SOLUTIONS TO EXERCISES WEEK 40

Additional exercise 1

In general the Aalen-Johansen estimator is given by

$$\hat{P}(s,t) = \prod_{s<T_j\leq t} \left(I + \Delta\hat{A}(T_j)\right)$$

where

$$\Delta\hat{A}_{gh}(T_j) = \frac{\Delta N_{gh}(T_j)}{Y_g(T_j)}$$

and

$$\Delta\hat{A}_{gg}(T_j) = -\sum_{h\neq g} \Delta\hat{A}_{gh}(T_j) = -\frac{\Delta N_{g\cdot}(T_j)}{Y_g(T_j)}$$

where

$$\Delta N_{g\cdot}(T_j) = \sum_{h\neq g} \Delta N_{gh}(T_j)$$

a) For the competing risks model with $k=2$ causes of death, we have

$$I + \Delta\hat{A}(T_j) = \begin{pmatrix} 1+\Delta\hat{A}_{00}(T_j) & \Delta\hat{A}_{01}(T_j) & \Delta\hat{A}_{02}(T_j) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For convenience, we label the transitions between s and t as $T_1 < T_2 < \cdots < T_d$.

We will show that (3.69) and (3.70) are fullfilled, i.e that

$$\hat{P}_{00}(s, T_d) = \prod_{j=1}^{d} \left(1 - \frac{\Delta N_{0\bullet}(T_j)}{Y_0(T_j)}\right)$$

$$= \prod_{j=1}^{d} \left(1 + \Delta\hat{A}_{00}(T_j)\right) \qquad (*)$$

and (for $h=1,2$)

$$\hat{P}_{0h}(s, T_d) = \sum_{j=1}^{d} \hat{P}_{00}(s, T_{j-1}) \Delta\hat{A}_{0h}(T_j) \qquad (**)$$

We have that

$$\hat{P}(s, T_1) = I + \Delta \hat{A}(T_1)$$

which is given as above with $j=1$

Further we have

$$\hat{P}(s, T_2) = (I + \Delta \hat{A}(T_1))(I + \Delta \hat{A}(T_2))$$

$$= \begin{pmatrix} \prod_{j=1}^{2} (1 + \Delta \hat{A}_{00}(T_j)) & \Delta \hat{A}_{01}(T_1) + (1 + \Delta \hat{A}_{00}(T_1)) \Delta \hat{A}_{01}(T_2) & \Delta \hat{A}_{02}(T_1) + (1 + \Delta \hat{A}_{00}(T_1)) \Delta \hat{A}_{02}(T_2) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{P}_{00}(s, T_2) & \hat{P}_{00}(s, T_1) \Delta \hat{A}_{01}(T_2) & \hat{P}_{00}(s, T_1) \Delta \hat{A}_{02}(T_2) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus we have shown that for $l=1$ and $l=2$ we have [using that $P_{00}(0,T_{j-1})=1$ when $j=1$]

$$\hat{P}(s,T_l) = \begin{pmatrix} \hat{P}_{00}(s,T_l) & \sum_{j=1}^{l} \hat{P}_{00}(s,T_{j-1})\Delta\hat{A}_{01}(T_j) & \sum_{j=1}^{l} \hat{P}_{00}(s,T_{j-1})\Delta\hat{A}_{02}(T_j) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\#)$$

We now assume that (#) holds for all $l \leq m$. Then by performing the matrix multiplication

$$\hat{P}(s,T_{m+1}) = \hat{P}(s,T_m)\left(I + \Delta\hat{A}(T_{m+1})\right)$$

we see that (#) holds for $l=m+1$

Then (#) holds for all m by induction, and this proves (*) and (**).

b) For the illness-death model without recovery, we have

$$I + \Delta A(T_j) = \begin{pmatrix} 1+\Delta \hat{A}_{00}(T_j) & \Delta \hat{A}_{01}(T_j) & \Delta \hat{A}_{02}(T_j) \\ 0 & 1+\Delta \hat{A}_{11}(T_j) & \Delta \hat{A}_{12}(T_j) \\ 0 & 0 & 1 \end{pmatrix}$$

For convenience, we label the transitions between s and t as $T_1 < T_2 < \cdots < T_d$

We will show that (3.74), (3.75) and (3.76) are fullfilled, i.e. that

$$\hat{P}_{00}(s, T_d) = \prod_{j=1}^{d} \left(1 - \frac{\Delta N_{0\cdot}(T_j)}{Y_0(T_j)}\right)$$

$$= \prod_{j=1}^{d} \left(1 + \Delta \hat{A}_{00}(T_j)\right) \qquad (1)$$

$$\hat{P}_{11}(s, T_d) = \prod_{j=1}^{d} \left(1 - \frac{\Delta N_{12}(T_j)}{Y_1(T_j)}\right) = \prod_{j=1}^{d} \left(1 + \Delta \hat{A}_{11}(T_j)\right) \qquad (2)$$

and

$$\hat{P}_{01}(s, T_d) = \sum_{j=1}^{d} \hat{P}_{00}(s, T_{j-1}) \Delta \hat{A}_{01}(T_j) \hat{P}_{11}(T_j, T_d) \qquad (3)$$

We have that

$$\hat{P}(s, T_1) = I + \Delta \hat{A}(T_1)$$

which is given as above with $j=1$.

Further we have

$$P(s, T_2)$$

$$= P(s, T_1)\,(I + \Delta \hat{A}(T_2))$$

(continued on next page)

$$
= \begin{pmatrix}
\prod_{j=1}^{2} \left(1+\Delta\hat{A}_{00}(T_j)\right) & \left(1+\Delta\hat{A}_{00}(T_1)\right)\Delta\hat{A}_{01}(T_2) + \Delta\hat{A}_{01}(T_1)\left(1+\Delta\hat{A}_{11}(T_2)\right) & \Delta\hat{A}_{02}(T_1) + \left(1+\Delta\hat{A}_{00}(T_1)\right)\Delta\hat{A}_{02}(T_2) + \Delta\hat{A}_{01}(T_1)\Delta\hat{A}_{12}(T_2) \\
0 & \prod_{j=1}^{2} \left(1+\Delta\hat{A}_{11}(T_j)\right) & \Delta\hat{A}_{12}(T_1) + \left(1+\Delta\hat{A}_{11}(T_1)\right)\Delta\hat{A}_{12}(T_2) \\
0 & 0 & 1
\end{pmatrix}
$$

Thus we have shown that for $l=1$ and $l=2$ (with "empty products" set equal to 1):

$$\hat{P}(s,T_l) = \begin{pmatrix} \hat{P}_{00}(s,T_l) & \sum_{j=1}^{l} \hat{P}_{00}(s,T_{j-1})\Delta\hat{A}_{01}(T_j)\hat{P}_{11}(T_j,T_l) & \sum_{j=1}^{l} \{\hat{P}_{00}(s,T_{j-1})\Delta\hat{A}_{02}(T_j) + \hat{P}_{01}(s,T_{j-1})\Delta\hat{A}_{12}(T_j)\} \\ 0 & \hat{P}_{11}(s,T_l) & \sum_{j=1}^{l} \hat{P}_{11}(s,T_{j-1})\Delta\hat{A}_{12}(T_j) \\ 0 & 0 & 1 \end{pmatrix}$$

We now assume that the result above holds for all $l \le m$. Then by performing the matrix multiplication $\hat{P}(s,T_{m+1}) = \hat{P}(s,T_m)(I+\Delta\hat{A}(T_{m+1}))$ we see that the result holds for $l=m+1$. Thus the result holds for all m by induction, and this proves (1), (2) and (3)