UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Eksamen i	STK4150 solutions — Environmental and spatial statistics
Eksamensdag:	Friday 8. June 2012.
Tid for eksamen:	09.00 – 13.00.
Oppgavesettet er på 3 sider.	
Vedlegg:	??
Tillatte hjelpemidler:	??

Kontroller at oppgavesettet er komplett før du begynner å besvare spørsmålene.

Oppgave 1

(a) For any finite m and any set of points $(\boldsymbol{s}_1, ..., \boldsymbol{s}_m)$ $\boldsymbol{Y} = (Y(\boldsymbol{s}_1), ..., Y(\boldsymbol{s}_m))^T$ follows a multivariate Gaussian distribution.

The main advantage is the ease in modelling dependence through a covariance function. Furthermore, simplicities in the covariance structure (sparseness in precision matrix or covariance matrix, separability) can give great computational savings.

(b) A multivariate Poisson distribution is difficult to specify directly. This is however easy in a hierarchical setting. Further, separating the observation model from the underlying "physical" model can make it much easier to specify the spatial structure involved.

(c) We have

$$
\begin{aligned}
E[Z(\boldsymbol{s}_i)] =& E[E[Z(\boldsymbol{s}_i)|\boldsymbol{Y}] = E[\exp(Y(\boldsymbol{s}_i))] \\
=& \exp(\mu(\boldsymbol{s}_i) + \tfrac{1}{2}C_y^0(\mathbf{0})) \\
&\text{Cov}[Z(\boldsymbol{s}_i), Z(\boldsymbol{s}_j)] \\
=& E[\text{Cov}[Z(\boldsymbol{s}_i), Z(\boldsymbol{s}_j)|\boldsymbol{Y}] + \text{Cov}[E[Z(\boldsymbol{s}_i)|\boldsymbol{Y}], E[Z(\boldsymbol{s}_j)|\boldsymbol{Y}]] \\
=& \text{Cov}[\exp(Y(\boldsymbol{s}_i)), \exp(Y(\boldsymbol{s}_j))] \\
=& E[\exp(Y(\boldsymbol{s}_i) + Y(\boldsymbol{s}_j))] - \exp(\mu(\boldsymbol{s}_i) + \mu(\boldsymbol{s}_j) + C_y^0(\mathbf{0})) \\
=& \exp(\mu(\boldsymbol{s}_i) + \mu(\boldsymbol{s}_j) + C_y^0(\mathbf{0}) + C_y^0(||\boldsymbol{s}_j - \boldsymbol{s}_i||)) - \\
& \exp(\mu(\boldsymbol{s}_i) + \mu(\boldsymbol{s}_j) + C_y^0(\mathbf{0})) \\
=& \exp(\mu(\boldsymbol{s}_i) + \mu(\boldsymbol{s}_j) + C_y^0(\mathbf{0})[\exp(C_y^0(||\boldsymbol{s}_j - \boldsymbol{s}_i||)) - 1]
\end{aligned}
$$

(Fortsettes på side 2.)

(d) Yes, by assuming $\{\mathcal{D} = (\boldsymbol{s}_1, ..., \boldsymbol{s}_m)\}$ and by defining $C_y(\boldsymbol{s}_i, \boldsymbol{s}_j) = \text{Cov}(b_i, b_j)$ we obtain the model.

Including both an independent part and a spatial part makes it possible to see how important the two components are.

(e) Model 3 contain both the other models. However, in this model, the independent part has very high precision corresponding to a very low variance indicating that this term is negligible and not important. Model 2 is therefore preferable given that we want to choose a parsimonious model.

Oppgave 2

(a) We have

$$\begin{aligned}\sigma_Y^2 =&\mathsf{var}Y_t = a^2\mathsf{var}Y_{t-1} + \sigma_\varepsilon^2\\ =&a^2\sigma_Y^2 + \sigma_\varepsilon^2\end{aligned}$$

giving $\sigma_Y^2 = \sigma_\varepsilon^2/(1-a^2)$. We need $|a| < 1$.

(b) We have that

$$f(\boldsymbol{y}) = f(y_0)\prod_{t=1}^T f(y_t|y_{t-1}) = \exp(\log f(y_0) + \sum_{t=1}^T \log f(y_t|y_{t-1}))$$

showing that

$$f(y_t|\boldsymbol{y}_{-t}) \propto f(y_t|y_{t-1}))f(y_{t+1}|y_t))$$

so $\mathcal{N}_t = \{t-1, t+1\}$ with obvious corrections on the borders.

(c) We need that the eigenvalues of $\boldsymbol{M}$ are less than one in absolute value.

(d) We have that

$$\begin{aligned}p(\boldsymbol{Y}) =&p(\boldsymbol{Y}_0)\prod_{t=1}^T p(\boldsymbol{Y}_t|\boldsymbol{Y}_{0:t-1})\\ \log p(\boldsymbol{Y}) =&\log p(\boldsymbol{Y}_0) + \sum_{t=1}^T \log p(\boldsymbol{Y}_t|\boldsymbol{Y}_{0:t-1})\\ =&\text{Const} - \frac{1}{2}(\boldsymbol{Y}_1 - \boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{Y}_1 - \boldsymbol{\mu}_0)\\ &- \frac{1}{2}\sum_{t=1}^T(\boldsymbol{Y}_t - \boldsymbol{M}\boldsymbol{Y}_{t-1})^T\boldsymbol{\Sigma}_\varepsilon^{-1}(\boldsymbol{Y}_t - \boldsymbol{M}\boldsymbol{Y}_{t-1})\end{aligned}$$

(Fortsettes på side 3.)

from the model definition and using the multivariate density formula.

Now we get elements $\boldsymbol{Y}_t\boldsymbol{Q}_\varepsilon\boldsymbol{Y}_t$, $\boldsymbol{Y}_t\boldsymbol{Q}_\varepsilon\boldsymbol{M}\boldsymbol{Y}_{t-1}$ and $\boldsymbol{Y}_{t-1}\boldsymbol{M}^T\boldsymbol{Q}_\varepsilon\boldsymbol{M}\boldsymbol{Y}_{t-1}$ in the exponent, showing that node (i,t) has neighbours $\{(i,t-1),(i,t+1),\{(j,t),j\in\mathcal{N}_i\}$.

(e) We have that

$$\begin{aligned}\mathsf{cov}[\boldsymbol{Y}_t(\boldsymbol{s}_i),\boldsymbol{Y}_{t+\tau}(\boldsymbol{s}_j)] =&\mathsf{cov}[\boldsymbol{Y}_t(\boldsymbol{s}_i),\boldsymbol{M}\boldsymbol{Y}_{t+\tau-1}(\boldsymbol{s}_j)]+\boldsymbol{\varepsilon}_t]\\ =&\mathsf{cov}[\boldsymbol{Y}_t(\boldsymbol{s}_i),\boldsymbol{Y}_{t+\tau-1}(\boldsymbol{s}_j)]\boldsymbol{M}'\\ =&\mathsf{cov}[\boldsymbol{Y}_t(\boldsymbol{s}_i),\boldsymbol{Y}_{t+\tau-2}(\boldsymbol{s}_j)]\boldsymbol{M}'\boldsymbol{M}'\\ &\vdots\\ =&\boldsymbol{\Sigma}_Y(\boldsymbol{M}')^\tau\end{aligned}$$

which is not a separable covariance structure. However, if we look at the structure, we have

$$\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_Y\times\begin{pmatrix}\boldsymbol{I} & \boldsymbol{M}' & (\boldsymbol{M}')^2 & (\boldsymbol{M}')^3 & \cdots & (\boldsymbol{M}')^{T-2} & (\boldsymbol{M}')^{T-1}\\ \boldsymbol{M}' & \boldsymbol{I} & \boldsymbol{M}' & (\boldsymbol{M}')^2 & \cdots & (\boldsymbol{M}')^{T-3} & (\boldsymbol{M}')^{T-2}\\ (\boldsymbol{M}')^2 & \boldsymbol{M}' & \boldsymbol{I} & \boldsymbol{M}' & \cdots & (\boldsymbol{M}')^{T-4} & (\boldsymbol{M}')^{T-3}\\ (\boldsymbol{M}')^3 & (\boldsymbol{M}')^2 & \boldsymbol{M}' & \boldsymbol{I} & \cdots & (\boldsymbol{M}')^{T-5} & (\boldsymbol{M}')^{T-4}\\ \vdots & & & & \ddots & & \vdots\\ (\boldsymbol{M}')^{T-2} & (\boldsymbol{M}')^{T-3} & (\boldsymbol{M}')^{T-4} & (\boldsymbol{M}')^{T-5} & \cdots & \boldsymbol{I} & \boldsymbol{M}'\\ (\boldsymbol{M}')^{T-1} & (\boldsymbol{M}')^{T-2} & (\boldsymbol{M}')^{T-3} & (\boldsymbol{M}')^{T-4} & \cdots & \boldsymbol{M}' & \boldsymbol{I}\end{pmatrix}$$

which has as inverse

$$\boldsymbol{Q}=\begin{pmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{A}_2 & \boldsymbol{A}_3 & \boldsymbol{A}_2 & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0}\\ 0 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \boldsymbol{A}_2 & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0}\\ 0 & 0 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \boldsymbol{A}_2 & \cdots & \boldsymbol{0} & \boldsymbol{0}\\ \vdots & & & & & \ddots & & \vdots\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{A}_3 & \boldsymbol{A}_2\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{A}_2 & \boldsymbol{A}_1\end{pmatrix}\times\boldsymbol{\Sigma}_Y^{-1}$$

where

$$\begin{aligned}\boldsymbol{A}_1 =&[\boldsymbol{I}-(\boldsymbol{M}')^2]^{-1}\\ \boldsymbol{A}_2 =&-[\boldsymbol{I}-(\boldsymbol{M}')^2]^{-1}\boldsymbol{M}\\ \boldsymbol{A}_3 =&\boldsymbol{I}+2[\boldsymbol{I}-(\boldsymbol{M}')^2]^{-1}\boldsymbol{M}\end{aligned}$$

We see $\boldsymbol{Q}$ is sparse, which simplifies computation.

END