SOLUTION TO EXERCISES WEEK 45

Exercise 5.1

We have counting processes $N_1(t), \ldots, N_n(t)$ with intensity processes $\lambda_i(t) = Y_i(t)\, e^{\beta}$ where $Y_i(t)$ is an at risk indicator $(i=1,\ldots,n)$.
We introduce $R_i(t) = \int_0^t Y_i(u)\,du$ and let
$N_\bullet(t) = \sum_{i=1}^n N_i(t)$ and $R_\bullet(t) = \sum_{i=1}^n R_i(t)$.

a) The log-likelihood is given by [cf. (5.5)]

$$l(\beta) = \sum_{i=1}^n \int_0^\tau \log \lambda_i(t)\, dN_i(t) - \int_0^\tau \lambda_\bullet(t)\,dt$$

By inserting $\lambda_i(t) = Y_i(t)\, e^{\beta}$ and $\lambda_\bullet(t) = Y_\bullet(t)\, e^{\beta}$ we find

$$l(\beta) = \sum_{i=1}^n \int_0^\tau \log(Y_i(t)\, e^{\beta})\, dN_i(t) - e^{\beta} \int_0^\tau Y_\bullet(t)\,dt$$

$$= \beta \sum_{i=1}^n \int_0^\tau dN_i(t) - e^{\beta} R_\bullet(\tau) + \text{konst}$$

$$= \beta\, N_\bullet(\tau) - e^{\beta} R_\bullet(\tau) + \text{konst}$$

The score function becomes

$$U(\beta) = l'(\beta) = N_\bullet(T) - e^{\beta} R_\bullet(T)$$

This gives for the MLE:

$$e^{\hat\beta} R_\bullet(T) = N_\bullet(T)$$

$$e^{\hat\beta} = \frac{N_\bullet(T)}{R_\bullet(T)} = \hat\nu$$

so $\hat\beta = \log \hat\nu$

b) By the general result for MLE for counting process models, we have that [cf section 5.1.4]

$$\hat\beta \overset{\text{approx}}{\sim} N(\beta, \sigma^2)$$

where $\hat\sigma^2 = 1/I(\hat\beta)$, where $I(\beta)$ is the observed information

$$I(\beta) = -U'(\beta) = -l''(\beta)$$

Thus $I(\beta) = e^{\beta} R_{\cdot}(r)$, so the variance may be estimated by

$$\hat{\sigma}^2 = \frac{1}{e^{\hat{\beta}} R_{\cdot}(r)} = \frac{1}{\frac{N_{\cdot}(r)}{R_{\cdot}(r)} R_{\cdot}(r)} = \frac{1}{N_{\cdot}(r)}$$

<u>Remark:</u> We may alternatively derive the result in questions a and b by the delta-method, which states that [cf. exercise 3.3] for a smooth function $g(\nu)$, we have that

$$g(\hat{\nu}) \approx g(\nu) + g'(\nu)(\hat{\nu} - \nu)$$

from where we obtain that

$$g(\hat{\nu}) \overset{\text{approx}}{\sim} N(g(\nu), [g'(\nu)]^2 \tau^2)$$

where τ^2 is the asymptotic variance of $\hat{\nu}$. By the result of example 5.1, we have that

τ^2 may be estimated by

$$\hat{\tau}^2 = \hat{\nu}^2 / N_0(\tau) = \hat{\nu} / R_0(\tau)$$

Thus we obtain that

$\hat{\beta} = \log \hat{\nu}$ is approximately normally distributed around $\beta = \log \nu$ with a variance that may be estimated by $[(\log \hat{\nu})']^2 \, \hat{\tau}^2 = \left(\frac{1}{\hat{\nu}}\right)^2 \frac{\hat{\nu}^2}{N_0(\tau)} = \frac{1}{N_0(\tau)}$

c) To obtain a confidence interval for $\nu = e^{\beta}$, we first derive a confidence interval for β, namely

$$\hat{\beta} \pm 1.96 \frac{1}{\sqrt{N_0(\tau)}}$$

By exponentiating the limits of this confidence interval, we obtain the following 95% confidence interval for ν:

$$e^{\hat{\beta} \pm 1.96/\sqrt{N_0(\tau)}} \quad \text{i.e} \quad \hat{\nu}\, e^{\pm 1.96/\sqrt{N_0(\tau)}}$$

Exercise 5.4

We have counting processes $N_1(t), \dots, N_n(t)$ with intensity processes $\lambda_i(t) = \nu Y_i(t)$

a) The log-likelihood is given by

$$l(\nu) = \sum_{i=1}^{n} \int_0^{\tau} \log \lambda_i(t)\, dN_i(t) - \int_0^{\tau} \lambda_\bullet(t)\, dt$$

$$= \sum_{i=1}^{n} \int_0^{\tau} \log(\nu Y_i(t))\, dN_i(t) - \nu \int_0^{\tau} Y_\bullet(t)\, dt$$

$$= \log \nu \cdot N_\bullet(\tau) - \nu R_\bullet(\tau) + \text{konst}$$

Hence the score function becomes

$$U(\nu) = l'(\nu) = \frac{N_\bullet(\tau)}{\nu} - R_\bullet(\tau)$$

b) At the true parameter value we have that $N_i(t) = \int_0^t \nu Y_i(t)\, dt + M_i(t)$ and hence

$$N_\bullet(\tau) = \nu R_\bullet(\tau) + M_\bullet(\tau)$$

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Thus the score evaluated at the true value of ν is given by

$$U(\nu) = \frac{N_0(T)}{\nu} - R_0(T)$$

$$= \frac{\nu R_0(T) + M_0(T)}{\nu} - R_0(T)$$

$$= R_0(T) + \frac{M_0(T)}{\nu} - R_0(T) = \frac{M_0(T)}{\nu}$$

From this it follows that

$$E(U(\nu)) = E\left(\frac{M_0(T)}{\nu}\right) = 0$$

and that the variance of the score equals

$$Var(U(\nu)) = \frac{1}{\nu^2} Var(M_0(T))$$

$$= \frac{1}{\nu^2} E\langle M_0\rangle(T)$$

$$= \frac{1}{\nu^2} E\left\{\int_0^T \lambda_0(t)dt\right\}$$

$$= \frac{1}{\nu^2} E\left\{ \nu \int_0^{\tau} Y_0(t)dt \right\}$$

$$= E(R_0(\tau))/\nu$$

c) The observed information is given by

$$i(\nu) = -U'(\nu) = \frac{N_0(\tau)}{\nu^2}$$

When evaluated at the true parameter value we have

$$i(\nu) = \frac{N_0(\tau)}{\nu^2} = \frac{\nu R_0(\tau) + M_0(\tau)}{\nu^2}$$

$$= \frac{R_0(\tau)}{\nu} + \frac{M_0(\tau)}{\nu^2}$$

and hence

$$E(i(\nu)) = E\left(\frac{R_0(\tau)}{\nu}\right)$$

i.e. the same as the variance of the score