SOLUTION TO EXERCISES WEEK 43

Additional exercise 2

We have counting processes $N_i(t)$ for $i=1,\dots,n$ with intensity processes

$$\lambda_i(t) = Y_i(t)\,\alpha_0(t)\, e^{\beta_0 x_i} \qquad (1)$$

a) By (4.7) in the ABG-book the partial likelihood is given by

$$L(\beta) = \prod_{T_j} \frac{e^{\beta x_{i_j}}}{\sum_{l\in R_j} e^{\beta x_l}}$$

where $R_j = \{l \mid Y_l(T_j)=1\}$ is the risk set at T_j. The logarithm of the partial likelihood becomes

$$l(\beta) = \log L(\beta) = \sum_{T_j} \Big\{ \beta x_{i_j} - \log\Big(\sum_{l\in R_j} e^{\beta x_l}\Big)\Big\}$$

Note that $\sum_{l\in R_j} e^{\beta x_l} = \sum_{l=1}^{n} Y_l(T_j)\, e^{\beta x_l} = S_0(\beta, T_j)$, where $S_0(\beta,t)$ is defined in the problem text. Hence we have that

$$l(\beta) = \sum_{T_j} \{ \beta X_{i_j} - \log S_0(\beta, T_j) \}$$

$$= \sum_{i=1}^{n} \int_0^{\tau} \{ \beta x_i - \log S_0(\beta, u) \} \, dN_i(u)$$

b) We note that $\frac{\partial}{\partial \beta} S_0(\beta,t) = S_1(\beta,t)$, where $S_1(\beta,t)$ is defined in the problem text. Thus we obtain

$$U(\beta) = l'(\beta) = \sum_{i=1}^{n} \int_0^{\tau} \{ x_i - \frac{S_1(\beta,u)}{S_0(\beta,u)} \} \, dN_i(u)$$

$$= \sum_{i=1}^{n} \int_0^{\tau} \{ x_i - E(\beta,u) \} \, dN_i(u) \qquad (2)$$

where $E(\beta,t)$ is defined in the text.

Differentiating once more and using that $\frac{\partial}{\partial \beta} S_1(\beta, t) = S_2(\beta, t)$ [cf. problem text] we obtain

$$I(\beta) = -U'(\beta) = \sum_{i=1}^{n} \int_0^{\tau} \frac{S_2(\beta,u)S_0(\beta,u) - S_1(\beta,u)^2}{S_0(\beta,u)^2} \, dN_i(u)$$

$$= \int_0^{\tau} \left\{ \frac{S_2(\beta,u)}{S_0(\beta,u)} - \left(\frac{S_1(\beta,u)}{S_0(\beta,u)} \right)^2 \right\} dN_{\bullet}(u)$$

$$= \int_0^{\tau} V(\beta,u) \, dN_{\bullet}(u) \qquad (3)$$

where $V(\beta, t)$ is given in the text and $N_{\bullet}(t) = \sum_{i=1}^{n} N_i(t)$.

c) For the counting process $N_i(t)$ we may use (1) to get the decomposition

$$dN_i(t) = \lambda_i(t)\,dt + dM_i(t)$$

$$= Y_i(t)\,\alpha_0(t)\, e^{\beta_0 x_i}\,dt + dM_i(t)$$

Note that for this decomposition to be valid, it is important that we use the true value β_0 of the regression coefficient in the intensity process.

If we use the decomposition in (2) we obtain

$$U(\beta_0) = \sum_{i=1}^{n} \int_0^{\tau} \{x_i - E(\beta_0, u)\} Y_i(u) e^{\beta_0 x_i} \alpha_0(u) du$$

$$+ \sum_{i=1}^{n} \int_0^{\tau} \{x_i - E(\beta_0, u)\} dM_i(u)$$

$$= \int_0^{\tau} \Big(\sum_{i=}^{n} Y_i(u) x_i e^{\beta_0 x_i} \Big) \alpha_0(u) du$$

$$- \int_0^{\tau} E(\beta_0, u) \Big(\sum_{i=1}^{n} Y_i(u) e^{\beta_0 x_i} \Big) \alpha_0(u) du$$

$$+ \sum_{i=1}^{n} \int_0^{\tau} \{x_i - E(\beta_0, u)\} dM_i(u)$$

$$= \int_0^\tau S_1(\beta_0, u) - \int_0^\tau \frac{S_1(\beta_0,u)}{S_0(\beta_0,u)} S_0(\beta_0,u)\,\alpha_0(u)\,du$$

$$+ \sum_{i=1}^n \int_0^\tau \{x_i - E(\beta_0,u)\}\, dM_i(u)$$

$$= \sum_{i=1}^n \int_0^\tau \{x_i - E(\beta_0,u)\}\, dM_i(u) \qquad (4)$$

d) We may write $U(\beta_0) = U(\tau, \beta_0)$, where

$$U(t,\beta_0) = \sum_{i=1}^n \int_0^t \{x_i - E(\beta_0,u)\}\, dM_i(u) \quad (5)$$

Note that $E(\beta_0,u) = S_1(\beta_0,u)/S_0(\beta,u)$ is a predictable process (since the at risk indicators are predictable) Hence (5) is a sum of stochastic integrals, and therefore a mean zero martingale.

By (2.48) in the ABG-book and (1), the predictable variation of (5) is given by

$$\langle U(\cdot,\beta_0)\rangle(t) = \sum_{i=1}^{n} \int_0^t \{x_i - E(\beta_0,u)\}^2 \lambda_i(u)\,du$$

$$= \sum_{i=1}^{n} \int_0^t \{x_i^2 - 2x_i E(\beta_0,u) + E(\beta_0,u)^2\} Y_i(u) e^{\beta_0 x_i} \alpha_0(u)\,du$$

$$= \int_0^t \Big(\sum_{i=1}^{n} Y_i(u) x_i^2 e^{\beta_0 x_i}\Big) \alpha_0(u)\,du$$

$$- 2\int_0^t \Big(\sum_{i=1}^{n} Y_i(u) x_i e^{\beta_0 x_i}\Big) E(\beta_0,u)\,\alpha_0(u)\,du$$

$$+ \int_0^t E(\beta_0,u)^2 \Big(\sum_{i=1}^{n} Y_i(u) e^{\beta_0 x_i}\Big) \alpha_0(u)\,du$$

$$= \int_0^t S_2(\beta_0,u)\alpha_0(u)\,du - 2\int_0^t S_1(\beta_0,u)\frac{S_1(\beta_0,u)}{S_0(\beta_0,u)}\alpha_0(u)\,du$$

$$+ \int_0^t \left(\frac{S_1(\beta_0,u)}{S_0(\beta_0,u)}\right)^2 S_0(\beta_0,u)\,\alpha_0(u)\,du$$

$$= \int_0^t \frac{S_2(\beta_0, u)}{S_0(\beta_0, u)} S_0(\beta_0, u) du$$

$$- \int_0^t \left(\frac{S_1(\beta_0, u)}{S_0(\beta_0, u)} \right)^2 S_0(\beta_0, u) du$$

$$= \int_0^t V(\beta_0, u) S_0(\beta_0, u) du$$

e) The aggregated counting process $N_\cdot(t) = \sum_{i=1}^n N_i(t)$ has intensity process [using (1)]

$$\lambda_\cdot(t) = \sum_{i=1}^n Y_i(t) e^{\beta_0 x_i} \alpha_0(t)$$

$$= S_0(\beta_0, t) \alpha_0(t)$$

We therefore have the decomposition

$$dN_\cdot(t) = \lambda_\cdot(t) dt + dM_\cdot(t) = S_0(\beta_0, t) \alpha_0(t) dt + dM_\cdot(t)$$

It therefore follows that the observed information (3) may be written

$$I(\beta_0) = \int_0^{\tau} V(\beta_0, u)\, dN_{\bullet}(u)$$

$$= \int_0^{\tau} V(\beta_0, u)\, S_0(\beta_0, u)\, \nu_0(u)$$

$$+ \int_0^{\tau} V(\beta_0, u)\, dM_{\bullet}(u)$$

$$= \langle U(\cdot, \beta_0) \rangle (\tau) + \int_0^{\tau} V(\beta_0, u)\, dM_{\bullet}(u)$$

f) We have that $V(\beta_0, t)$ is a predictable process. Hence $\int_0^{\tau} V(\beta_0, u)\, dM_{\bullet}(u)$ is a stochastic integral, and as such it has mean zero. By taking expectations, we therefore obtain from

The result in question e that

$$E\{I(\beta_0)\} = E\langle U(\cdot,\beta_0)\rangle(\tau) + 0$$

But by a general result [cf (2.24) in the ABG-book] we have that

$$E\langle U(\cdot,\beta_0)\rangle(\tau) = \mathrm{Var}\, U(\tau,\beta_0)$$

$$= \mathrm{Var}\, U(\beta_0)$$

Thus we have

$$E\{I(\beta_0)\} = \mathrm{Var}\, U(\beta_0),$$

i.e the expected information equals the variance of the score. This is a similar result as for the classical case of iid random variables; see e.g Section 7.4 in the book by Devore & Berk (the book used in STK 2120).

By the martingale central limit theorem one may show that $U(\beta_0)$ is approximately normally distributed with mean zero and a variance that is approximately equal to $I(\beta_0)$

g) We have that $\hat{\beta}$ is the value of β that maximizes $l(\beta)$. The value can also be found by solving $U(\beta) = 0$, so we have $U(\hat{\beta}) = 0$. We then obtain by a Taylor expansion of $U(\hat{\beta})$ around β_0 that

$$0 = U(\hat{\beta}) \approx U(\beta_0) + U'(\beta_0)(\hat{\beta} - \beta_0)$$

Now $I(\beta_0) = -U'(\beta_0)$, and we obtain

$$U(\beta_0) - I(\beta_0)(\tilde{\beta} - \beta_0) \approx 0 \qquad (6)$$

b) From the approximation (6) we have that

$$I(\beta_0)(\tilde{\beta} - \beta_0) \approx U(\beta_0)$$

or

$$\tilde{\beta} - \beta_0 \approx \frac{U(\beta_0)}{I(\beta_0)} \qquad (7)$$

We know that if $X \sim N(0, \sigma^2)$, then $X/\sigma^2 \sim N(0, 1/\sigma^2)$. Therefore we have that $U(\beta_0)/I(\beta_0)$ is approximately normal with mean zero and a variance that may be estimated by $1/I(\beta_0)$. By (7) this holds for $\tilde{\beta} - \beta_0$ as well.