Additional exercise 1 Show the identity

$$\frac{\hat{S}(t)}{S^\star(t)} - 1 = -\int_0^t \frac{\hat{S}(s-)}{S^\star(s)} \frac{J(s)}{Y(s)} dM(s) \tag{1}$$

where $\hat{S}(t)$ is the Kaplan-Meier estimator for the survival function $S(t)$, $J(t) = I(Y(t) > 0$, $S^\star(t) = \exp(-\int_0^t J(s)\alpha(s)ds$ and notation otherwise as in the lectures and book.

The equation is a special case of Duhamel's equation (See Appendix A.1 of ABG, equation (A.12) on pg. 461).

Hint: It is sufficient to show that

$$\frac{\hat{S}(t-)}{S^\star(t-)} - \frac{\hat{S}(t)}{S^\star(t)} = +\frac{\hat{S}(t-)}{S^\star(t)} \frac{J(t)}{Y(t)} dM(t).$$

Solution: It is in fact sufficient to show the hint. Note that both function start in 0. Then we only need to show that the change is the same on both side of the equality.

For a function $V(t) = -\int_0^t h(s)ds$ we have that the change is $V(t-) - V(t) = -\int_0^{t-} h(s)ds - (-\int_0^t h(s)ds) = h(t)dt$ (and this explains the typo).

We consider the change in the left hand side. First:

$$[\frac{\hat{S}(t-)}{S^\star(t-)} - 1] - [\frac{\hat{S}(t)}{S^\star(t)} - 1] = \frac{\hat{S}(t-)}{S^\star(t-)} - \frac{\hat{S}(t)}{S^\star(t)}$$

and then

$$\begin{aligned}
\frac{\hat{S}(t-)}{S^\star(t-)} - \frac{\hat{S}(t)}{S^\star(t)} &= \frac{\hat{S}(t-)}{S^\star(t)} \left[\frac{S^\star(t)}{S^\star(t-)} - \frac{\hat{S}(t)}{\hat{S}(t-)}\right] \\
&= \frac{\hat{S}(t-)}{S^\star(t)} \left[\frac{\exp(-\int_0^t J(s)\alpha(s)ds)}{\exp(-\int_0^{t-} J(s)\alpha(s)ds)} - \frac{\hat{S}(t-)(1-dN(t)/Y(t))}{\hat{S}(t-)}\right] \\
&= \frac{\hat{S}(t-)}{S^\star(t)} \left[\exp(-J(t)\alpha(t)dt) - (1 - \frac{dN(t)}{Y(t)})\right] \\
&= \frac{\hat{S}(t-)}{S^\star(t)} \left[(1 - J(t)\alpha(t)dt) - (1 - \frac{dN(t)}{Y(t)})\right] \\
&= \frac{\hat{S}(t-)}{S^\star(t)} \frac{J(t)}{Y(t)} [dN(t) - Y(t)\alpha(t)dt] = \frac{\hat{S}(t-)}{S^\star(t)} \frac{J(t)}{Y(t)} dM(t)
\end{aligned}$$

where $J(t)\alpha(t)dt$ is an infinitesimal and hence we can replace $\exp(-J(t)\alpha(t)dt)$ by (the first order Taylor expansion) $1 - J(t)\alpha(t)dt$.