SOLUTIONS TO EXERCISES WEEK 38

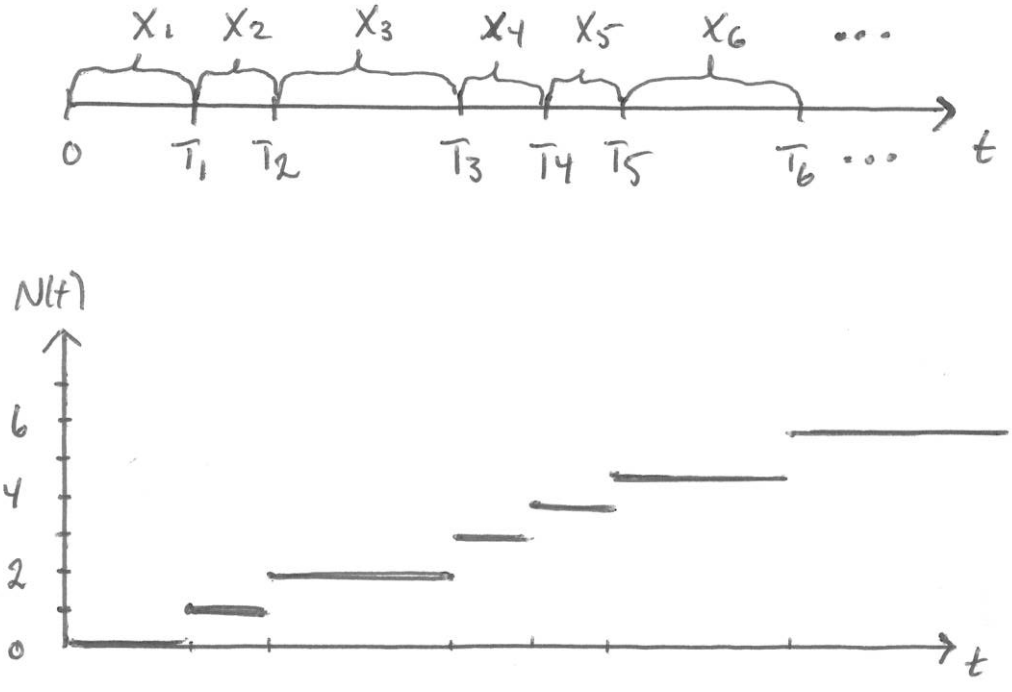
Exercise 1.8

$X_1, X_2, \dots$ are iid random variables with hazard rate $h(x)$. We consider the renewal process $T_n = X_1 + X_2 + \dots + X_n$; $n = 1, 2, \dots$

and let

$$N(t) = \sum_{n \geq 1} I(T_n \leq t)$$

Illustration:

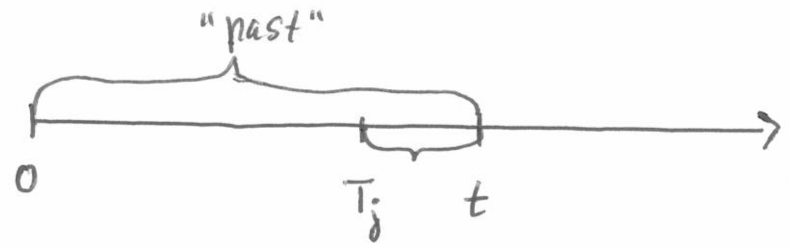


Intensity process $\lambda(t)$ of $N(t)$ is given by

$$\lambda(t)dt = P(dN(t) = 1 \mid \text{past})$$

From the past of time t we know the time elapsed since the last renewal. Illustration when

$T_j < t \leq T_{j+1}$:



We have that

$$P(dN(t) = 1 \mid \text{past},\ T_j < t \leq T_{j+1})$$

$$= P(t - T_j \leq X_{j+1} < t - T_j + dt \mid T_j < t \leq T_{j+1})$$

$$= h(t - T_j)\,dt$$

This gives

$$P(dN(t) = 1 \mid \text{past}) = h(t - T_{N(t-)})\,dt$$

and hence we have

$$\lambda(t) = h(t - T_{N(t-)})$$

Exercise 1.10

We have n independent survival times $T_1, T_2, .., T_n$, where T_i has hazard rate $\alpha_i(t)$. We introduce the counting processes $N_i(t) = I(T_i \le t)$.

a) We let $\mu_i(t)$ for $i=1,..,n$ be known hazard functions (corresponding to known population hazards) and introduce the aggregated counting process $N(t) = \sum_{i=1}^{n} N_i(t)$

By the result of example 1.17 on page 29 in the ABG-book, we have that $N_i(t)$ has intensity process

$$\lambda_i(t) = \alpha_i(t) \, I(T_i \ge t)$$

and that $N(t)$ has intensity process

$$\lambda(t) = \sum_{i=1}^{n} \lambda_i(t)$$

We then consider three choices of $\alpha_i(t)$

(i) $\alpha_i(t) = \alpha(t)$

Here we have

$$\lambda(t) = \alpha(t) \sum_{i=1}^{n} I(T_i \geq t)$$

(ii) $\alpha_i(t) = \mu_i(t)\,\alpha(t)$

Here we have

$$\lambda(t) = \alpha(t) \sum_{i=1}^{n} \mu_i(t)\, I(T_i \geq t)$$

(iii) $\alpha_i(t) = \mu_i(t) + \alpha(t)$

Here we have

$$\lambda(t) = \sum_{i=1}^{n} \mu_i(t)\, I(T_i \geq t) + \alpha(t) \sum_{i=1}^{n} I(T_i \geq t)$$

b) We see that situations (i) and (ii) of question a satisfies the multiplicative intensity model $\lambda(t) = \alpha(t) Y(t)$, while this is not the case for situation (iii).

Exercise 2.12

$W(t)$ is a Wiener process.
The increment $W(t)-W(s)$ over $(s,t]$ is normally distributed with mean zero and variance $t-s$, and increments over disjoint intervals are independent.

$V(t)$ is a strictly increasing continuous function with $V(0)=0$.

Define $U(t) = W(V(t))$ and let $\mathcal{F}_t$ be generated by $U(s)$ for $s \le t$.

a) We note that:

$$U(t) - U(s) = W(V(t)) - W(V(s))$$

is the increment of $W(\cdot)$ over the interval $(V(s), V(t)]$. Hence $U(t)-U(s)$ is normally distributed with mean zero and variance $V(t)-V(s)$.
Further the increments of $U(t)$ over disjoint intervals are independent.

Hence we have $(s<t)$

$$E(U(t) \mid \mathcal{F}_s) = E(U(t) \mid U(s))$$

$$= E(U(s) + U(t) - U(s) \mid U(s))$$

$$= U(s) + E(U(t) - U(s))$$

$$= U(s) + 0 = U(s)$$

Thus $U(t)$ is a martingale

b) We have

$$E(U^2(t) - V(t) \mid \mathcal{F}_s)$$

$$= E((U(s) + U(t) - U(s))^2 \mid U(s)) - V(t)$$

$$= E\{U^2(s) + 2U(s)(U(t) - U(s)) + (U(t) - U(s))^2 \mid U(s))$$

$$- V(t)$$

$$= U^2(s) + 2U(s)E(U(t) - U(s)) + E((U(t) - U(s))^2)$$
$$- V(t)$$

$$= U^2(s) + 2U(s) \cdot 0 + \mathrm{Var}(U(t) - U(s)) - V(t)$$

$$= U^2(s) + V(t) - V(s) + V(t)$$

$$= U^2(s) - V(s)$$

This shows that $U^2(t) - V(t)$ is a martingale, and this implies that $\langle U \rangle (t) = V(t)$.

Exercise 3.3

From section 3.1.6 we have that $\hat{A}(t)$ is approximately normally distributed with mean $A(t)$ and a variance that may be estimated by $\hat{\sigma}^2(t)$ given by (3.5).

a) Let g be a strictly increasing and continuously differentiable function. A Taylor series expansion gives that

$$g(\hat{A}(t)) \approx g(A(t)) + g'(A(t))(\hat{A}(t) - A(t))$$

Thus the distribution of $g(\hat{A}(t))$ is approximately the same as the distribution of

$$g(A(t)) + g'(A(t))(\hat{A}(t) - A(t))$$

Using the approximate normality of $\hat{A}(t)$, this is seen to be approximately normally distributed with mean $g(A(t))$ and variance $(g'(A(t))^2 \operatorname{Var} \hat{A}(t)$. The latter is estimated by $(g'(\hat{A}(t))^2 \hat{\sigma}^2(t)$. This shows the result.

b) By the above result we have that

$$\frac{g(\hat{A}(t)) - g(A(t))}{g'(\hat{A}(t))\,\hat{\sigma}(t)}$$

is approximately $N(0,1)$-distributed

It then follows by a standard argument that

$$g(\hat{A}(t)) \pm z_{1-\alpha/2}\, g'(\hat{A}(t))\,\hat{\sigma}(t)$$

is an approximate $100(1-\alpha)\%$ confidence interval for $g(A(t))$

c) With $g(x) = \log x$, we have $g'(x) = 1/x$. Thus an approximate $100(1-\alpha)\%$ confidence interval for $\log A(t)$ is

$$\log \hat{A}(t) \pm z_{1-\alpha/2} \frac{1}{\hat{A}(t)} \hat{\sigma}(t)$$

By exponentiating the lower and upper limit of this confidence interval, we get the following confidence interval for $A(t)$:

$$\exp\{\log \hat{A}(t) \pm z_{1-\alpha/2} \hat{\sigma}(t)/\hat{A}(t)\}$$

i.e.

$$\hat{A}(t) \cdot \exp\{\pm z_{1-\alpha/2} \hat{\sigma}(t)/\hat{A}(t)\}$$

cf (3.7) in ABG.