## Aalen's additive hazards model

$$\alpha(t|x) = \beta_0(t) + \beta(t)'x = \beta_0(t) + \sum_{j=1}^{p} \beta_j(t) x_j$$

Estimates of cumulative regression functions

$$B_j(t) = \int_0^t \beta_j(s) \mathrm{d}s$$

In counting process notation

$$dN_i(t) = Y_i(t)\beta_0(t)dt + Y_i(t) \sum_{j=1}^{p} \beta_j(t) x_{ij} dt + dM_i(t)$$

Interpreted as a linear regression model at each time $t$ with $dN_i(t)$ as responses, $dB_j(t) = \beta_j(t)dt$ as regression coefficients, $Y_i(t)x_{ij}$ as covariates

## Estimator i Aalen additive model, explicitly

Let the responses at time $t$ be given by

$$d\tilde{N}(t)^\top = (dN_1(t), dN_2(t), \ldots, dN_n(t))$$

and the design-matrix at time $t$

$$\tilde{X}(t) = \begin{bmatrix} Y_1(t) & Y_1(t)x_{11} & Y_1(t)x_{12} & \cdots & Y_1(t)x_{1p} \\ Y_2(t) & Y_2(t)x_{21} & Y_2(t)x_{22} & \cdots & Y_2(t)x_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_n(t) & Y_n(t)x_{n1} & Y_n(t)x_{n2} & \cdots & Y_n(t)x_{np} \end{bmatrix}$$

The increments $dB(t)$ are (usually) estimated by

$$d\hat{B}(t) = (\tilde{X}(t)^\top \tilde{X}(t))^{-1} \tilde{X}(t)^\top d\tilde{N}(t)$$

for $t$ such that $\tilde{X}(t)$ has full rank.

# Simple example

Two groups: $x = 0, 1$ in group 0 and 1 resp.

Calculates Nelson-Aalen estimates $\hat{A}_x(t)$ for both groups.

Under the additive model $\alpha_x(t) = \beta_0(t) + \beta_1(t)x$ a special case of the Aalen additive hazards estimator becomes

$$\hat{B}_0(t) = \hat{A}_0(t) \qquad \text{og} \qquad \hat{B}_1(t) = \hat{A}_1(t) - \hat{A}_0(t)$$

and we may also estimate Var$[\hat{B}_x(t)]$ by

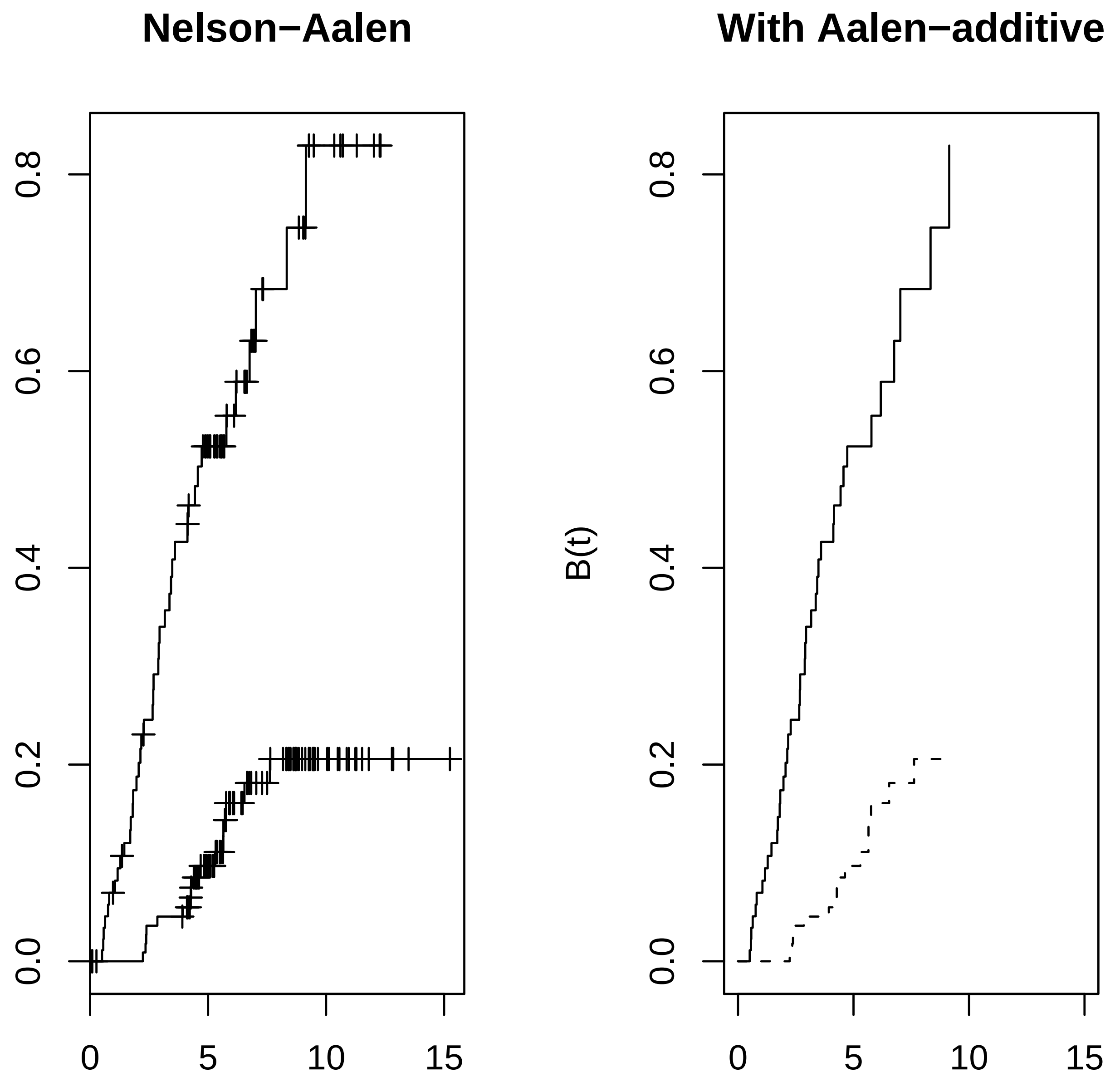
$$\widehat{\text{Var}}[\hat{B}_0(t)] = \int_0^t \frac{dN_{\bullet 0}(t)}{Y_{\bullet 0}(t)^2}$$

and

$$\widehat{\text{Var}}[\hat{B}_1(t)] = \int_0^t \frac{dN_{\bullet 0}(t)}{Y_{\bullet 0}(t)^2} + \int_0^t \frac{dN_{\bullet 1}(t)}{Y_{\bullet 1}(t)^2}$$

where $N_{\bullet x}(t)$ and $Y_{\bullet x}(t)$ are counting processes and number under risk in group $x = 0, 1$.

# On the melanoma-data with $x$ indicator for ulceration



# Exercise 4.4 ABG: Algebra for simple example

Design-matrix

$$\tilde{X}(t) = \begin{bmatrix} Y_1(t) & Y_1(t)x_{11} \\ Y_2(t) & Y_2(t)x_{21} \\ \vdots & \vdots \\ Y_n(t) & Y_n(t)x_{n1} \end{bmatrix}$$

which gives

$$\tilde{X}(t)^\top \tilde{X}(t) = \begin{bmatrix} \sum_{i=1}^n Y_i(t)^2 & \sum_{i=1}^n x_{i1} Y_i(t)^2 \\ \sum_{i=1}^n x_{i1} Y_i(t)^2 & \sum_{i=1}^n x_{i1}^2 Y_i(t)^2 \end{bmatrix}$$

$$= \begin{bmatrix} Y_{\bullet 0}(t) + Y_{\bullet 1}(t) & Y_{\bullet 1}(t) \\ Y_{\bullet 1}(t) & Y_{\bullet 1}(t) \end{bmatrix}$$

## Exercise. 4.4 ABG: Algebra for simple example, contd.

Determinant $= (Y_{\bullet 0}(t) + Y_{\bullet 1}(t))Y_{\bullet 1}(t) - Y_{\bullet 1}(t)^2 = Y_{\bullet 0}(t)Y_{\bullet 1}(t)$,

$$(\tilde{X}(t)^\top \tilde{X}(t))^{-1} = \begin{bmatrix} Y_{\bullet 1}(t) & -Y_{\bullet 1}(t) \\ -Y_{\bullet 1}(t) & Y_{\bullet 0}(t) + Y_{\bullet 1}(t) \end{bmatrix} \frac{1}{Y_{\bullet 0}(t)Y_{\bullet 1}(t)}$$

$$= \begin{bmatrix} \frac{1}{Y_{\bullet 0}(t)} & \frac{-1}{Y_{\bullet 0}(t)} \\ \frac{-1}{Y_{\bullet 0}(t)} & \frac{1}{Y_{\bullet 0}(t)} + \frac{1}{Y_{\bullet 1}(t)} \end{bmatrix}$$

Furthermore

$$\tilde{X}(t)^\top d\tilde{N}(t) = \begin{bmatrix} dN_{\bullet 0}(t) + dN_{\bullet 1}(t) \\ dN_{\bullet 1}(t) \end{bmatrix}$$

giving

$$d\hat{B}(t) = \begin{bmatrix} \frac{dN_{\bullet 0}(t)+dN_{\bullet 1}(t)}{Y_{\bullet 0}(t)} - \frac{dN_{\bullet 1}(t)}{Y_{\bullet 0}(t)} \\ \frac{-dN_{\bullet 0}(t)-dN_{\bullet 1}(t)}{Y_{\bullet 0}(t)} + \frac{dN_{\bullet 1}(t)}{Y_{\bullet 0}(t)} + \frac{dN_{\bullet 1}(t)}{Y_{\bullet 1}(t)} \end{bmatrix} = \begin{bmatrix} \frac{dN_{\bullet 0}(t)}{Y_{\bullet 0}(t)} \\ \frac{dN_{\bullet 1}(t)}{Y_{\bullet 1}(t)} - \frac{dN_{\bullet 0}(t)}{Y_{\bullet 0}(t)} \end{bmatrix}$$

## Ex. 4.4: Algebra for simple example, conclusion.

Thus, with the increments of Nelson-Aalen $d\hat{A}_x(t) = \frac{dN_{\bullet x}(t)}{Y_{\bullet x}(t)}$ in group $x$,

$$d\hat{B}(t) = \begin{bmatrix} d\hat{B}_0(t) \\ d\hat{B}_1(t) \end{bmatrix} = \begin{bmatrix} d\hat{A}_0(t) \\ d\hat{A}_1(t) - d\hat{A}_0(t) \end{bmatrix}$$

and Nelson-Aalen and the estimates in the Aalen additive hazards model coincides in this simple situation.

## Exercise 4.3

Model $\lambda_i(t) = Y_i(t)\beta_0(t)$. This is an additive regression model with just an intercept function and then regression functions $\beta_1(t) = \beta_2(t) = \cdots = \beta_p(t)$.

The model is also equivalent to a iid-model with hazard $\alpha(t) = \beta_0(t)$, thus the machinery of Aalen-additive regression should give the usual Nelson-Aalen estimator - and it does as shown below.

The (transpose of the) design matrix becomes just

$$\tilde{X}(t) = [Y_1(t), Y_2(t), \ldots, Y_n(t)]$$

and so

$$\tilde{X}(t)^\top \tilde{X}(t) = \sum_{i=1}^{n} Y_i(t)Y_i(t) = \sum_{i=1}^{n} Y_i(t) = Y_\bullet(t)$$

## Exercise 4.3, cont.

Furthermore

$$\tilde{X}(t)^\top d\tilde{N}(t) = \sum_{i=1}^{n} Y_i(t) dN_i(t) = \sum_{i=1}^{n} dN_i(t) = dN_\bullet(t),$$

thus the increment of $\hat{B}_0(t)$ equals

$$d\hat{B}_0(t) = d\hat{B}(t) = (\tilde{X}(t)^\top \tilde{X}(t))^{-1} \tilde{X}(t)^\top d\tilde{N}(t) = \frac{1}{Y_\bullet(t)} dN_\bullet(t),$$

i.e. the increment of the Nelson-Aalen estimator.