Exercise 10 (Solution)

(e). We have that

$$p(\boldsymbol{\theta}|\mathbf{z}) \propto p(\boldsymbol{\theta})p(\mathbf{z}|\boldsymbol{\theta}) = p(\boldsymbol{\theta})g(\mathbf{T};\boldsymbol{\theta})h(\mathbf{z})$$

showing that the posterior only depend on $\mathbf{z}$ through $\mathbf{T}$.

(f). We have

$$\begin{aligned} f(\mathbf{z};\boldsymbol{\theta}) =& \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\tfrac{1}{2\sigma^2}(z_i-\mu)^2\} \\ =& \frac{1}{(2\pi)^{m/2}\sigma^m} \exp\{-\tfrac{1}{2\sigma^2}\sum_{i=1}^{m}(z_i-\bar{z}+\bar{z}-\mu)^2\} \\ =& \frac{1}{(2\pi)^{m/2}\sigma^m} \exp\{-\tfrac{1}{2\sigma^2}[\sum_{i=1}^{m}(z_i-\bar{z})^2+m(\bar{z}-\mu)^2]\} \\ =& \frac{1}{(2\pi)^{m/2}\sigma^m} \exp\{-\tfrac{m}{2\sigma^2}[\hat{\sigma}^2+(\bar{z}-\mu)^2]\} \end{aligned}$$

which only depend on $(\bar{z}, \hat{\sigma}^2)$

(g). $p(\mu,\tau|\mathbf{z}) = p(\mu|\tau,\mathbf{z})p(\tau|\mathbf{z})$ is given directly by rules from probability. Since we in the first part considered σ^2 and therefore τ fixed, we have that $p(\mu|\tau,\mathbf{z})$ is as we showed in (d).

Also $p(\tau|\mathbf{z}) \propto p(\tau)p(\mathbf{z}|\tau)$ is given directly by probability rules.

(h). We have that $z_i = \mu + \varepsilon_i$ so that

$$\begin{aligned} E[z_i] =& E[\mu] + E[\varepsilon_i] = \mu_0 \\ \text{Var}[z_i] =& \text{Var}[\mu] + \text{Var}[\varepsilon_i] = k\sigma^2 + \sigma^2 = (k+1)\sigma^2 \\ \text{Cov}[z_i, z_j] =& \text{Cov}[\mu+\varepsilon_i, \mu+\varepsilon_j] = \text{Cov}[\mu,\mu] + \text{Cov}[\varepsilon_i,]\varepsilon_j] = k\sigma^2 \end{aligned}$$

which, together with that all the components of $\mathbf{z}$ are normal, gives the result.

(i). We have

$$\begin{aligned} [\mathbf{I}+k\mathbf{1}\mathbf{1}^T][\mathbf{I}-\tfrac{k}{1+km}\mathbf{1}\mathbf{1}^T] =& \mathbf{I} - \tfrac{k}{1+km}\mathbf{1}\mathbf{1}^T + k\mathbf{1}\mathbf{1}^T - \tfrac{k^2m}{1+km}\mathbf{1}\mathbf{1}^T \\ =& \mathbf{I} + \frac{-k+k+k^2m-k^2m}{1+km}\mathbf{1}\mathbf{1}^T = \mathbf{I} \end{aligned}$$

This gives directly the density given using the multivariate normal distribution from the previous question.

(j). We have that

$$\begin{aligned} p(\tau|\mathbf{z}) \propto& \tau^{a-1}e^{-b\tau}\tau^{m/2}\exp\{-0.5\tau(\mathbf{z}-\mu_0\mathbf{1})^T[\mathbf{I}-\tfrac{k}{1+km}\mathbf{1}\mathbf{1}^T](\mathbf{z}-\mu_0\mathbf{1})\} \\ \propto& \tau^{a+m/2-1}\exp\{-[b+0.5(\mathbf{z}-\mu_0\mathbf{1})^T[\mathbf{I}-\tfrac{k}{1+km}\mathbf{1}\mathbf{1}^T](\mathbf{z}-\mu_0\mathbf{1})]\tau\} \\ \propto& \tau^{\bar{a}-1}\exp\{-\bar{b}\tau\} \end{aligned}$$

with

$$\begin{aligned} \bar{a} =& a+m/2 \\ \bar{b} =& b+0.5(\mathbf{z}-\mu_0\mathbf{1})^T[\mathbf{I}-\tfrac{k}{1+km}\mathbf{1}\mathbf{1}^T](\mathbf{z}-\mu_0\mathbf{1}) \end{aligned}$$

which is proportional to the Gamma distribution with parameters $\bar{a}$ and $\bar{b}$. Since both $p(\tau|\mathbf{z})$ and this Gamma distributions need to integrate to one we get that $p(\tau|\mathbf{z})$ is equal to this Gamma distribution.