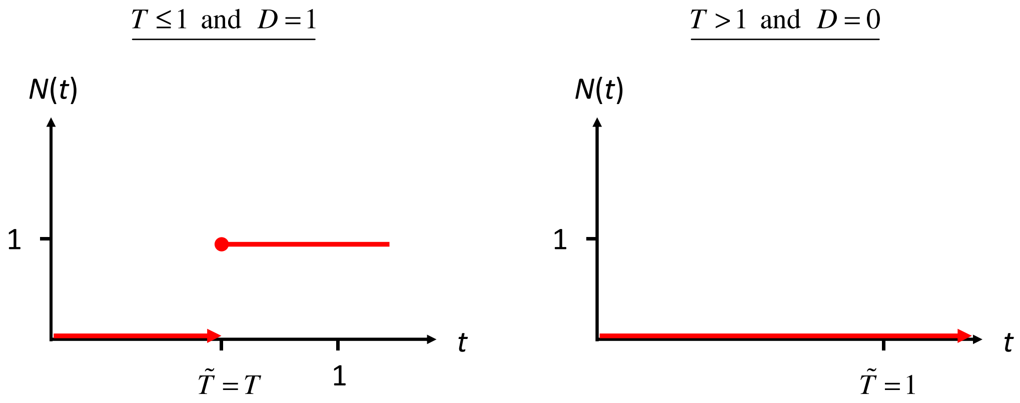
Solutions to exercises to Week 36

Exercise 1.6

T is exponentially distributed with hazard rate $\alpha(t) = 2$. We introduce the censored survival time $\widetilde{T} = \min(T, 1)$, the event indicator $D = I\{T \leq 1\}$, and the counting process $N(t) = I\{\widetilde{T} \leq t, D = 1\}$.

a



b) The intensity process $\lambda(t)$ is given by

$$\lambda(t)\,\mathrm{d}t = P(\,\mathrm{d}N(t) = 1 \,|\, \text{past}) = \begin{cases} \alpha(t)\,\mathrm{d}t & \text{for } \widetilde{T} \geq t \\ 0 & \text{for } \widetilde{T} < t \end{cases}$$

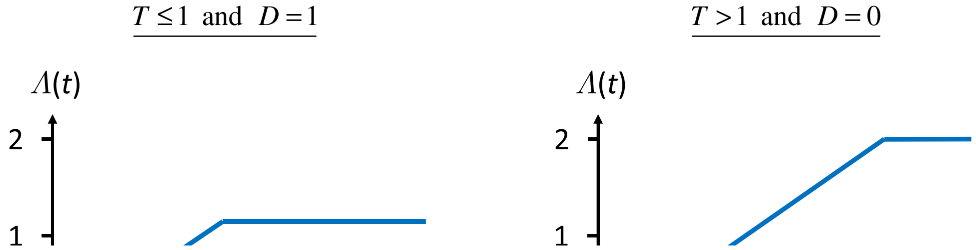
Thus we have

$$\lambda(t) = \alpha(t) I\{\widetilde{T} \geq t\} = 2 \cdot I\{\widetilde{T} \geq t\}$$

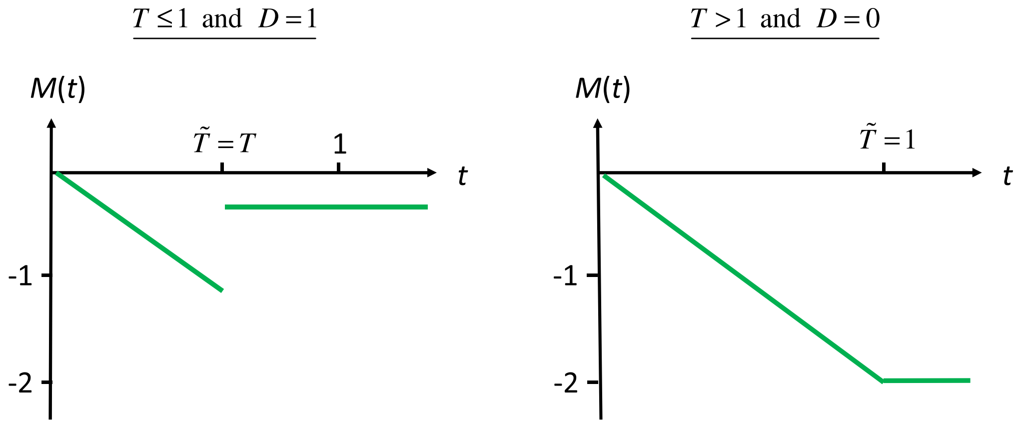
which gives

$$\Lambda(t) = \begin{cases} 2t & \text{for } t \leq \widetilde{T} \\ 2\widetilde{T} & \text{for } t > \widetilde{T} \end{cases}$$

Sketch of $\Lambda(t)$:



c)



Exercise 1.7

$X(t)$ is a birth- and death process with $X(0) = x_0 > 0$. Note that x_0 is the initial population size and $X(t)$ is the population size at time t. The birth intensity is ϕ and death intensity is μ, so that

$$P(X(t+dt) = k \,|\, X(t-) = j\,) = \begin{cases} j\phi\,\mathrm{d}t & k = j+1 \\ 1 - j(\phi+\mu)\,\mathrm{d}t & k = j \\ j\mu\,\mathrm{d}t & k = j-1 \end{cases}$$

We introduce the counting processes

$$\begin{aligned} N_b(t) &= \text{number of births in } [0,t] \\ N_d(t) &= \text{number of deaths in } [0,t] \end{aligned}$$

The intensity process $\lambda_b(t)$ of $N_b(t)$ is given by

$$\lambda_b(t)\,\mathrm{d}t = P(\,\mathrm{d}N_b(t) = 1 \,|\, \text{past})$$

From the past at time t, we know the value of $X(t-)$. Further since a birth- and death-process is a Markov process, once we know $X(t-)$ all the other information on the past is irrelevant, i.e.

$$\begin{aligned} & P\{\,\mathrm{d}N_b(t) = 1 \,|\, \text{past}, X(t-) = j\} \\ &= P\{\,\mathrm{d}N_b(t) = 1 \,|\, X(t-) = j\} \\ &= P\{X(t+\,\mathrm{d}t) = j+1 \,|\, X(t-) = j\} \\ &= j\phi\,\mathrm{d}t \end{aligned}$$

This gives

$$P(\,\mathrm{d}N_b(t) = 1 \,|\, \text{past}) = X(t-)\phi\,\mathrm{d}t$$

and hence

$$\lambda_b(t) = X(t-)\phi$$

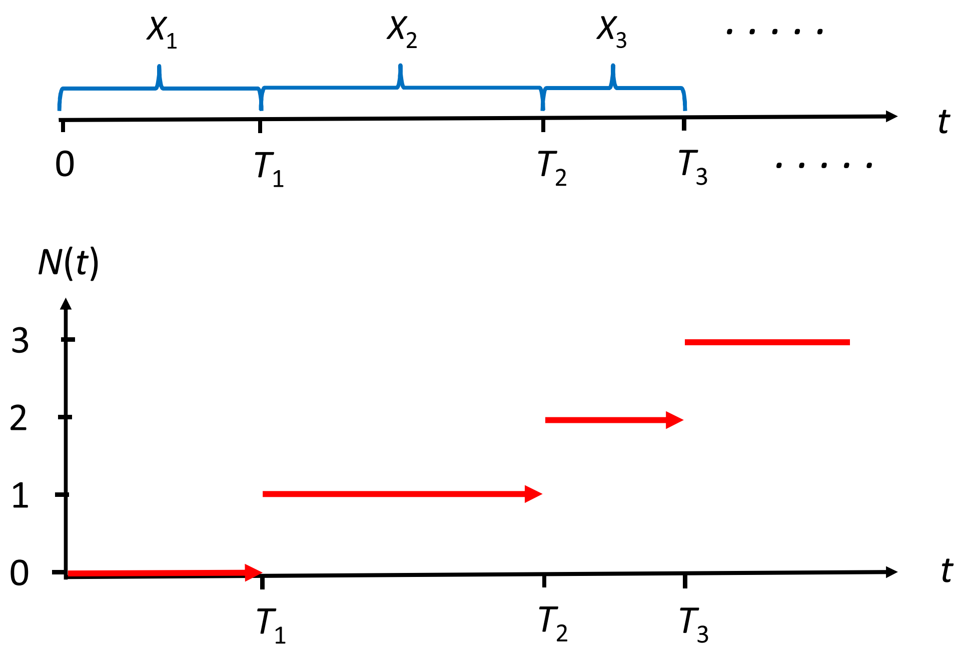
In a similar manner we find that the intensity process $\lambda_d(t)$ of $N_d(t)$ takes the form

$$\lambda_d(t) = X(t-)\mu$$

Exercise 1.8

$X_1, X_2, \ldots$ are *iid* random variables with hazard rate $h(x)$. We let $T_n = X_1 + \cdots + X_n$ for $n = 1, 2, \ldots$, and consider the renewal process $T = \{T_0, T_1, T_2, \ldots\}$, where $T_0 = 0$. Corresponding to the renewal process, we may define the counting process:

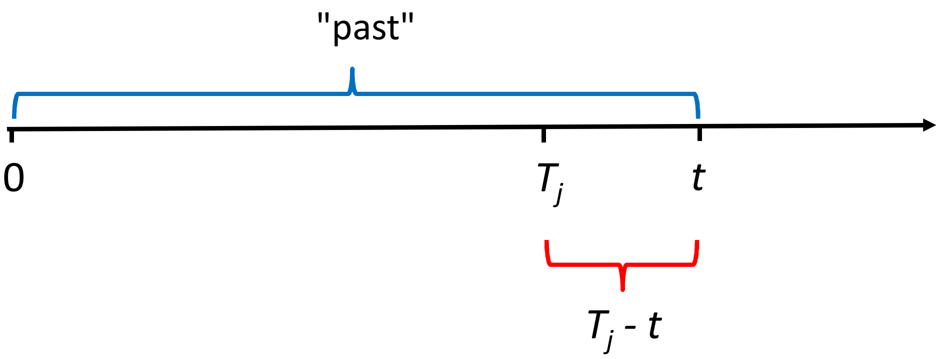
$$N(t) = \sum I\{T_n \leq t\}$$



The intensity process $\lambda(t)$ of $N(t)$ is given by

$$\lambda(t)\,\mathrm{d}t = P(\,\mathrm{d}N(t) = 1 \,|\, \text{past})$$

Note that from the past at time t we know the time of the last event (renewal), and hence the time elapsed since this event. Further, since the times between events are



Thus we have that

$$\begin{aligned}
&P(\,\mathrm{d}N(t) = 1 \,|\, \text{past}, T_j < t \leq T_{j+1}) \\
&\quad = \quad P(t \leq T_{j+1} \leq t + \,\mathrm{d}t \,|\, T_j, T_j < t \leq T_{j+1}) \\
&\quad = \quad P(t \leq T_j + X_{j+1} < t + \,\mathrm{d}t \,|\, T_j, X_{j+1} \geq t - T_j) \\
&\quad = \quad P(t - T_j \leq X_{j+1} < t - T_j + \,\mathrm{d}t \,|\, T_j, X_{j+1} \geq t - T_j) \\
&\quad = \quad h(t - T_j)\,\mathrm{d}t
\end{aligned}$$

This gives

$$P(\,\mathrm{d}N(t) = 1 \,|\, \text{past}) = h(t - T_{Nt-)})\,\mathrm{d}t$$

and hence we have

$$\lambda(t) = h(t - T_{Nt-)})$$

Exercise 1.9

T is a survival time with hazard rate $\alpha(t)$ and $v > 0$ is a constant. If we consider T conditional on $T > v$, we say that the survival time is *left-truncated*.

a) When $t > v$, the conditional survival function of T takes the form:

$$\begin{aligned} S(t \mid v) &= P(T > t \mid T > v) \\ &= \frac{P(T > t, T > v)}{P(T > v)} \\ &= \frac{P(T > t)}{P(T > v)} \\ &= \frac{\exp\left(-\int_0^t \alpha(u)\,\mathrm{d}u\right)}{\exp\left(-\int_0^v \alpha(u)\,\mathrm{d}u\right)} \\ &= \exp\left(-\int_v^t \alpha(u)\,\mathrm{d}u\right) \end{aligned}$$

If $t \leq v$, then $S(t \mid v) = 1$.

Thus the hazard rate of T, given $T > v$, becomes [cf. (1.4) in the ABG-book]:

$$\alpha(t \mid v) = -\frac{S'(t \mid v)}{S(t \mid v)} = \begin{cases} 0 & \text{for } t \leq v \\ \alpha(t) & \text{for } t > v \end{cases}$$

By the argument on page 28 in the ABG-book, it follows that conditional on $T > v$, the counting process $N(t) = I\{v < T \leq t\}$ has intensity process

$$\lambda(t) = \alpha(t \mid v)\, I\{T \geq t\} = \alpha(t)\, I\{v < t \leq T\}$$

Note that the intensity process is derived from the conditional distribution of T given $T > v$.

b) We then consider the left-truncated and right-censored survival time $\widetilde{T} = T \wedge u = \min(T, u)$ obtained by censoring the left-truncated survival time at $u > v$, and let $D = I\{\widetilde{T} = T\}$.

By the argument on page 31 in the ABG-book, it follows that the counting process $N(t) = I\{v < \widetilde{T} \leq t, D = 1\}$ has intensity process (derived from the conditional distribution of T given $T > v$):

$$\lambda(t) = \alpha(t \mid v)\, I\{\widetilde{T} \geq t\} = \alpha(t)\, I\{v < t \leq \widetilde{T}\}$$

Note that this is of the same form as (1.22) in the ABG-book, but with the at risk indicator given by $Y(t) = I\{v < t \leq \widetilde{T}\}$, not by (1.23).

Exercise 1.10

We have n independent survival times $T_1, \ldots, T_n$, where T_i has hazard rate $\alpha_i(t)$. We introduce the counting processes $N_i(t) = I\{T_i \leq t\}$; $i = 1, \ldots, n$.

a) By the result of example 1.17 on page 29 in the ABG-book, the counting processes $N_i(t)$ have intensity processes $(i = 1, \ldots, n)$

$$\lambda_i(t) = \alpha_i(t) I\{T_i \geq t\}$$

and the aggregated counting process $N(t) = \sum_{i=1}^n N_i(t)$ has intensity process

$$\lambda(t) = \sum_{i=1}^n \lambda_i(t) = \sum_{i=1}^n \alpha_i(t) I\{T_i \geq t\}$$

We let $\mu_i(t)$; $i = 1, \ldots, n$; be known hazard functions (corresponding to known population hazards).

(i) When $\alpha_i(t) = \alpha(t)$, we have

$$\lambda(t) = \sum_{i=1}^n \alpha(t) I\{T_i \geq t\} = \alpha(t) \sum_{i=1}^n I\{T_i \geq t\}$$

(ii) When $\alpha_i(t) = \mu_i(t)\alpha(t)$, we have

$$\lambda(t) = \sum_{i=1}^n \mu_i(t)\alpha(t) I\{T_i \geq t\} = \alpha(t) \sum_{i=1}^n \mu_i(t) I\{T_i \geq t\}$$

(iii) When $\alpha_i(t) = \mu_i(t) + \alpha(t)$, we have

$$\lambda(t) = \sum_{i=1}^n \{\mu_i(t) + \alpha(t)\} I\{T_i \geq t\} = \sum_{i=1}^n \mu_i(t) I\{T_i \geq t\} + \alpha(t) \sum_{i=1}^n I\{T_i \geq t\}$$

b) The aggregated counting process $N(t)$ satisfies the multiplicative intensity model if there exist an observable left-continuous process $Y(t)$ such that $\lambda(t) = \alpha(t)Y(t)$.

(i) The multiplicative intensity model is satisfied with $Y(t) = \sum_{i=1}^n I\{T_i \geq t\}$.

(ii) The multiplicative intensity model is satisfied with $Y(t) = \sum_{i=1}^n \mu_i(t) I\{T_i \geq t\}$.

(iii) The multiplicative intensity model is not satisfied.