Exercise 2.11

This exercise generalizes the results (2.36) and (2.37) in the ABG-book to inhomogeneous Poisson processes.

Let $N(t)$ be an inhomogeneous Poisson process with intensity $\lambda(t)$. For $s < t$ we then have that $N(t) - N(s)$ is Poisson distributed with parameter $\int_s^t \lambda(u)\,\mathrm{d}u$ and that the number of events in disjoint time intervals are independent. It follows that $N(t)$ is a Markov process, so all information on the history $\mathcal{F}_s$ is contained in $N(s)$.

We introduce the process $M(t) = N(t) - \int_0^t \lambda(u)\,\mathrm{d}u$.

a) For $s < t$ we have that

$$\begin{aligned}
\mathrm{E}\left\{M(t) \,|\, \mathcal{F}_s\right\} &= \mathrm{E}\left\{N(t) - \int_0^t \lambda(u)\,\mathrm{d}u \,\middle|\, \mathcal{F}_s\right\} \\
&= \mathrm{E}\{N(s) + N(t) - N(s) \,|\, N(s)\} - \int_0^t \lambda(u)\,\mathrm{d}u \\
&= N(s) + \mathrm{E}\{N(t) - N(s)\} - \int_0^t \lambda(u)\,\mathrm{d}u \\
&= N(s) + \int_s^t \lambda(u)\,\mathrm{d}u - \int_0^t \lambda(u)\,\mathrm{d}u \\
&= N(s) - \int_0^s \lambda(u)\,\mathrm{d}u \\
&= M(s)
\end{aligned}$$

which shows that $M(t)$ is a martingale.

b) We want to show that $\mathrm{E}[M^2(t) - \int_0^t \lambda(s)ds|\mathcal{F}_s] = M^2(s) - \int_0^s \lambda(s)ds$.

But $\mathrm{E}[M^2(t) - \int_0^t \lambda(s)ds|\mathcal{F}_s]$ will equal

$\mathrm{E}[(M(t) - M(s) + M(s))^2 - \int_0^t \lambda(u)du|\mathcal{F}_s]$

$= \mathrm{E}[(M(t) - M(s))^2 + M(s)^2 - 2(M(t) - M(s))M(s) - \int_0^s \lambda(u)du - \int_s^t \lambda(u)du|\mathcal{F}_s]$

$= M^2(s) - \int_0^s \lambda(u)du$

because $\mathrm{E}[(M(t) - M(s))M(s)|\mathcal{F}_s] = M(s)\mathrm{E}[(M(t) - M(s))|\mathcal{F}_s] = 0$ since $M(t)$ is a martingale and because $\mathrm{E}[(M(t) - M(s))^2|\mathcal{F}_s] - \int_s^t \lambda(u)du = 0$ since

$$\begin{aligned}
\mathrm{E}[(M(t) - M(s))^2|\mathcal{F}_s] &= \mathrm{E}[(M(t) - M(s))^2] \\
&= \mathrm{Var}[(M(t) - M(s))] + (\mathrm{E}[(M(t) - M(s))])^2 \\
&= \textstyle\int_s^t \lambda(u)du + 0,
\end{aligned}$$

where the first equality follows since $M(t) - M(s)$ is independent of $\mathcal{F}_s$, the second from the general $\mathrm{E}(Z^2) = \mathrm{Var}(Z) + (\mathrm{E}Z)^2$ and the third from $\mathrm{E}[(M(t) - M(s))] = 0$ and $\mathrm{Var}[(M(t) - M(s))] = \mathrm{Var}[(N((t) - N(s))] = \int_s^t \lambda(u)du$.