a martingale