Add. ex. 2: Aalen-Johansen & Competing risk

Assume only two competing risks with states 0 for alive, 1 for death by cause 1 and 2 for death by cause 2. Then the competing risk model estimator of the prob. of staying in state 0 at time t after starting in state 0 at $t = 0$ equals (3.69)

$$\hat{P}_{00}(t) = \prod_{s \leq t} [1 - \frac{dN_{01}(s) + dN_{02}(s)}{Y_0(s)}] = \prod_{s \leq t} [1 - \frac{dN_{0\bullet}(s)}{Y_0(s)}]$$

and the estimator of the probability of having moved to state j is given by (3.70)

$$\hat{P}_{0h}(t) = \int_s^t \hat{P}_{00}(s-) \frac{dN_{0h}(s)}{Y_0(s)}$$

Add. ex. 2: Aalen-Johansen & Competing risk, II

An estimated transition matrix is then given as the 3×3

$$\mathbf{P}^{\star}(0,t) = \begin{bmatrix} \hat{P}_{00}(t) & \hat{P}_{01}(t) & \hat{P}_{02}(t) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

since states 1 and 2 are absorbing.

We want to show that $\mathbf{P}^{\star}(0,t)$ is identical to the Aalen-Johansen estimator, with event times $t_1 < t_2 < \cdots t_k \leq t < t_{k+1}$

$$\hat{\mathbf{P}}(0,t) = \hat{\mathbf{P}}(0,t_1) \times \hat{\mathbf{P}}(t_1,t_2) \times \cdots \times \hat{\mathbf{P}}(t_{k-1},t_k)$$

where

$$\hat{\mathbf{P}}(t_{j-1},t_j) = \mathbf{I} + d\hat{\mathbf{A}}(t_j) = \begin{bmatrix} 1 - \frac{dN_{0\bullet}(t_j)}{Y_0(t_j)} & \frac{dN_{01}(t_j)}{Y_0(t_j)} & \frac{dN_{02}(t_j)}{Y_0(t_j)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Add. ex. 2: Aalen-Johansen & Competing risk, III

But for $t_1 \leq t < t_2$ we have $\hat{P}_{00}(t_1-) = 1$ and

$$\hat{P}_{00}(t) = 1 - \frac{dN_{0\bullet}(t_1)}{Y_0(t_1)}$$
$$\hat{P}_{0h}(t) = \hat{P}_{00}(t_1-)\frac{dN_{0h}(t_1)}{Y_0(t_1)} = \frac{dN_{0h}(t_1)}{Y_0(t_1)}$$

thus $\mathbf{P}^\star(t_1) = \hat{\mathbf{P}}(0, t_1)$. Furthermore, if $\mathbf{P}^\star(t_{k-1}) = \hat{\mathbf{P}}(0, t_{k-1})$,
$\hat{\mathbf{P}}(0, t_k) = \hat{\mathbf{P}}(0, t_{k-1}) \times \hat{\mathbf{P}}(t_{k-1}, t_k) = \hat{\mathbf{P}}(0, t_k-) \times \hat{\mathbf{P}}(t_k-, t_k) =$

$$\begin{bmatrix} \hat{P}_{00}(t_k-) & \hat{P}_{01}(t_k-) & \hat{P}_{02}(t_k-) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 - \frac{dN_{0\bullet}(t_k)}{Y_0(t_k)} & \frac{dN_{01}(t_k)}{Y_0(t_k)} & \frac{dN_{02}(t_k)}{Y_0(t_k)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Add. ex. 2: Aalen-Johansen & Competing risk, IV

Multiplying out this gives, $t_k \leq t < t_{k+1}$,

$$\hat{P}_{00}(t_k-)(1 - \frac{dN_{0\bullet}(t_k)}{Y_0(t_k)}) = \prod_{s \leq t}[1 - \frac{dN_{0\bullet}(s)}{Y_0(s)}] = \hat{P}_{00}(t_k) = \hat{P}_{00}(t)$$

and

$$\hat{P}_{00}(t_k-)\frac{dN_{0h}(t_k)}{Y_0(t_k)} + \hat{P}_{0h}(t_k-) = \int_0^{t_k} \hat{P}_{00}(s-)\frac{dN_{0h}(s)}{Y_0(s)} = \hat{P}_{0h}(t_k),$$

For $g > 1$ we will still have $\hat{P}_{gh}(t) = I(g = h)$.

Hence the matrix $\mathbf{P}^{\star}(t)$ consisting of ABG formulas (3.69) and (3.70) in the first line and furthermore one along and zeros outside the diagonal equals the Aalen-Johansen-estimator $\hat{\mathbf{P}}(t)$ for competing risks.

Add. ex. 2: Aalen-Johansen & Illness-Death, I

The formulas for the transition probabilities for the illness-death setting (ID) are given in (3.74-3.76) in ABK (well, some of them). They can also be written

$$\hat{P}_{00}(0,t) = \prod_{u \le t}[1 - \tfrac{dN_{0\bullet}(u)}{Y_0(u)}]$$
$$\hat{P}_{11}(0,t) = \prod_{u \le t}[1 - \tfrac{dN_{12}(u)}{Y_1(u)}]$$
$$\hat{P}_{01}(0,t) = \int_0^t \hat{P}_{00}(0,u-)\tfrac{dN_{01}(u)}{Y_0(u)}\hat{P}_{11}(u,t)$$
$$\hat{P}_{02}(0,t) = \int_0^t \hat{P}_{00}(0,u-)\tfrac{dN_{02}(u)}{Y_0(u)} + \int_0^t \hat{P}_{01}(0,u-)\tfrac{dN_{12}(u)}{Y_1(u)}$$
$$\hat{P}_{12}(t) = 1 - \hat{P}_{11}(t)$$

We want to verify that these transition probabilities are the same as those obtained with the Aalen-Johansen estimator

$$\hat{\mathbf{P}}(0,t) = \hat{\mathbf{P}}(0,t_1) \times \hat{\mathbf{P}}(t_1,t_2) \times \cdots \times \hat{\mathbf{P}}(t_{k-1},t_k)$$

Add. ex. 3: Aalen-Johansen & ID, II

The ID-transition probabilities on matrix form becomes, since death is absorbing and $\hat{P}_{22}(t) = 1$,

$$\mathbf{P}^{\star}(0,t) = \begin{bmatrix} \hat{P}_{00}(0,t) & \hat{P}_{01}(0,t) & \hat{P}_{02}(0,t) \\ 0 & \hat{P}_{11}(0,t) & \hat{P}_{12}(0,t) \\ 0 & 0 & 1 \end{bmatrix}$$

We may proceed as for the competing risk situation

- First we note that $\mathbf{P}^{\star}(0,0) = \mathbf{I} = \hat{\mathbf{P}}(0,0)$
- Assuming that $\mathbf{P}^{\star}(0,t_{k-1}) = \hat{\mathbf{P}}(0,t_{k-1})$ we want to establish that $\mathbf{P}^{\star}(0,t_k) = \hat{\mathbf{P}}(0,t_k)$

Since both matrices can change at event times t_k it is sufficient to consider only these.

Add. ex. 2: Aalen-Johansen & ID, III

As for competing risk $\hat{\mathbf{P}}(0,t_k) = \hat{\mathbf{P}}(0,t_{k-1}) \times \hat{\mathbf{P}}(t_{k-1},t_k) =$

$$\begin{bmatrix} \hat{P}_{00}(0,t_{k-1}) & \hat{P}_{01}(0,t_{k-1}) & \hat{P}_{02}(0,t_{k-1}) \\ 0 & \hat{P}_{11}(0,t_{k-1}) & \hat{P}_{12}(0,t_{k-1}) \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 - \frac{dN_{0\bullet}(t_k)}{Y_0(t_k)} & \frac{dN_{01}(t_k)}{Y_0(t_k)} & \frac{dN_{02}(t_k)}{Y_0(t_k)} \\ 0 & 1 - \frac{dN_{12}(t_k)}{Y_1(t_k)} & \frac{dN_{12}(t_k)}{Y_(t_k)} \\ 0 & 0 & 1 \end{bmatrix}$$

We only need to verify for component (1,1): $\hat{P}_{00}(0,t_k)$, component (1,3): $\hat{P}_{02}(0,t_k)$ and component (2,2): $\hat{P}_{11}(0,t_k)$ since the probabilities over each row will sum to 1.

For component (1,1) we get

$$\hat{P}_{00}(0,t_{k-1})(1 - \frac{dN_{0\bullet}(t_k)}{Y_0(t_k)}) = \prod_{0<s\leq t_k} [1 - \frac{dN_{0\bullet}(t_k)}{Y_0(t_k)}] = \hat{P}_{00}(t_k)$$

Add. ex. 2: Aalen-Johansen & ID, IV

Similarly for component (2,2):

$$\hat{P}_{11}(0,t_{k-1})(1-\frac{dN_{12}(t_k)}{Y_1(t_k)}) = \prod_{0<s\leq t_k}[1-\frac{dN_{12}(t_k)}{Y_1(t_k)}] = \hat{P}_{11}(0,t_k)$$

Then component (1,3) becomes

$$\hat{P}_{00}(0,t_{k-1})\tfrac{dN_{02}(t_k)}{Y_0(t_k)} + \hat{P}_{01}(t_{k-1})\tfrac{dN_{12}(t_k)}{Y(t_k)} + \hat{P}_{02}(t_{k-1})$$

$$= \int_0^{t_{k-1}} \hat{P}_{00}(0,u-)\tfrac{dN_{02}(u)}{Y_0(u)} + \hat{P}_{00}(t_{k-1})\tfrac{dN_{02}(t_k)}{Y_0(t_k)}$$

$$+ \int_0^{t_{k-1}} \hat{P}_{01}(0,u-)\tfrac{dN_{12}(u)}{Y_1(u)} + \hat{P}_{01}(t_{k-1})\tfrac{dN_{12}(t_k)}{Y(t_k)}$$

$$= \int_0^{t_k} \hat{P}_{00}(0,u-)\tfrac{dN_{02}(u)}{Y_0(u)} + \int_0^{t_k} \hat{P}_{01}(s,u)\tfrac{dN_{12}(u)}{Y_1(u)} = \hat{P}_{00}(0,t_k)$$