# Chapter 4.2 - Lattice processes
# Lecture notes

Odd Kolbjørnsen Geir Storvik

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# Last Time

- Precision matrices
- Lattice process
- AR(1) as lattice process
- MRF – Markov random field
- Neighborhood
- Clique
- Undirected graph (MRF)
- Gaussian CAR – Conditional auto regressive (intro)
- Besag's lemma (conditional vs joint distribution)

# Today

- Lattice model and other models
- Neighborhood
- Clique
- Negpotential function
- Hammersley- Clifford theorem
- Auto spatial models
  - CAR model
  - Latent Gaussian process
  - Auto logistic model (Ising model)
  - auto Poisson model

Next time:

- How to construct CAR from scratch
- Examples of Models in INLA (disease mapping, etc)
- Examples of models outside INLA (Potts model)

## Smooth model

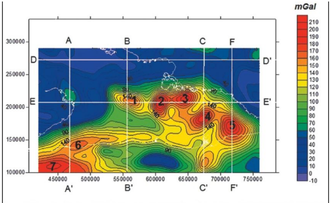
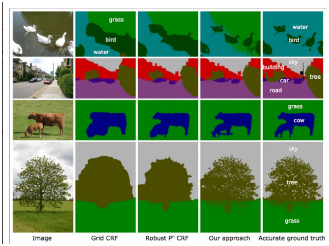
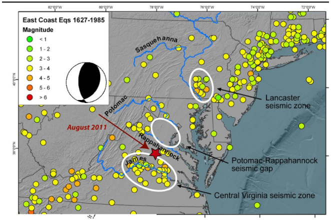



**Figure 3.** Complete Bouguer anomaly map for Southeastern Cuba island. Gravity anomaly highs are named; 1 Levingstone, 2 La Guira, 3 Piloto, 4 El Salvador, 5 La Perrera, 6 Eje Magmático Sur, 7 New oceanic crust coming from the Cayman dispersion center. Capital letters and lines indicate the six 2D cross-sections made to the 3D density model.

## Discrete Model (segmentation)




## Event observations



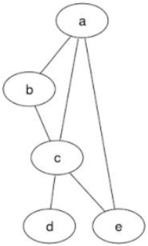
# Spatial models

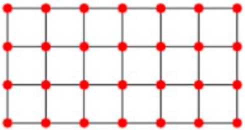
- Process model (What is modeled?)
  - Smooth phenomena (e.g. Gaussian models)
  - Discrete phenomena (e.g. Potts model)
  - Event models (e.g. Poisson process)
- Modeling approach (How is it modeled?)
  - Joint distribution (Continuous domain),
    e.g. Gaussian random field with a given Covariance function
  - Sequential approach ( Directed acyclic graph)
    e.g. Gaussian AR(1) model
  - Conditional approach/ Lattice model (Undirected graph)
    e.g. Gaussian Markov random fields on a lattice (Precission matrix)

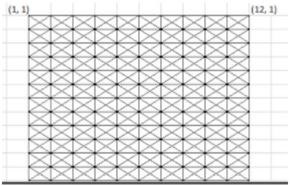
Want to make a statistical model of the process which is 1) consistent, 2) models important features in a good way, 3) we can do inference in.

# Lattice processes

- Aim: Construct model based on "local modeling" (Markov models)
- Neighbors are illustrated in undirected graphs






- Motivation:
  - Fast computations on large Gaussian models
  - Model non-Gaussian dependency in spatial models.
- Basic question. Does a joint distribution with the specified conditional distributions exist?

# Making a joint distribution from the conditionals

- Last time : Besag's Lemma = First piece of the puzzle. Relation between the joint distribution and the conditional approach. (Assuming the joint distribution exist)

$$\frac{\Pr(\mathbf{y})}{\Pr(\mathbf{w})} = \prod_{i=1}^{n} \frac{\Pr(y_i|y_1, ..., y_{i-1}, w_{i+1}, ..., w_n)}{\Pr(w_i|y_1, ..., y_{i-1}, w_{i+1}, ..., w_n)}$$

- Now: Hammersley - Clifford theorem. On the graph $(G)$ Graph = Nodes (=Lattice) + Edges (=Neighbors)

$$\Pr(\mathbf{y}|\boldsymbol{\theta}) = \frac{1}{c(\boldsymbol{\theta})} \exp\{Q(\boldsymbol{y};\boldsymbol{\theta})\} = \frac{1}{c(\boldsymbol{\theta})} \exp\left\{\sum_{c \in C_G} \psi_c(y_c, \boldsymbol{\theta})\right\}$$

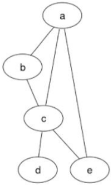
with $C_G$ being the set of all cliques, $Q(\mathbf{0}, \boldsymbol{\theta}) = 0$, and $c(\boldsymbol{\theta})$ is a normalizing constant.

# Neighbor/Neighborhood

On a countable set of indexes, i=1,2,...
(corresponding to the random variables on the lattice $Y_i$, $i = 1, 2, ...$)

- $i$ is not a neighbor to itself $i \notin \mathcal{N}(i)$
- if $j$ is a neighbor of $i$ then $i$ is a neighbor of $j$ , $j \in \mathcal{N}(i) \Leftrightarrow i \in \mathcal{N}(j)$

A neighborhood system defines an undirected graph, where there is a line form each (node) to each of its neighbors.




The maximum number of edges for a finite set of $n$ nodes are $\frac{n(n-1)}{2}$.
often it is $kn$, with $k << \frac{n-1}{2}$

# Cliques

- Clique: A set of locations (e.g. nodes, points in lattice) that consists either of a single site or a set of sites that are *all* neighbors to each other
- Example: $n = 3, \mathcal{N}(\mathbf{s}_1) = \{\mathbf{s}_2, \mathbf{s}_3\}, \mathcal{N}(\mathbf{s}_2) = \{\mathbf{s}_1\}, \mathcal{N}(\mathbf{s}_3) = \{\mathbf{s}_1\}$,
  Cliques: $\{\mathbf{s}_1\}, \{\mathbf{s}_2\}, \{\mathbf{s}_3\}, \{\mathbf{s}_1, \mathbf{s}_2\}, \{\mathbf{s}_1, \mathbf{s}_3\}$

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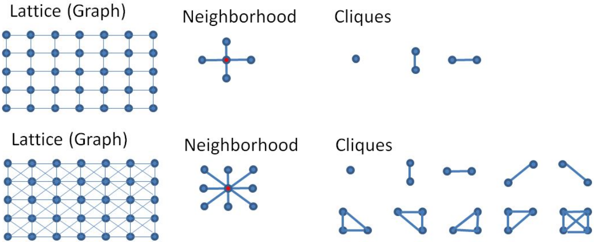
- Example: Lattice



# Negpotential function

- Assume $P(\mathbf{0}) > 0$ (always possible by defining "$\mathbf{0}$" appropriately)
- Define *Negpotential function*: $Q(\mathbf{y}) \equiv \log\{\Pr(\mathbf{y})/\Pr(\mathbf{0})\}$

# Negpotential function

- Assume $P(\mathbf{0}) > 0$ (always possible by defining "$\mathbf{0}$" appropriately)
- Define *Negpotential function*: $Q(\mathbf{y}) \equiv \log\{\Pr(\mathbf{y})/\Pr(\mathbf{0})\}$
- and expand it as:

$$Q(\mathbf{y}) = \sum_i y_i G_i(y_i) + \sum_{ij} y_i y_j G_{ij}(y_i, y_j) + \sum_{ijk} y_i y_j y_k G_{ijk}(y_i, y_j, y_k) + \cdots + y_1 y_2 \cdots y_m G_{1,\dots,m}(y_1, \dots, y_m)$$

# Terms in Negpotential function - expansion

The $G$-functions are defined/constructed through

$$\begin{aligned}
\mathbf{y} =&(0,\dots,0,y_i,0,\dots,0),\; y_i\neq 0\\
&\Rightarrow G_i(y_i) = Q(0,\dots,0,y_i,0,\dots,0)/y_i\\
&\Rightarrow Q(0,\dots,0,y_i,0,\dots,0) = y_iG_i(y_i) \qquad (*)\\
\\
\mathbf{y} =&(0,\dots,0,y_i,0,\dots,0,y_j,0,\cdots,0),\; y_i,y_j\neq 0\\
&\Rightarrow G_{i,j}(y_i,y_j) = [Q(0,\dots,0,y_i,0,\dots,0,y_j,0,\cdots,0)-y_iG_i(y_i)-y_jG_j(y_j)]/(y_iy_j)\\
&\Rightarrow Q(0,\dots,0,y_i,0,\dots,0,y_j,0,\cdots,0) = y_iG_i(y_i)+y_jG_j(y_j)+y_iy_jG_{i,j}(y_i,y_j)\\
&\vdots
\end{aligned}$$

# Hammersley-Clifford theorem

$$Q(\mathbf{y}) = \sum_i y_i G_i(y_i) + \sum_{ij} y_i y_j G_{ij}(y_i, y_j) + \sum_{ijk} y_i y_j y_k G_{ijk}(y_i, y_j, y_k) + \cdots + y_1 y_2 \cdots y_m G_{1,\ldots,m}(y_1, \ldots, y_m)$$

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Theorem: If $\{i_1, i_2, ..., i_k\}$ is not a clique, then $G_{i_1,i_2,\ldots,i_k}(\cdot) \equiv 0$

# Hammersley-Clifford theorem

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The hard thing here is to get all the definitions straight...

# Hammersley-Clifford theorem proof

If a set is not a clique: Then there exist (at least) two indices not being neighbors in this set. Assume these are $i_1 = i$ and $i_2 = l$.
Define $\mathbf{y}_i = (y_1, ..., y_{i-1}, 0, y_{i+1}, ..., y_m)$.

$$\frac{\Pr(\mathbf{y})}{\Pr(\mathbf{y}_i)} = \frac{\Pr(y_i|\mathbf{y}_{-i})}{\Pr(0|\mathbf{y}_{-i})} \qquad \text{do not depend on } y_l$$

$$\frac{\Pr(\mathbf{y})}{\Pr(\mathbf{y}_i)} = \exp(Q(\mathbf{y}) - Q(\mathbf{y}_i))$$

so $Q(\mathbf{y}) - Q(\mathbf{y}_i)$ do not depend on $y_l$, since this was the way we constructed it by the way we picked indices

# Proof (cont)

$$Q(\mathbf{y}) - Q(\mathbf{y}_i) = y_i G_i(y_i) + \sum_{j \neq i} y_i y_j G_{i,j}(y_i, y_j) + \sum_{j,k \neq i} y_i y_j y_k G_{i,j,k}(y_i, y_j, y_k) + \cdots$$

- The sum only include terms where $y_i$ is included since the others cancel out in the subtraction

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- The sum only include terms where $y_i$ is included since the others cancel out in the subtraction
- Putting $y_j = 0, j \neq i, l$ we get:
  - $y_i y_l G_{i,l}(y_i, y_l) = 0$ since this is only term depending on $y_l$

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  - Imply $y_i y_l G_{i,l}(y_i, y_l) = 0$ also for $y_l \neq 0$ and any $y_i$ ( since $l \notin \mathcal{N}(i)$ )

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  - Imply $G_{i,l}(y_i, y_l) = 0$

# Higher order

$$Q(\mathbf{y}) - Q(\mathbf{y}_i) = y_i G_i(y_i) + \sum_{j \neq i} y_i y_j G_{i,j}(y_i, y_j) + \sum_{j,k \neq i} y_i y_j y_k G_{i,j,k}(y_i, y_j, y_k) + \cdots$$

- Showed for all second order effects
- Similarly for triplets: Putting $y_j = 0, j \neq i, k, l$
- Using that $G_{i,l}(y_i, y_l) = 0$ for the non neighboring pair.
- ...
- implies $G_{i,k,l}(y_i, y_k, y_l) = 0$
- Iterating argument expanding to higher order 4th, 5th,... using at all of orders below are zero.
- Imply all $G$'s containing $i, l$ are zero.

# Existence of simultaneous distribution

Besag's lemma, Hammersley-Clifford: Results assuming simultaneous distribution exist.

Assume conditional distributions and

1. $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m$
2. The G-functions are invariant under permutation
3. $\sum_{\mathbf{t} \in \Omega} \exp\{Q(\mathbf{t})\} < \infty$

Then simultaneous distribution exist.

Condition 1 can be relaxed, see eq (4.116) in the book

# Auto-spatial models

- Auto-spatial models: Only second order terms

$$Q(\mathbf{y}) = \sum_i y_i G_i(y_i) + \sum_{ij} y_i y_j G_{ij}(y_i, y_j)$$

  - The CAR model
  - Latent Gaussian processes
  - auto Poisson model

Note: $\{i, j, k\}$ might be a clique but still $G_{ijk}(y_i, y_j, y_k) = 0$.

# CAR model, $\mu = 0$

$$\begin{aligned}
\log P(\mathbf{y}) =& -\frac{m}{2}\log(2\pi) - \frac{1}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}\mathbf{y}^T\boldsymbol{\Sigma}^{-1}\mathbf{y}\\
=& -\frac{m}{2}\log(2\pi) + \frac{1}{2}\log|\mathbf{Q}| - \frac{1}{2}\mathbf{y}^T\mathbf{Q}\mathbf{y}\\
=& -\frac{m}{2}\log(2\pi) + \frac{1}{2}\log|\mathbf{Q}| - \frac{1}{2}\sum_{ij} y_i Q_{ij} y_j\\
=& -\frac{m}{2}\log(2\pi) + \frac{1}{2}\log|\mathbf{Q}| - \frac{1}{2}\sum_{i} y_i^2 Q_{ii} - \frac{1}{2}\sum_{i}\sum_{j\in\mathcal{N}_i} y_i Q_{ij} y_j
\end{aligned}$$

giving

$$\begin{aligned}
\log P(\mathbf{0}) =& -\frac{m}{2}\log(2\pi) + \frac{1}{2}\log|\mathbf{Q}|\\
Q(\mathbf{y}) =& -\frac{1}{2}\sum_{i} y_i^2 Q_{ii} - \frac{1}{2}\sum_{i}\sum_{j\in\mathcal{N}_i} y_i Q_{ij} y_j\\
G_i(y_i) =& -\frac{1}{2}y_i Q_{ii} \quad G_{ij}(y_i, y_j) = -\frac{1}{2}Q_{ij}, \quad G_{ijk}(y_i, y_j, y_k) = 0
\end{aligned}$$

# CAR model, $\mu = 0$

$$
\begin{aligned}
P(\mathbf{y}) =& P(\mathbf{0})e^{\mathbf{Q}(\mathbf{y})} \\
Q(\mathbf{y}) =& -\frac{1}{2}\sum_i y_i^2 Q_{ii} - \frac{1}{2}\sum_{i\neq j} y_i Q_{ij} y_j \\
=& -\frac{1}{2}\sum_i y_i^2 Q_{ii} - \frac{1}{2}\sum_i \sum_{j\in\mathcal{N}_i} y_i Q_{ij} y_j
\end{aligned}
$$

Assume $Q_{ij} = 0$. Then

- $Q_{ji} = 0$
- $i \notin \mathcal{N}_j$ and $j \notin \mathcal{N}_i$
- If $\mathcal{N}_i$ are small for all $i$, then $\mathbf{Q}$ is sparse

# Latent Gaussian processes

Assume

$$\mathbf{y} \sim \text{MVN}(\mathbf{0}, \mathbf{Q}^{-1}) \qquad \mathbf{Q} \text{ sparse}$$

$$z_i|\mathbf{y} \sim f(z_i|y_i), \qquad \text{Cond. independent}$$

Then

$$p(\mathbf{y}|\mathbf{z}) \propto p(\mathbf{y})p(\mathbf{z}|\mathbf{y}) = p(\mathbf{y})\prod_i p(z_i|\mathbf{y}_i)$$

$$\log p(\mathbf{y}|\mathbf{z}) = \text{Const} - \frac{m}{2}\log(2\pi) + \frac{1}{2}\log|\mathbf{Q}| - \frac{1}{2}\sum_{ij} y_i Q_{ij} y_j + \sum_i \log f(z_i|y_i)$$

$$\begin{aligned}
Q(\mathbf{y}) &= \log p(\mathbf{y}|\mathbf{z}) - \log p(\mathbf{0}|\mathbf{z}) \\
&= -\frac{1}{2}\sum_{ij} y_i Q_{ij} y_j + \sum_i \log f(z_i|y_i) - \sum_i \log f(z_i|0) \\
&= -\frac{1}{2}\sum_i y_i^2 Q_{ii} + \sum_i \log f(z_i|y_i) - \sum_i \log f(z_i|0) - \frac{1}{2}\sum_{ij,i\neq j} y_i Q_{ij} y_j
\end{aligned}$$

$$y_i G_i(y_i) = -\frac{1}{2} y_i Q_{ii} y_i + \log f(z_i|y_i) - \log f(z_i|0)$$

$$G_{ij}(y_i) = -\frac{1}{2} Q_{ij}$$

# Finding $G$ functions and simultaneous distribution

Assume simultaneous distribution exist
For MRF

$$P(y_i|\mathbf{y}_{-i}) = \frac{P(\mathbf{y})}{P(\mathbf{y}_{-i})} \propto P(\mathbf{y}) \propto \exp(Q(\mathbf{y}))$$
$$\propto \exp\{y_i G_i(y_i) + y_i \sum_j y_j G_{ij}(y_i, y_j) + \cdots\}$$

Assume conditional distributions are specified.
Put $y_j = 0, j \neq i$ for finding $G_i(y_i)$
Put $y_k = 0, k \neq i, j$ for finding $G_{ij}(y_i, y_j)$
etc
$P(\mathbf{y}) \propto \exp(Q(\mathbf{y}))$

# Auto logistic

For auto-logistic, $y_i \in \{0, 1\}$

$$\Pr(y_i|\mathbf{y}_{-i}) = \frac{\exp\{\alpha_i y_i + \sum_{j\neq i}\theta_{ij}y_i y_j\}}{1 + \exp\{\alpha_i + \sum_{j\neq i}\theta_{ij}y_j\}}$$
$$\propto \exp\{\alpha_i y_i + \sum_{j\neq i}\theta_{ij}y_i y_j\}$$
$$G_i(y_i) = \alpha_i$$
$$G_{ij}(y_i, y_j) = \theta_{ij}$$
$$Q(\mathbf{y}) = \sum_i \alpha_i y_i + \sum_{i,j,i\neq j} y_i y_j \theta_{ij}$$
$$P(\mathbf{y}) \propto \exp\{\sum_i \alpha_i y_i + \sum_{i,j,i\neq j} y_i y_j \theta_{ij}\}$$

On regular lattice: *Ising model*

- $i = (u, v), \mathcal{N}_i = \{(u, v-1), (u, v+1), (u-1, v), (u+1, v)\}$
- $\theta_{ij} = 0$ for non-neighbors

# Auto Poisson

For auto-Poisson (misprint in eq (4.114) in the book)

$$
\begin{aligned}
P(y_i|y_{-i}) =& \text{Poisson}(\exp\{\alpha_i + \sum_{j=1}^{m} \theta_{ij} y_j\}) \qquad \theta_{ij} = 0, j \notin \mathcal{N}_i \\
=& \text{Poisson}(\exp\{\alpha_i + \sum_{j \in \mathcal{N}_i} \theta_{ij} y_j\}) \\
=& \exp\{y_i(\alpha_i + \sum_{j \in \mathcal{N}_i} \theta_{ij} y_j) - \log y_i! - e^{\alpha_i + \sum_{j \in \mathcal{N}_i} \theta_{ij} y_j}\} \\
\propto& \exp\{y_i \alpha_i - \log y_i! + y_i \sum_{j \in \mathcal{N}_i} \theta_{ij} y_j\}
\end{aligned}
$$

$$
\begin{aligned}
G_i(y_i) =& \alpha_i - y_i^{-1} \log y_i! \\
G_{ij}(y_i, y_j) =& \theta_{ij} \\
Q(\mathbf{y}) =& \sum_i (\alpha_i y_i - \log y_i!) + \sum_{i,j} y_i y_j \theta_{ij}
\end{aligned}
$$

# Auto models - general

Exponential family:

$$f_Y(y|\theta) = \exp\{(y\tilde{\theta} - a(\theta))/\phi + \log c(y;\phi)\}$$

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$$f_Y(y|\theta) = \exp\{(y\tilde{\theta} - a(\theta))/\phi + \log c(y;\phi)\}$$

Auto models (a bit more general in the book)

$$[Y_i|\mathbf{Y}_{-i}] \equiv \exp(A_i(\mathbf{Y}_{-i})(Y_i - B_i) + C_i(Y_i) + D_i(\mathbf{Y}_{-i}))$$

$$\tilde{\theta} = A_i(\mathbf{Y}_{-i}) = \alpha_i + \sum_{j \neq i} \theta_{ij}(Y_j - B_j)$$

$$Q(\mathbf{y}) = \sum_i [\alpha_i(y_i - B_i) + C_i(y_i)] + \sum_{i<j} \theta_{ij}(y_i - B_i)(y_j - B_j)$$

- auto Gaussian (CAR)
- auto gamma model
- Winsorized auto Poisson model
- auto beta model

# Auto models - general

Exponential family:

$$f_Y(y|\theta) = \exp\{(y\tilde{\theta} - a(\theta))/\phi + \log c(y;\phi)\}$$

Auto models ( in the book)

$$[Y_i|\mathbf{Y}_{-i}] \equiv \exp(A_i(\mathbf{Y}_{-i})(B_i(Y_i)) + C_i(Y_i) + D_i(\mathbf{Y}_{-i}))$$

$$\tilde{\theta} = A_i(\mathbf{Y}_{-i}) = \alpha_i + \sum_{j\neq i} \theta_{ij}(B_j(Y_j) - B_j(0))$$

$$Q(\mathbf{y}) = \sum_i [\alpha_i(B_i(y_i) - B_i(0)) + C_i(y_i)] + \sum_{i<j} \theta_{ij}(B_i(y_i) - B_i(0))(B_j(y_j) - B_j(0))$$

- auto Gaussian (CAR)
- auto gamma model
- Winsorized auto Poisson model (truncated at $y_i > t_i$)
- auto beta model