Chapter 6 - spatio-temporal models

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27. March 2017

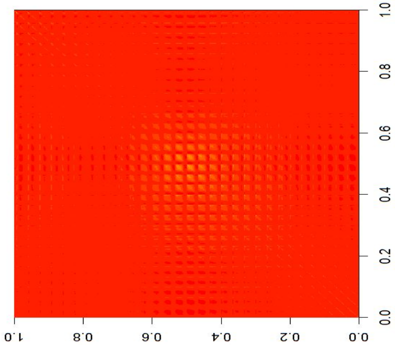
Last time Exploratory methods

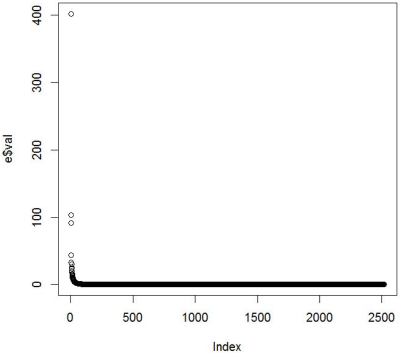
- Visualization
- Presentation of results
- Empirical Orthogonal functions
  $\mathbf{C}_Z = \mathbf{\Psi}\mathbf{\Lambda}^2\mathbf{\Psi}^T, \quad \mathbf{\Lambda}^2 = \text{diag}\{\lambda_i^2\} \quad \mathbf{\Psi} = [\psi_i, ..., \psi_m]$
  - Why ...
  - Efficient computation
  - Estimation of space functions (time coefficients)
  - Estimation of time functions (space coefficients)
- Space-Time Index (STI) used for permutation test in time space dependency.
- Conditional simulation using Kriging equations
- Conditioning to nonlinear functions of a random field
- Application of marked point processes

Empirical Orthogonal functions Why

Empirical Orthogonal functions

$$\mathbf{C}_Z = \mathbf{\Psi}\mathbf{\Lambda}^2\mathbf{\Psi}^T, \quad \mathbf{\Lambda}^2 = \text{diag}\{\lambda_i^2\} \quad \mathbf{\Psi} = [\psi_i, ..., \psi_m]$$





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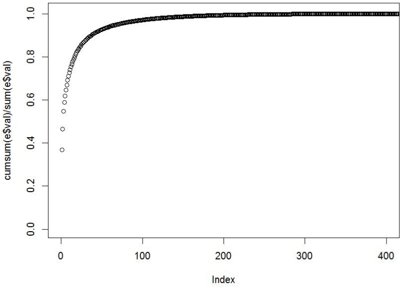

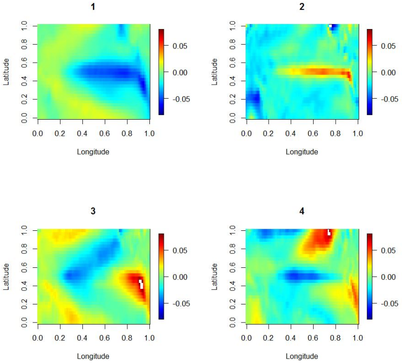

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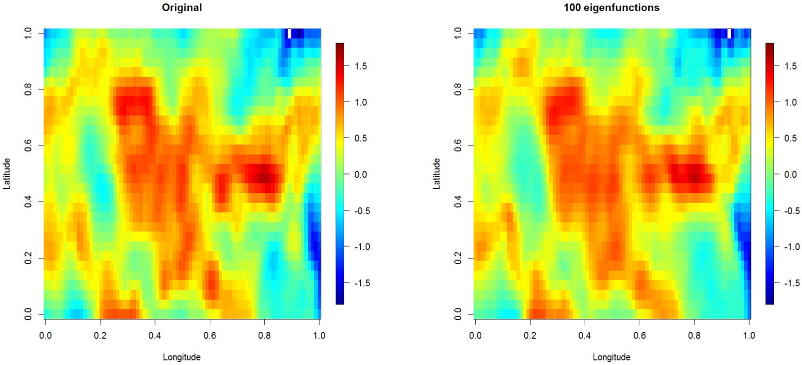

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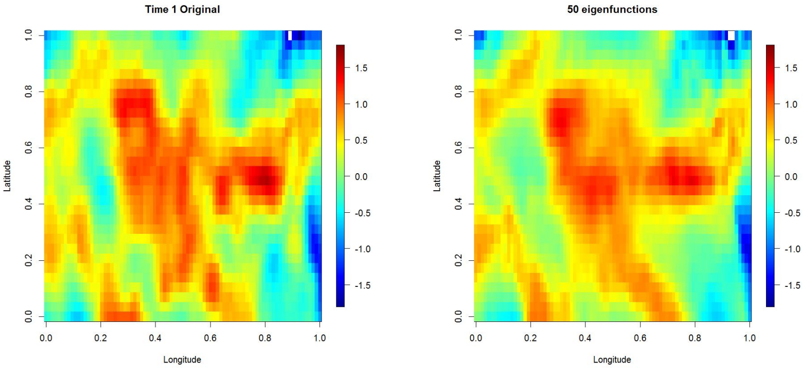

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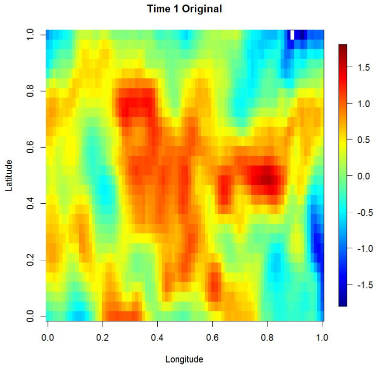

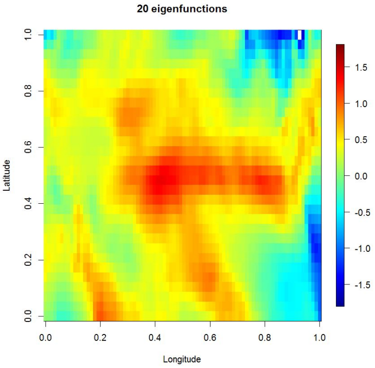

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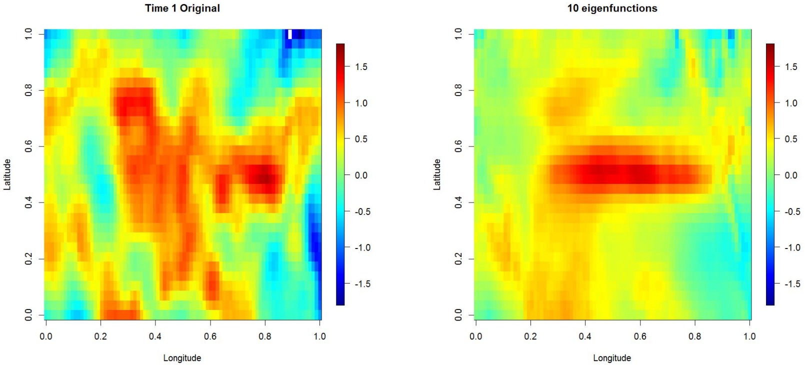

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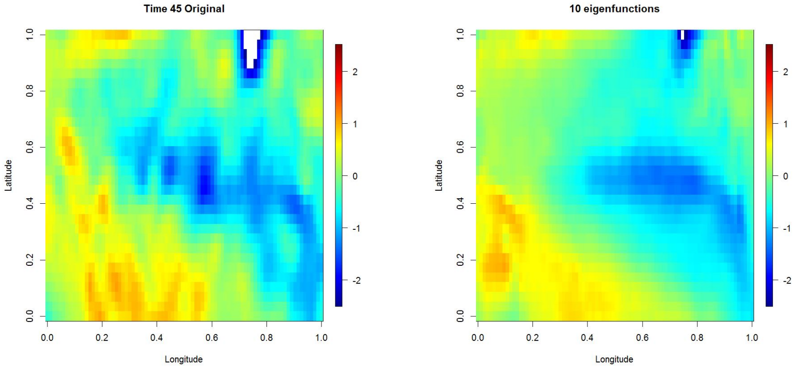

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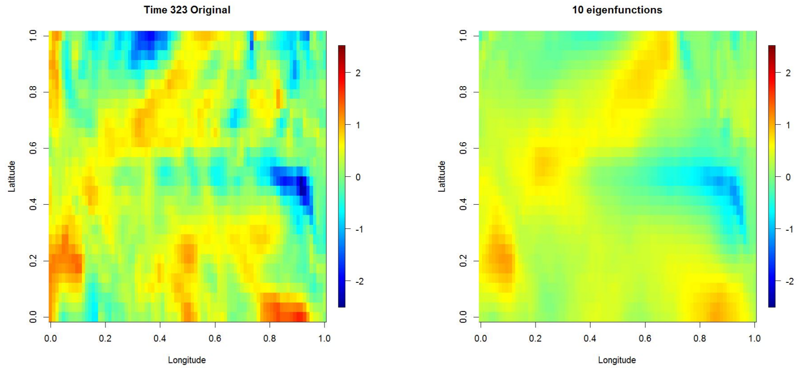

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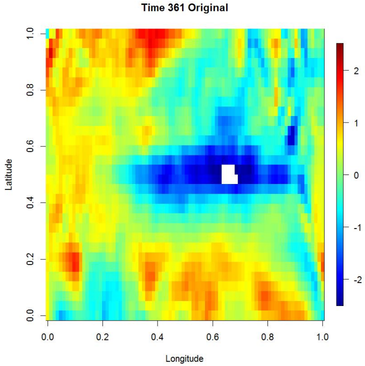

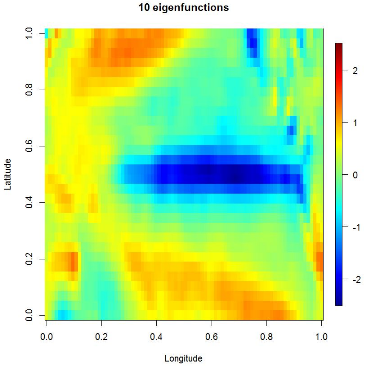

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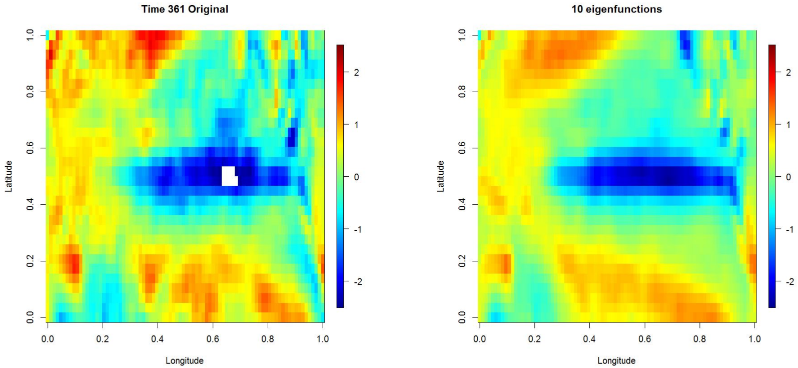
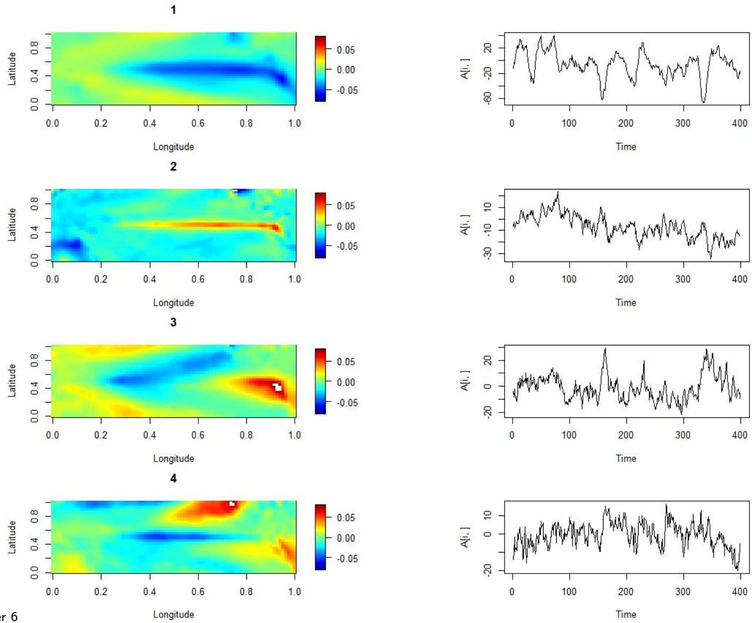
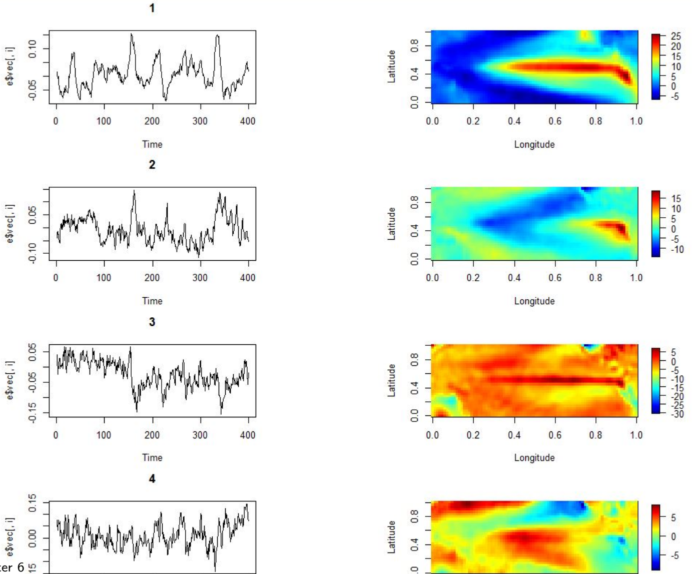


EOF: Coefficients of spatial eigenvectors create timeseries



EOF: Coefficients of temporal eigenvectors create spatial variations



Spatio-temporal data

In mathematical terms there is no difference between $\mathcal{R}^{d+1}$ "Spatial" random fields and $\mathcal{R}^d \times [0, \ \infty)$ "Time-space". The modeling will still be different because the features have different characteristics in time and space.

- Similar: Nearby values tend to be more alike than those far apart. (Near both in time and space)
- Difference: Order in time, no general order in space
- Time series: Two options
  - Describing covariance structure
  - Specifying dynamical model
- Same possibilities with spatio-temporal models
- Book: Emphasis on dynamical models
  - Use *conditional* distributions

$$[Y(\mathbf{s};t)|\{Y(\mathbf{x};r):\mathbf{x}\in D_s, r<t\}\cup\{Y(\mathbf{x},t):\mathbf{x}\neq s\}$$

Time ahead

Today modeling strategies:

- Spatio-temporal covariance functions (sec 6.1,6.2)
- Stochastic differential/difference equations (sec 6.3)
- Time series of spatial processes (sec 6.4)

3. April Ch 7 & 8

- Hierarchical spatio-temporal processes
- Data inference

24. April Ch 8 & 9

- Hierarchical spatio-temporal processes
- Data inference
- Example

8.May Ch 9. Project assignment available online.

- Hierarchical spatio-temporal processes
- Example

Time ahead

8.May Project assignment available online.
15.May no lecture
22.May Project assignment due.
22.May Course summary

30 May at 09:00 (4 hours). Written Exam

Spatio-temporal covariance functions

General model

$$\begin{aligned}
Y(\mathbf{s};t) &\equiv \mu(\mathbf{s};t) + \beta(\mathbf{s}) + \gamma(t) + \kappa(\mathbf{s};t) + \delta(\mathbf{s};t), \quad \mathbf{s}\in D_s, t\in D_t\\
\mu(\mathbf{s};t) &\equiv \text{fixed (covariate) term}\\
\text{cov}[\beta(\mathbf{s}),\beta(\mathbf{x})] &\equiv C_\beta(\mathbf{s},\mathbf{x})\\
\text{cov}[\gamma(t),\gamma(r)] &\equiv C_\gamma(t,r)\\
\text{cov}[\kappa(\mathbf{s};t),\kappa(\mathbf{x};r)] &\equiv C_\kappa(\mathbf{s},\mathbf{x};t,r)\\
\text{cov}[\delta(\mathbf{s};t),\delta(\mathbf{x};r)] &\equiv \sigma_\delta^2 I(\mathbf{s}=\mathbf{x}, t=r)
\end{aligned}$$

Many important special cases

Main focus now: Specification of $C_\kappa(\mathbf{s},\mathbf{x};t,r)$.

Spatio-temporal covariance functions

- Positive definiteness:
  - $\text{var}[\sum_i a_i Y(\mathbf{s}_i, t_i)] \geq 0$ imply
  - requirement $\sum_i \sum_j a_i a_j C((\mathbf{s}_i, t_i), (\mathbf{s}_j, t_j)) \geq 0$

  A function $C()$ satisfying this for any collection $\{Y(\mathbf{s}_i, t_i)\}$ and all $\{a_i\}$ is positive definite.
- Stationarity:

$$\begin{aligned} \text{cov}[Y(\mathbf{s};t), Y(\mathbf{x};r)] =& C(\mathbf{s}-\mathbf{x}, t, r), && \text{spatial stationarity} \\ \text{cov}[Y(\mathbf{s};t), Y(\mathbf{x};r)] =& C(\mathbf{s}, \mathbf{x}, t-r), && \text{temporal stationarity} \\ \text{cov}[Y(\mathbf{s};t), Y(\mathbf{x};r)] =& C(\mathbf{s}-\mathbf{x}, t-r), && \text{spatio-temporal stationarity} \end{aligned}$$

- Constructing positive definite functions are difficult (even under stationarity)

Separability

$$\text{cov}[Y(\mathbf{s};t),Y(\mathbf{x};r)] = C(\mathbf{s},t,\mathbf{x},r) = C^{(s)}(\mathbf{s},\mathbf{x})\cdot C^{(t)}(t,r)$$

$C^{(s)}(\mathbf{s},\mathbf{x})$ is legal spatial covariance function and $C^{(t)}(t,r)$ is a legal temporal covariance function imply product also legal
(exercise)
Assuming stationarity

$$C(\mathbf{s},t,\mathbf{x},r) = C(\mathbf{s}-\mathbf{x},t-r) = C^{(s)}(\mathbf{s}-\mathbf{x})\cdot C^{(t)}(t-r)$$

gives

$$\rho(\mathbf{h};\tau) = C(\mathbf{h};\tau)/C(\mathbf{0};0) = \rho(\mathbf{h};0)\rho(\mathbf{0};\tau)$$

so $\rho(\cdot;\tau)$ is proportional to $\rho(\cdot;0)$ for all τ.
Much used because of computational simplifications.

Separability and computation

Assume $C(\mathbf{s}-\mathbf{t},t-r)=C^{(s)}(\mathbf{s}-\mathbf{x})\cdot C^{(t)}(t-r), C^{(t)}(0)=1$
Assume observed $\{Y(s_i,t_j), i=1,...,m, j=1,...,n\}$.
$\mathbf{Y}_j=(Y(s_1,t_j),...,Y(s_m,t_j))'$, $\mathbf{Y}'=(\mathbf{Y}_1',...,\mathbf{Y}_n')$: We have

$$\begin{aligned}\text{Cov}[\mathbf{Y}] =&\boldsymbol{\Sigma}_0\otimes\boldsymbol{\Gamma}_0\\ \boldsymbol{\Sigma}_0 =&\text{Var}[\mathbf{Y}_j]\\ \boldsymbol{\Gamma}_0 =&\text{Var}[(Y(s_i,t_1),...,Y(s_i,t_n))']\end{aligned}$$

Gives

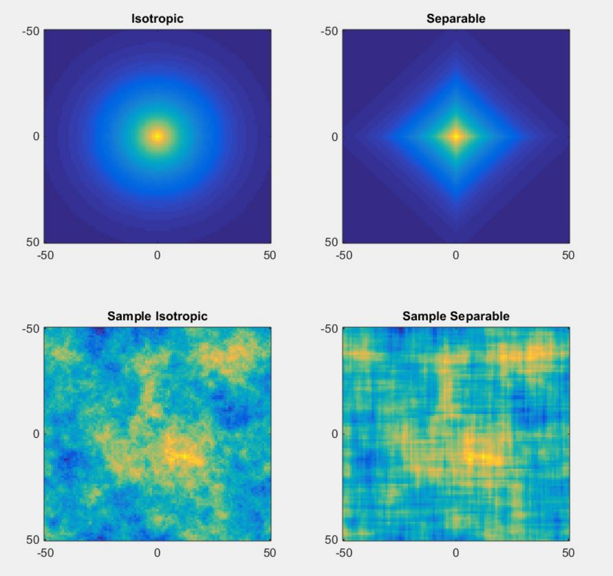
$$\mathbf{Q}=\boldsymbol{\Sigma}_0^{-1}\otimes\boldsymbol{\Gamma}_0^{-1}$$

Only need to invert

- $\boldsymbol{\Sigma}_0$, an $m\times m$ matrix
- $\boldsymbol{\Gamma}_0$, an $n\times n$ matrix

Direct: Need to work with an $nm\times nm$ matrix

Isotropic vs separable Exponential correlation , $\mathcal{R}^2$



Additive structures

Additive independent parts:

$$\begin{aligned} Y(\mathbf{s};t) =& \beta(\mathbf{s}) + \gamma(t) \\ \text{cov}[Y(\mathbf{s};t), Y(\mathbf{x};r)] =& C_\beta(\mathbf{s},\mathbf{x}) + C_\gamma(t,r) \end{aligned}$$

Typically used as part of a larger model

$$Y(\mathbf{s};t) = \mu(\mathbf{s};t) + \beta(\mathbf{s}) + \gamma(t) + \kappa(\mathbf{s};t) + \delta(\mathbf{s};t),$$

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General:

$$\sum_j w_j C^j(\mathbf{s},t;\mathbf{x},r), w_j > 0$$

is a valid covariance function if each $C^j(\mathbf{s},t;\mathbf{x},r)$ are.

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Example: Adding two separable covariance functions

$$pC^{s1}(\mathbf{h})C^{t1}(\tau) + qC^{s2}(\mathbf{h})C^{t2}(\tau)$$

Gives non-separable covariance function

Multiplication of correlation functions

Given two valid stationary correlation functions : $C_1(\mathbf{h}, \delta)$ and $C_2(\mathbf{h}, \delta)$ Then the product: $C_{\text{prod}}(\mathbf{h}, \delta)) = C_1(\mathbf{h}, \delta)C_2(\mathbf{h}, \delta)$ is a legal correlation function as well.

Pf:
$\mathcal{F}$ Denote Fourier transform
$\mathcal{F}(C_{\text{prod}}) = \mathcal{F}(C_1) * \mathcal{F}(C_2)$
Since $\mathcal{F}(C_1)$ and $\mathcal{F}(C_2)$ are non negative then also the convolution of these functions is non negative. Thus Bochner's Theorem holds.

Iterate the argument such that the product of valid correlation functions is a valid correlation function.

Spatio-temporal kriging

Assume observed $\{Z(\mathbf{s}_i; t_{ij}, i = 1, ..., m, j = 1, ..., T_i\}$

$$Z(\mathbf{s}_i; t_{ij}) = Y(\mathbf{s}_i; t_{ij}) + \varepsilon(\mathbf{s}_i; t_{ij})$$

Aim: Predict $Y(\mathbf{s}_0; t_0)$

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- Mathematics as for pure spatial processes
- Notionally more complex
- Matrices involved much larger

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In general

$$[Y(\mathbf{s}_0; t_0)|\mathbf{Z}] \sim N(\mu(\mathbf{s}_0; t_0) + \mathbf{c}_0'\mathbf{C}_Z^{-1}(\mathbf{Z} - \boldsymbol{\mu}), C_{0,0} - \mathbf{c}_0'\mathbf{C}_Z^{-1}\mathbf{c}_o)$$

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Separable covariance functions: $\mathbf{C}_Z$ simplifies considerable (if $T_i = T$ and $t_{ij} = t_j$)

- Similar extensions when parameters are to be estimated

Time-dynamic modelling

- Integro-difference equation models (AR-type)
- Using (partial) differential equations

Common question:

- What covariance structures do we obtain?
- How can we make the models operational (do computation)?

Integro-difference equation models

Assume

$$Y_t(\mathbf{s}) = \int_{D_s} m(\mathbf{v}) Y_{t-1}(\mathbf{s}+\mathbf{v}) d\mathbf{v} + \eta_t(\mathbf{s})$$

Storvik et al(2002)

$$f_{Y,\tau}(\boldsymbol{\omega}) \text{ :Spatial Fourier transform of } C_Y(\mathbf{h},\tau)$$

$$\begin{aligned} f_{Y,0}(\boldsymbol{\omega}) =& \frac{f_\eta(\boldsymbol{\omega})}{1 - f_m(\boldsymbol{\omega}) f_m(-\boldsymbol{\omega})} \\ f_{Y,\tau}(\boldsymbol{\omega}) =& f_{Y,0}(\boldsymbol{\omega}) f_m(-\boldsymbol{\omega})^{|\tau|} \end{aligned}$$

- Non-separable covariance function
- Separable in Fourier domain!
- AR-process in Fourier domain (all frequencies independent)!
- Can also look at properties when time-difference gets smaller

Modelling through (partial) differential equations

Diffusion-injection models (1D in space)

$$\frac{\partial Y(s;t)}{\partial t} = \beta \frac{\partial^2 Y(s;t)}{\partial s^2} - \alpha Y(s;t) + \delta(s;t)$$

Rate of change in Y equal to

- "spread" of Y in space (diffusion) and
- an offset (loss) proportional to Y

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Example:

- $Y(s;t)$: concentration of some nutrient in soil
- $\beta \frac{\partial^2 Y(s;t)}{\partial s^2}$: physical diffusion
- $-\alpha Y(s;t)$: Removement due to oxidation, leaching or extraction by plants
- $\delta(s;t)$: Injection part, small random additions/removals of the nutrient

Diffusion-injection models

$$\frac{\partial Y(s;t)}{\partial t} = \beta \frac{\partial^2 Y(s;t)}{\partial s^2} - \alpha Y(s;t) + \delta(s;t)$$

If $\{\delta(s;t)\}$ is white noise, one can show:

$$\begin{aligned}\rho_Y(|h|;\tau) =& \text{Cor}[Y(s;t), Y(s+h;t+\tau)]\\ =& 0.5 e^{-h(\alpha/\beta)^{1/2}} \text{Erfc}\left(\frac{2\tau(\alpha/\beta)^{1/2} - h/\beta}{2(\tau/\beta)^{1/2}}\right) +\\ & 0.5 e^{h(\alpha/\beta)^{1/2}} \text{Erfc}\left(\frac{2\tau(\alpha/\beta)^{1/2} + h/\beta}{2(\tau/\beta)^{1/2}}\right)\\ \text{Erfc}(z) \equiv& (2/\pi)^{1/2} \int_z^\infty e^{-v^2} dv, \quad z \geq 0\\ \text{Erfc}(z) =& 2 - \text{Erfc}(-z), \quad z < 0\end{aligned}$$

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\end{aligned}$$

- SPDE: α, β easily interpretable
- Covariance function: Make density for observations computable

Discretization of diffusion-injection models

Model

$$\frac{\partial Y(s;t)}{\partial t} = \beta\frac{\partial^2 Y(s;t)}{\partial s^2} - \alpha Y(s;t)$$

Discretization:

$$\begin{aligned}\frac{Y(s;t+\Delta_t)-Y(s,t)}{\Delta_t} =&\beta\left\{\frac{Y(s+\Delta_s;t)-2Y(s;t)+Y(s-\Delta_s;t)}{\Delta_s^2}\right\}\\&-\alpha Y(s;t)\end{aligned}$$

or

$$\begin{aligned}Y(s;t+\Delta_t) =&\theta_1 Y(s;t)+\theta_2 Y(s+\Delta_s;t)+\theta_2 Y(s-\Delta_s;t)\\ \theta_1 =&(1-\alpha\Delta_t-2\beta\Delta_t/\Delta_s^2)\\ \theta_2 =&\beta\Delta_t/\Delta_s^2\end{aligned}$$

Discretization of diffusion-injection models

$$Y(s;t+\Delta_t) = \theta_1 Y(s;t) + \theta_2 Y(s+\Delta_s;t) + \theta_2 Y(s-\Delta_s;t)$$

Let $D_s = \{s_0, s_1, ..., s_{n+1}\}, s_j = s_0 + j\Delta_s$. Then

$$\begin{aligned}
Y(s_1;t+\Delta_t) =& \theta_1 Y(s_1;t) + \theta_2 Y(s_2;t) + Y(s_0;t)\\
Y(s_2;t+\Delta_t) =& \theta_1 Y(s_2;t) + \theta_2 Y(s_3;t) + Y(s_1;t)\\
&\vdots\\
Y(s_n;t+\Delta_t) =& \theta_1 Y(s_n;t) + \theta_2 Y(s_{n+1};t) + Y(s_{n-1};t)
\end{aligned}$$

Defining $\mathbf{Y}_t = (Y(s_1;t), ..., Y(s_n;t))', \mathbf{Y}_t^{(b)} = (Y(s_0;t), Y(s_{n+1};t))'$ gives

$$\mathbf{Y}_{t+\Delta_t} = \mathbf{M}\mathbf{Y}_t + \mathbf{M}^{(b)}\mathbf{Y}_t^{(b)}$$

$\mathbf{M}$ sparse, $\mathbf{Y}_t^{(b)}$ are boundary conditions

Discr of diffusion-injection models - adding stochasticity

$$\mathbf{Y}_{t+\Delta_t} = \mathbf{M}\mathbf{Y}_t + \mathbf{M}^{(b)}\mathbf{Y}_t^{(b)}$$

Stochastic version

$$\mathbf{Y}_{t+\Delta_t} = \mathbf{M}\mathbf{Y}_t + \mathbf{M}^{(b)}\mathbf{Y}_t^{(b)} + \boldsymbol{\delta}_t$$

Multivariate AR(1) process (assuming $\{\mathbf{Y}_t^{(b)}\}$ given)

$$\mathbf{\Sigma}_Y^{(k)} = \text{Cov}(\mathbf{Y}_{t+k}, \mathbf{Y}_t) = \mathbf{M}^k\mathbf{\Sigma}_Y, \mathbf{\Sigma}_Y = \mathbf{\Sigma}_Y^{(0)} = \text{Var}(\mathbf{Y}_t)$$

Note:

$$\mathbf{\Sigma}_Y = \mathbf{M}\mathbf{\Sigma}_Y\mathbf{M}' + \mathbf{\Sigma}_\delta$$

Book: Plots showing that discretization typically quite good.
Inference:

- Use Kalman filter if observations also Gaussian
- Other sequential methods possible in more general cases

SPDE's and blurring - space $\mathcal{R}^p$

$$\frac{\partial Y(\mathbf{s};t)}{\partial t} = -\frac{1}{2}\left[2\boldsymbol{\mu}'\frac{\partial Y(\mathbf{s};t)}{\partial \mathbf{s}} - \text{tr}\left\{\left\{\frac{\partial^2 Y(\mathbf{s};t)}{\partial \mathbf{s}\partial \mathbf{s}'}\right\}\boldsymbol{\Sigma}\right\} + 2\lambda Y(\mathbf{s};t)\right] + \delta(\mathbf{s};t)$$

$$\frac{\partial Y(\mathbf{s};t)}{\partial \mathbf{s}} = \left(\frac{\partial Y(\mathbf{s};t)}{\partial s_1}, \cdots, \frac{\partial Y(\mathbf{s};t)}{\partial s_p}\right)'$$

$$\frac{\partial^2 Y(\mathbf{s};t)}{\partial \mathbf{s}\partial \mathbf{s}'} = \left\{\frac{\partial^2 Y(\mathbf{s};t)}{\partial s_i \partial s_j}\right\}$$

$\{\delta(\mathbf{s};t)\}$:Gaussian, independent in time, stationary in space

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$$\frac{\partial^2 Y(\mathbf{s};t)}{\partial \mathbf{s}\partial \mathbf{s}'} = \left\{\frac{\partial^2 Y(\mathbf{s};t)}{\partial s_i \partial s_j}\right\}$$

$\{\delta(\mathbf{s};t)\}$:Gaussian, independent in time, stationary in space

Brown et al (2000):

$$Y(\mathbf{s};t) = \int_0^\infty \exp(-\lambda v)\int_x \phi(\mathbf{x};v\boldsymbol{\mu},v\boldsymbol{\Sigma})\delta(\mathbf{s}-\mathbf{x};t-v))d\mathbf{x}dv$$

Gaussian since $\{\delta(\mathbf{s},t)\}$ Gaussian, nonseparable

$$C_Y(\mathbf{h};\tau) = \int_0^\infty \exp(-\lambda(2v+|\tau|)\int \phi(\mathbf{x};\tau\boldsymbol{\mu},(2v+|\tau|\boldsymbol{\Sigma}))C_\delta(\mathbf{h}-\mathbf{x})d\mathbf{x}dv$$

Fourier transforms

Complicated covariance function:

$$C_Y(\mathbf{h};\tau) = \int_0^\infty \exp(-\lambda(2v + |\tau|) \int \phi(\mathbf{x}; \tau\boldsymbol{\mu}, (2v + |\tau|\boldsymbol{\Sigma}))C_\delta(\mathbf{h} - \mathbf{x})d\mathbf{x}dv$$

Simpler in Fourier domain

$$f_Y(\boldsymbol{\omega};\xi) = |f_G(\boldsymbol{\omega};\xi)|^2 f_\delta(\boldsymbol{\omega})$$

$$\begin{aligned}
f_Y(\boldsymbol{\omega};\xi) &: \text{Fourier transform of } C_Y(\mathbf{h};\tau)\\
f_\delta(\boldsymbol{\omega}) &: \text{Fourier transform of } C_\delta(\mathbf{h})\\
f_G(\boldsymbol{\omega};\xi) &: \text{Fourier transform of } G(\mathbf{s};t)\\
G(\mathbf{s};t) &= \exp(-\lambda t)\phi(\mathbf{s}; t\boldsymbol{\mu}, t\boldsymbol{\Sigma})I(0 \le t < \infty)
\end{aligned}$$

Discretization

Discretizing in time:

$$\frac{\partial Y(\mathbf{s};t)}{\partial t} = -\frac{1}{2}\left[2\boldsymbol{\mu}'\frac{\partial Y(\mathbf{s};t)}{\partial \mathbf{s}} - \text{tr}\left\{\left\{\frac{\partial^2 Y(\mathbf{s};t)}{\partial \mathbf{s}\partial \mathbf{s}'}\right\}\boldsymbol{\Sigma}\right\} + 2\lambda Y(\mathbf{s};t)\right] + \delta(\mathbf{s};t)$$

leads to

$$Y(\mathbf{s};t) \approx \{(\mathbf{I} - (\Delta_t/2)\mathbf{A}_{\Delta_t})Y(\cdot;t-\Delta_t)\}(\mathbf{s}) + \delta(\mathbf{s};t)$$

where $\mathbf{A}_{\Delta_t}$ is linear operator defined by

$$\{(\mathbf{I} - (\Delta_t/2)\mathbf{A}_{\Delta_t})X(\cdot)\}(\mathbf{s}) = \exp(-\lambda\Delta_t)\int \phi(\mathbf{x};\Delta_t\boldsymbol{\mu},\Delta_t\boldsymbol{\Sigma})X(\mathbf{s}-\mathbf{x})d\mathbf{x}$$

Discretizing spatial domain as well:

$$\begin{aligned}
\mathbf{Y}_t \approx& \mathbf{M}\mathbf{Y}_{t-\Delta_t} + \boldsymbol{\delta}_t\\
\mathbf{Y}_t =& (Y(\mathbf{s}_1,t),...,Y(\mathbf{s}_n;t))'\\
\boldsymbol{\delta}_t =& (\delta(\mathbf{s}_1,t),...,\delta(\mathbf{s}_n;t))'
\end{aligned}$$

More general PDE's

(u, v): Spatial or spatio-temporal coordinates
Partial differential equations (PDE)s (nonstochastic):

$$\left\{a\frac{\partial^2}{\partial u^2} + 2h\frac{\partial^2}{\partial u \partial v} + b\frac{\partial^2}{\partial v^2} + 2g\frac{\partial}{\partial u} + 2f\frac{\partial}{\partial v} + c\right\} Y(u, v) = 0$$

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Stochastic PDE

$$\left\{a\frac{\partial^2}{\partial u^2} + 2h\frac{\partial^2}{\partial u \partial v} + b\frac{\partial^2}{\partial v^2} + 2g\frac{\partial}{\partial u} + 2f\frac{\partial}{\partial v} + c\right\} Y(u, v) = \delta(u, v)$$

$\{\delta(u, v)\}$ a zero-mean, two-dimensional stochastic process,

- random impulses from smaller-order contributions
- uncertainty in the scientific relationship expressed through the PDE

Stochastic PDE

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Question: Given model above, what is

- $E[Y(u,v)]$?
- $\text{var}[Y(u,v)]$?
- $\text{cov}[Y(u,v), Y(u',v')]$?

Zero mean δ and linearity of PDE: $E[Y(u,v)] = 0$
(Can always add expectation structure)
Covariance structure difficult to derive in general

Whittle (1963)

$Y(\mathbf{s}), \mathbf{s} \in \mathcal{R}^d$, $\{\delta(\mathbf{s})\}$ zero-mean white noise:

$$\left\{\frac{\partial^2}{\partial s_1^2} + \cdots + \frac{\partial^2}{\partial s_d^2} - \alpha^2\right\}^p Y(\mathbf{s}) = \delta(\mathbf{s})$$

has covariance function

$$\begin{aligned} C_Y(\mathbf{h}) \propto & \{||\mathbf{h}||/\theta_1\}^{\theta_2} K_{\theta_2}(||\mathbf{h}||/\theta_1) \\ \theta_1 =& 1/\alpha > 0 \\ \theta_2 =& 2p - d/2 > 0 \end{aligned}$$

$K_{\theta_2}(\cdot)$ modified Bessel function of the second kind
The Matern covariance function

Geostatistical and SPDE models

- Equivalent models in some cases

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- Different interpretations
  - of model
  - of parameters

Geostatistical and SPDE models

- Equivalent models in some cases
- Different interpretations
  - of model
  - of parameters
- Different ways of doing inference
  - Geostatistical models: Density directly given in Gaussian case
  - For large models: SPDEs can have computational benefits

Time series of spatial processes

Assume $D_t = \{0, 1, 2, ...\}$, discrete-time model
$\mathbf{Y}_t(\cdot) = \{Y(\mathbf{s}; t), \mathbf{s} \in D_s\}$ (possible infinite dimension)
AR-type model:

$$\mathbf{Y}_t(\mathbf{s}) = \mathcal{M}_t(\mathbf{s}, \mathbf{Y}_{t-1}) + \delta_t(\mathbf{s})$$

$\{\delta_t(\mathbf{s})\}$ independent in time.

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Special case

$$\mathbf{Y}_t(\mathbf{s}) = \int_{R^d} m(\mathbf{s}, \mathbf{x})\mathbf{Y}_{t-1}(\mathbf{x}) + \delta_t(\mathbf{s})$$

- Similar to discretized models considered earlier
- Now: Do modelling directly as AR-type models

Linear spatial time series

$$\mathbf{Y}_t = \mathbf{M}_t \mathbf{Y}_{t-1} + \boldsymbol{\delta}_t$$

gives

$$\begin{aligned}\boldsymbol{\Sigma}_{Y,t} =& \mathrm{Var}(\mathbf{Y}_t) = \mathbf{M}_t \boldsymbol{\Sigma}_{Y,t-1} \mathbf{M}_t' + \boldsymbol{\Sigma}_{\delta,t} \\ \mathrm{cov}(\mathbf{Y}_{t+\tau} \mathbf{Y}_t) =& \mathbf{M}_{t+\tau} \mathbf{M}_{t+\tau-1} \cdots \mathbf{M}_{t+1} \boldsymbol{\Sigma}_{Y,t}\end{aligned}$$

Allow for

- nonstationarity in time ($\mathbf{M}_t$ change with time)
- nonstationarity in space ($M_t(\mathbf{s}, \mathbf{x})$ depend on spatial locations)

Example

$D_s = [0, 40], \Delta_s = 1$ and $D_t = [0, 0.8], \Delta_t = 0.01$

$$\mathbf{M}_t = h_t \mathbf{M},$$

$$\mathbf{M} = \begin{pmatrix} \widetilde{\mathbf{M}}_1 & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{M}}_2 \end{pmatrix} \qquad h_t = \begin{cases} 1, & \text{if } 1\Delta_t \leq t \leq 30\Delta_t \\ -1, & \text{if } 31\Delta_t \leq t \leq 60\Delta_t \\ 1, & \text{if } 61\Delta_t \leq t \leq 80\Delta_t \end{cases}$$

$$\widetilde{\mathbf{M}}_1 = \begin{pmatrix} .90 & .01 & 0 & \cdots & 0 \\ .01 & .90 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & & .90 & .01 \\ 0 & 0 & & .01 & .90 \end{pmatrix}, \quad \widetilde{\mathbf{M}}_2 = \begin{pmatrix} .20 & .01 & 0 & \cdots & 0 \\ .01 & .20 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & & .20 & .01 \\ 0 & 0 & & .01 & .20 \end{pmatrix}$$

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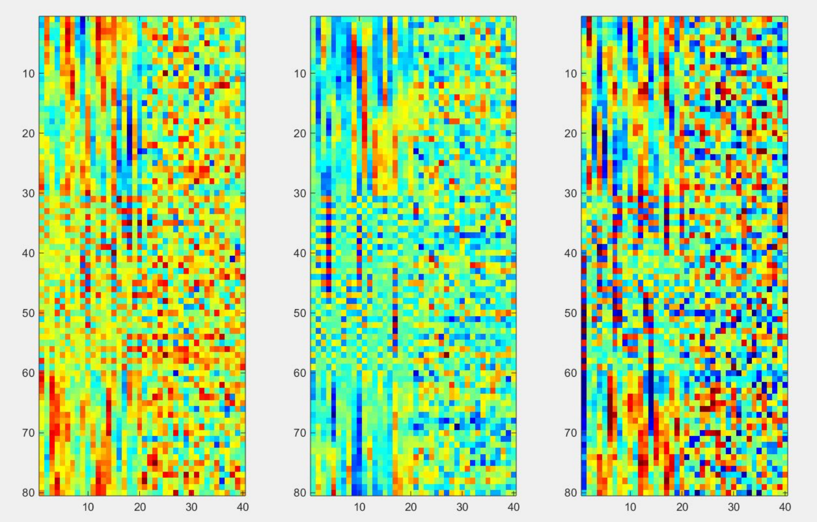
- 6 distinct “regions”
- Can specify $\mathbf{\Sigma}_{\delta,t}$ such that $\mathbf{\Sigma}_{Y,t} = \mathbf{\Sigma}$, i.e. $\mathbf{\Sigma}_{\delta,t} = \mathbf{\Sigma} - \mathbf{M}_t \mathbf{\Sigma} \mathbf{M}_t'$

Example - simulations



Higher order models

In general

$$[\mathbf{Y}_0, \mathbf{Y}_1, ..., \mathbf{Y}_T] = [\mathbf{Y}_0][\mathbf{Y}_1|\mathbf{Y}_0][\mathbf{Y}_2|\mathbf{Y}_0, \mathbf{Y}_1]\cdots[\mathbf{Y}_T|\mathbf{Y}_0, ..., \mathbf{Y}_{T-1}]$$

Need simplifications for

- simpler modelling
- possibility of estimating parameters
- computation

Typical simplifications:

- $[\mathbf{Y}_t|\mathbf{Y}_{t-1}, ..., \mathbf{Y}_0] = [\mathbf{Y}_t|\mathbf{Y}_{t-1}, ..., \mathbf{Y}_{t-r}]$, Vector AR-structure
- $[\mathbf{Y}_t|\mathbf{Y}_{t-1}, ..., \mathbf{Y}_{t-r}] = [\mathbf{Y}_r|\mathbf{Y}_{r-1}, ..., \mathbf{Y}_0]$, stationarity in time
- spatial sparsity in transitions
- stationarity in space

Vector autoregressive model

Linear model

$$\mathbf{Y}_t = \mathbf{M}_t\mathbf{Y}_{t-1} + \boldsymbol{\delta}_t$$

In general $T \times n^2$ parameters in $\mathbf{M}$'s

Simplifications

- $\mathbf{M}_t = \mathbf{M}$: Vector autoregressive model
- $M_{ij} = 0$ for $||\mathbf{s}_i - \mathbf{s}_j|| > h$

Similarly:

- $\boldsymbol{\Sigma}_{\delta,t} = \boldsymbol{\Sigma}_\delta$
- $\boldsymbol{\Sigma}_\delta$ sparse