Exercise 6 (Solution)

(a). Directly given by definition

(b). We have

$$\begin{aligned} l(\mu,\sigma^2) =& -\tfrac{m}{2}\log(2\pi) - \tfrac{1}{2}\log|\sigma^2\mathbf{C}| - \frac{1}{2\sigma^2}(\mathbf{z}-\mu\mathbf{1})^T\mathbf{C}^{-1}(\mathbf{z}-\mu\mathbf{1}) \\ =& -\tfrac{m}{2}\log(2\pi) - \tfrac{1}{2}\log|\sigma^2\mathbf{I}| - \tfrac{1}{2}\log|\mathbf{C}| - \frac{1}{2\sigma^2}(\mathbf{z}-\mu\mathbf{1})^T\mathbf{C}^{-1}(\mathbf{z}-\mu\mathbf{1}) \\ =& -\tfrac{m}{2}\log(2\pi) - \tfrac{m}{2}\log(\sigma^2) - \tfrac{1}{2}\log|\mathbf{C}| - \frac{1}{2\sigma^2}(\mathbf{z}-\mu\mathbf{1})^T\mathbf{C}^{-1}(\mathbf{z}-\mu\mathbf{1}) \end{aligned}$$

giving

$$\frac{\partial}{\partial\mu}l(\mu,\sigma^2) = -\frac{1}{\sigma^2}(\mathbf{z}-\mu\mathbf{1})^T\mathbf{C}^{-1}\mathbf{1} = -\frac{1}{\sigma^2}\left[\mathbf{z}^T\mathbf{C}^{-1}\mathbf{1} - \mu\mathbf{1}\mathbf{C}^{-1}\mathbf{1}\right]$$

which, when putting to zero gives

$$\hat{\mu} = \frac{\mathbf{z}^T\mathbf{C}^{-1}\mathbf{1}}{\mathbf{1}\mathbf{C}^{-1}\mathbf{1}}$$

Further,

$$\frac{\partial}{\partial\sigma^2}l(\hat{\mu},\sigma^2) - \frac{m}{2\sigma^2} + \frac{1}{2\sigma^4}(\mathbf{z}-\hat{\mu}\mathbf{1})^T\mathbf{C}^{-1}(\mathbf{z}-\hat{\mu}\mathbf{1})$$

which, putting to zero gives

$$\hat{\sigma}^2 = \frac{1}{m}(\mathbf{z}-\hat{\mu}\mathbf{1})^T\mathbf{C}^{-1}(\mathbf{z}-\hat{\mu}\mathbf{1})$$

(c). For independent data, $\mathbf{C}=\mathbf{I}$, in which case

$$\begin{aligned} \hat{\mu} =& \frac{\mathbf{z}\mathbf{1}}{\mathbf{1}\mathbf{1}} = \frac{1}{m}\sum_i z_i = \bar{z} \\ \hat{\sigma}^2 =& \frac{1}{m}(\mathbf{z}-\bar{z}\mathbf{1})^T(\mathbf{z}-\bar{z}\mathbf{1}) = \frac{1}{m}\sum_i (z_i-\bar{z})^2 \end{aligned}$$

(d). Since $\mathbf{Z} \sim \text{MVN}(\mu\mathbf{1},\sigma^2\mathbf{C})$, we have $\widetilde{\mathbf{Z}} \sim \text{MVN}(\mu\mathbf{L}\mathbf{1},\sigma^2\mathbf{L}\mathbf{C}\mathbf{L}^T)$ and by using that $\mathbf{L}\mathbf{C}\mathbf{L}^T = \mathbf{L}\mathbf{L}^{-1}(\mathbf{L}^T)^{-1}\mathbf{L}^T = \mathbf{I}$, we obtain the result. Then we have independent data with expectations dependening on one regression parameter μ.

(e). Using $\mathbf{X}=\mathbf{L}\mathbf{1}$, we have ordinary estimates

$$\begin{aligned} \hat{\mu} =& (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\widetilde{\mathbf{Z}} = (\mathbf{1}^T\mathbf{L}^T\mathbf{L}\mathbf{1})^{-1}\mathbf{1}^T\mathbf{L}^T\mathbf{L}\mathbf{Z} \\ =& (\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1})^{-1}\mathbf{1}^T\mathbf{C}^{-1}\mathbf{Z} \\ \hat{\sigma}^2 =& \frac{1}{m}(\widetilde{\mathbf{Z}}-\hat{\mu}\mathbf{L}\mathbf{1})^T(\widetilde{\mathbf{Z}}-\hat{\mu}\mathbf{L}\mathbf{1}) = \frac{1}{m}(\mathbf{L}\mathbf{Z}-\hat{\mu}\mathbf{L}\mathbf{1})^T(\mathbf{L}\mathbf{Z}-\hat{\mu}\mathbf{L}\mathbf{1}) \\ =& \frac{1}{m}(\mathbf{L}\mathbf{Z}-\hat{\mu}\mathbf{1})^T\mathbf{L}^T\mathbf{L}(\mathbf{Z}-\hat{\mu}\mathbf{1}) = \frac{1}{m}(\mathbf{L}\mathbf{Z}-\hat{\mu}\mathbf{1})^T\mathbf{C}^{-1}(\mathbf{Z}-\hat{\mu}\mathbf{1}) \end{aligned}$$

(f). Weights 0.12718610.12718610.12657110.12657110.4924856. Reasonable that the close points are downweighted