Spatial point process

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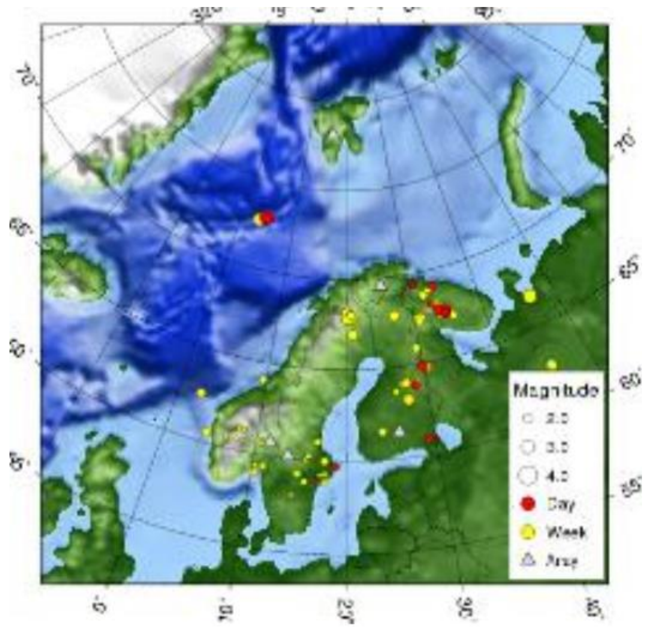
Today

- Point process
- Poisson process
  - Intensity
    - Homogeneous
    - Inhomogeneous
- Other processes
  - Random intensity (Cox process)
    - Log Gaussian cox process (LGCP)
  - Clustering
    - Parent – Child
    - Neyman-Scott
  - Repulsion (regularity)
    - Markov point process
    - Strauss
- Inference
  - Estimation of intensity
  - Detection of clustering and repulsion
    - Pair correlation function
    - K-function/L-function

Point process

- Stochastic process $Z(.)$ describing the location of events (points) in a region $D_s \subset \mathbb{R}^d$

- Both the number of points and the location of points are random
- e.g The number of earth quakes and their locations



http://www.norsardata.no/NDC/recenteq/lastweek.html

Marked point process:

- Additional parameters related to the event are denoted marks
- e.g magnitude

Poisson point process
(homogeneous)

Poisson distribution:

- The number of points in a region A has a Poisson distribution with intensity: $\lambda|A|$

$$p(N(A) = n) = \frac{(\lambda|A|)^n \exp(-\lambda|A|)}{n!}$$

Complete independence:

- The number of points in two non-overlapping regions are independent

- $\lambda|\mathrm{A}| = \int_A \lambda \, d\boldsymbol{s}$
- $\lambda > 0$

CSR= Complete spatial randomness

Poisson point process
(inhomogeneous)

Poisson distribution:

- The number of points in a region A has a Poisson distribution with intensity: $\Lambda(A)$

$$p(N(A) = n) = \frac{(\Lambda(A)\,)^n \exp(-\Lambda(A)\,)}{n!}$$

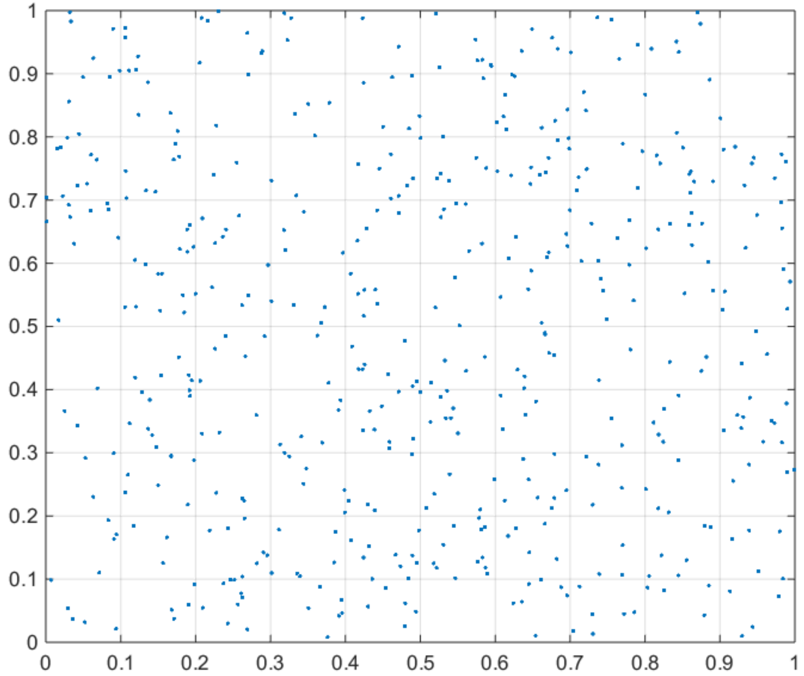
Complete independence:

- The number of points in two non-overlapping regions are independent

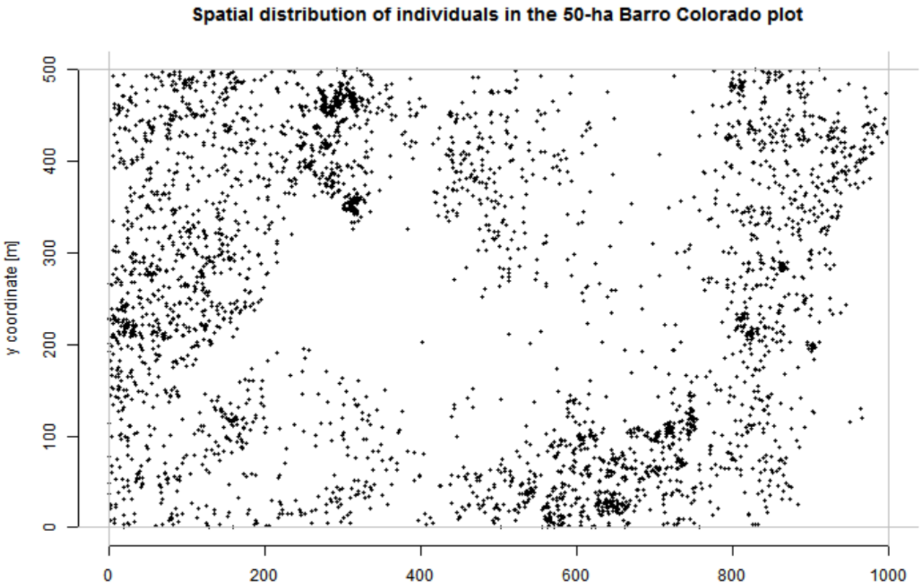
- $\Lambda(A) = \int_A \lambda(\boldsymbol{s}) d\boldsymbol{s}$
- $\lambda(\boldsymbol{s}) > 0 \quad \forall \boldsymbol{s}$

Homogeneous versus inhomogeneous

Homogeneous



Inhomogeneous



Homogeneous does not mean regular, common to see areas which appears as inhomogeneity but can equally well be caused by random fluctuations

Density of Poisson process

- In the Poisson process what is the probability for the point pattern $\{\boldsymbol{s}_i\}_{i=1}^{m}$?

- $\Pr(Z(d\boldsymbol{s}_1) = 1, \ldots, Z(d\boldsymbol{s}_m) = 1, Z(A \backslash \{\boldsymbol{s}_i\}_{i=1}^{m}) = 0)$

$$= \prod_{i=1}^{m} \lambda(\boldsymbol{s}_i) d\boldsymbol{s}_i \exp\{-\lambda(\boldsymbol{s}_i) d\boldsymbol{s}_{\boldsymbol{i}}\} \exp\{-\Lambda(A \backslash \{\boldsymbol{s}_i\}_{i=1}^{m})\}$$

$$= \prod_{i=1}^{m} \lambda(\boldsymbol{s}_i) \exp\{-\Lambda(A)\} \prod_{i=1}^{m} d\boldsymbol{s}_i$$

$$p(N = n|\Lambda) = \frac{\Lambda^n \exp(-\Lambda\ )}{n!}$$

Density of point process

- In the general case what is the probability for the point pattern $\{\boldsymbol{s}_i\}_{i=1}^{m}$?

- $\Pr(Z(d\boldsymbol{s}_1) = 1, \dots, Z(d\boldsymbol{s}_m) = 1 | Z(A) = m)$

$$\propto \lambda_m(\boldsymbol{s}_1, \boldsymbol{s}_2, \dots \boldsymbol{s}_m) \prod_{i=1}^{m} d\boldsymbol{s}_i$$

- *Full distribution:*

$$\Pr(Z(d\boldsymbol{s}_1) = 1, \dots, Z(d\boldsymbol{s}_m) = 1 | Z(A) = m) \Pr(Z(A) = m)$$

$$= \lambda_m(\boldsymbol{s}_1, \boldsymbol{s}_2, \dots \boldsymbol{s}_m) \prod_{i=1}^{m} d\boldsymbol{s}_i \Pr(Z(A) = m)$$

Higher order intensity functions

- $\lambda_m(\boldsymbol{s}_1, \boldsymbol{s}_2, \dots \boldsymbol{s}_m)$
  - Sequence m=1,2,3,…
  - Nonnegative
  - Permutation invariant

For fixed number of points this is a
Joint multivariate distribution,
but the number of samples m vary.

Other processes

- Cox model:
  - Random intensity

$$p(N(A) = n|\Lambda) = \frac{(\Lambda(A)\ )^n \exp(-\Lambda(A)\ )}{n!}$$

$$p(\Lambda) = \text{ gives different models}$$

  - Log Gaussian Cox Model
- Clustering
  - Parent child process
  - Neyman-Scott
- Repulsion
  - Markov point process
  - Strauss process

Log Gaussian Cox process

$$p(N(A) = n|\Lambda) = \frac{(\Lambda(A)\,)^n \exp(-\Lambda(A)\,)}{n!}$$

$$\lambda(\boldsymbol{s}) = \exp\big(Y(\boldsymbol{s})\big),$$

$$\Lambda(A) = \int_A \lambda(\boldsymbol{s})d\boldsymbol{s}$$

$$Y(\boldsymbol{s}) \; = \beta^T \boldsymbol{x}(\boldsymbol{s}) + R(\boldsymbol{s})$$

$$R(s) \sim N(0, C_R(\boldsymbol{h}))$$

Recall:

Spatial disease mapping

$$Z_i|Y_i \overset{ind}{\sim} \text{Poisson}(E_i \exp(Y_i))$$

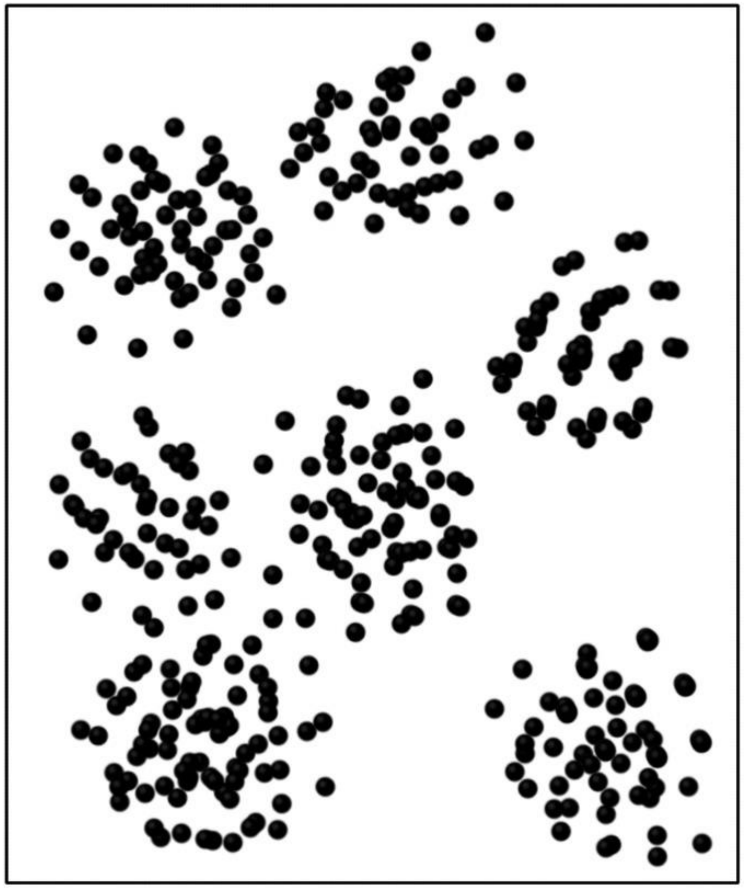
Z_i =Observed disease count

E_i =Expected count (known), and

$Y_i = \mathbf{x}_i^T \beta + \delta_i + \varepsilon_i$

Clustering point process

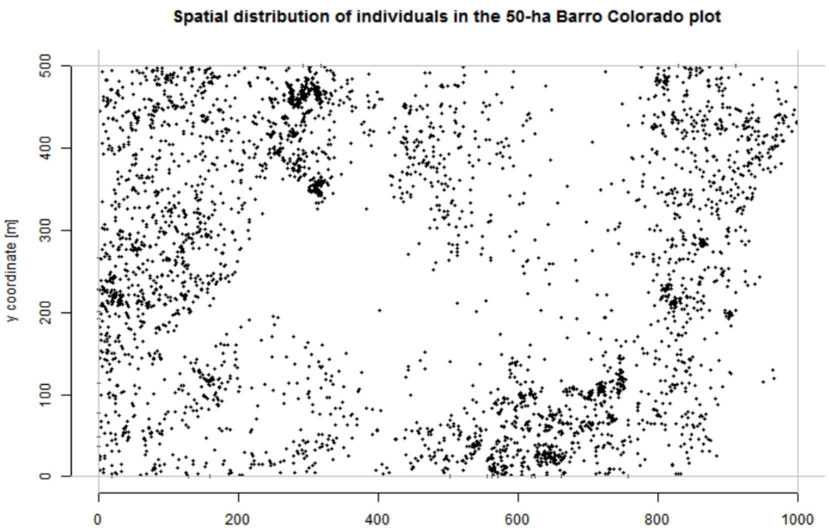
- In some occasions Data points tend to group together
- Stars in the universe cluster in galaxies
- The clusters are "self organizing" i.e. no external force

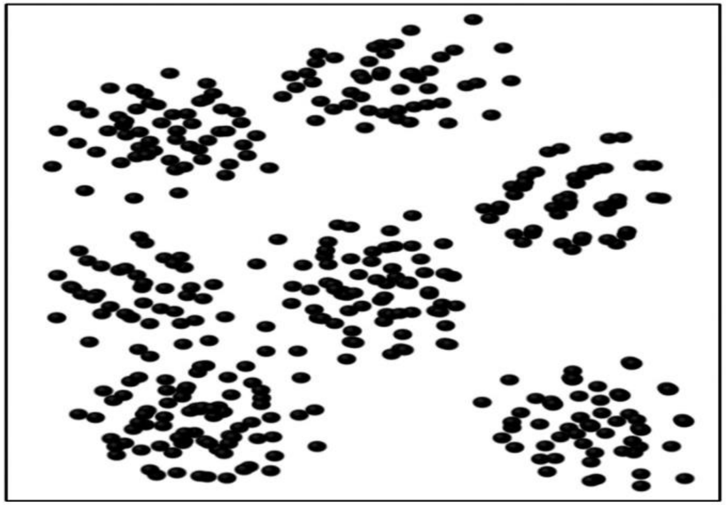


Informal definition of cluster:

- A cluster is a group of points whose intra-point distance is below the average distance in the pattern as a whole.

Clustering versus inhomogeneous





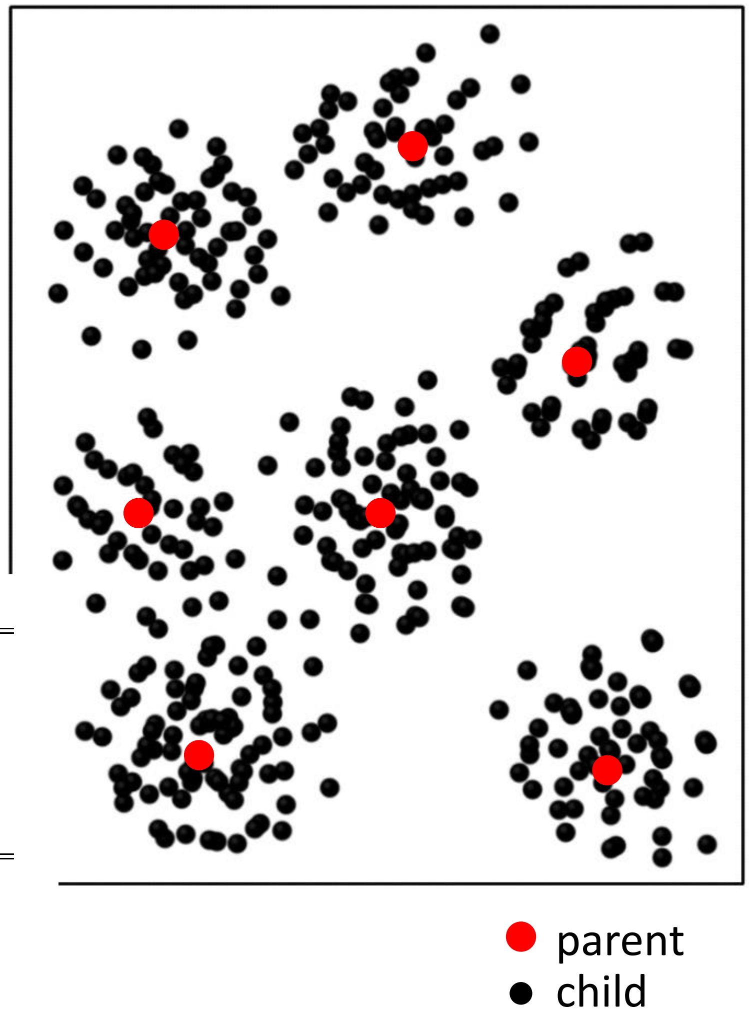
Inhomogeneous : model there is an external feature which causes the inhomogeneity in the model

Clustering model:
Then the point process "organize" , the cause is internal

Not always clear distinction: The term clusters are sometimes used for describing high intensity areas in inhomogeneous distributions
"Earth quakes cluster around tectonic boundaries"

Clustering processes

- Parent-child process
  - Parent events are distributed according to one process
  - Children events are distributed conditioned to parent distribution
  - The final process is just the children



Name of Process	Clusters	Parents
Independent cluster process	general	general
Poisson cluster process	general	Poisson
Cox cluster process	Poisson	general
Neyman-Scott process	Poisson	Poisson
Matérn cluster process	Poisson (uniform in ball)	Poisson (homog.)
Modified Thomas Process	Poisson (Gaussian)	Poisson (homog.)

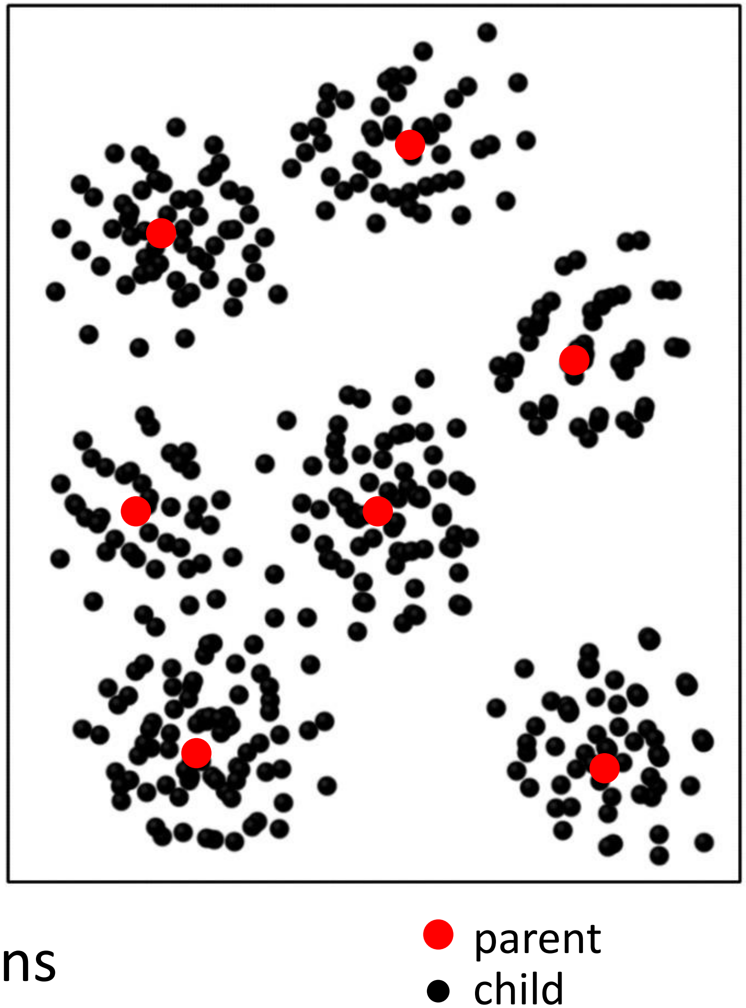
Table 2: Nomenclature for independent cluster processes.

Neyman-Scott process

- Parent $Z_p(s)$ is Poisson
- Children given parents $\{s_{p,i}\}_{i=1}^{N_p}$ is inhomogeneous Poisson:

$$\lambda(s|Z_p) = \sum_{i=1}^{N_p} \alpha \cdot \frac{1}{\sigma} f((s - s_p)/\sigma)$$

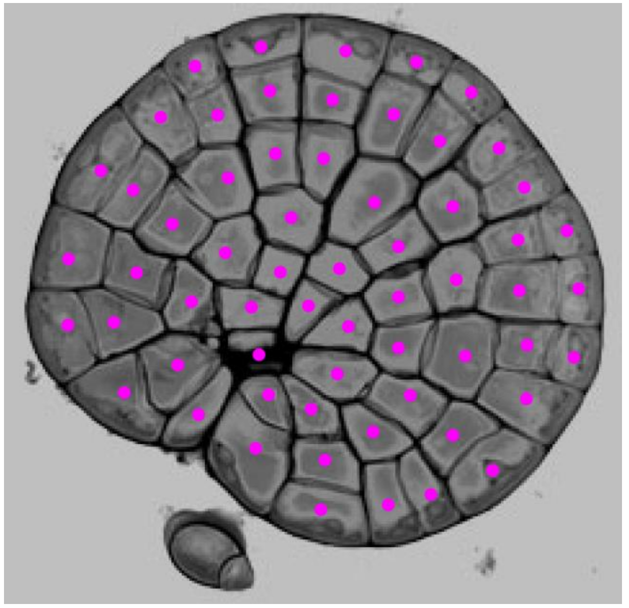
  - α : cluster intensify
  - σ : cluster spread
  - $f(s)$: probability density
- Neyman-Scott is also a Cox process (random intensity)
- Many special cases and generalizations



Repulsion

- In some occasions data points tend to be more regular than the Poisson process
- Cell centers in tissue repels each other
- The repulsion is "self organizing"(not due to external forces)

Coleochaete



Organism with multicellular organization

Repulsion is often modelled using Markov point process

$$\lambda_m(\boldsymbol{s}_1, \boldsymbol{s}_2, \dots \boldsymbol{s}_m)$$
$$\propto \exp\left\{\sum_{i=1}^{m} g_i(\boldsymbol{s}_i) + \sum_{i<j} g_{ij}(\boldsymbol{s}_i, \boldsymbol{s}_j) + \cdots + g_{12\dots m}(\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_m)\right\}$$

Strauss process:

$$g_i(\boldsymbol{s}_i) = \log \lambda$$
$$g_{ij}(\boldsymbol{s}_i, \boldsymbol{s}_j) = \log \gamma \, I(\|\boldsymbol{s}_i - \boldsymbol{s}_j\| < R)$$
$$g_{\text{Higher order}} \equiv 0$$

Strauss process

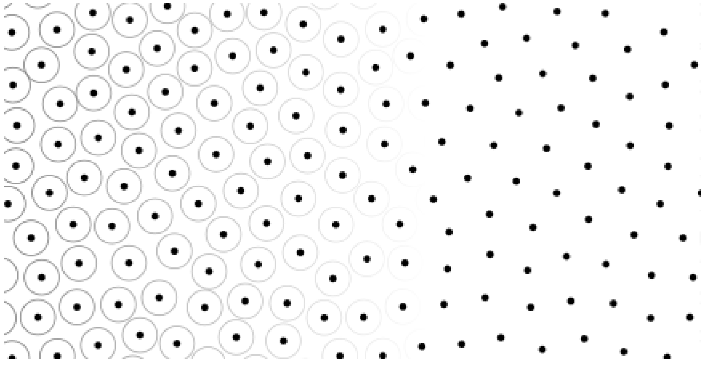
$$\lambda_m(\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots \boldsymbol{s}_m) = \frac{\lambda^m \gamma^{Y_m(R)}}{c(\lambda, \gamma, R)}$$

With $Y_m(R) = \Sigma_{i<j} I(\|\boldsymbol{s}_i - \boldsymbol{s}_j\| < R)$
number of pairs separated by less than R.

$c(\lambda, \gamma, R)$ is independent of m, thus gives the global scaling
(i.e sum over all m)

$\lambda > 0$
$R > 0$
0$\leq \gamma \leq 1$
- $\gamma > 1$ is impossible to normalize
- $\gamma = 0$ Hard core model
- $\gamma = 1$ Poisson process

Hard core model:
No points within a distance R



Estimation

- Inhomogeneous intensity
- Non parametric estimator : (Kernel estimator)

$$\hat{\lambda}(\boldsymbol{s}) = \frac{1}{p_{\boldsymbol{b}}(s)} \sum_{i=1}^{m} K_b(\boldsymbol{s} - \boldsymbol{s}_i)$$

- $p_{\boldsymbol{b}}(s)$ account for edge effect
- K_b probability density
- $b > 0$ bandwidth

Pair correlation function

- $g(s, x) = \lambda_2(\boldsymbol{s}, \boldsymbol{x}) / (\lambda(\boldsymbol{s})\lambda(\boldsymbol{x}))$
- Intensity

$$\lambda(\boldsymbol{s}) = E(Z(d\boldsymbol{s}))/d\boldsymbol{s}$$

- Second order intensity

$$\lambda_2(\boldsymbol{s}, \boldsymbol{x}) = E(Z(d\boldsymbol{s})Z(d\boldsymbol{x}))/(|d\boldsymbol{s}||d\boldsymbol{x}|)$$

If Z(.) is stationary *:* $\lambda_2(\boldsymbol{s}, \boldsymbol{x}) = \lambda_2(\boldsymbol{s} - \boldsymbol{x})$

and also isotropic *:* $\lambda_2(\boldsymbol{s}, \boldsymbol{x}) = \lambda_2(\|\boldsymbol{s} - \boldsymbol{x}\|)$

Pair correlation function

- Poisson distribution:
$$\lambda_2(\boldsymbol{s}, \boldsymbol{x}) = 1$$
- Co occurrence more frequent than Poisson
$$\lambda_2(\boldsymbol{s}, \boldsymbol{x}) > 1$$
- Co occurrence less often than Poisson
$$\lambda_2(\boldsymbol{s}, \boldsymbol{x}) < 1$$

- Estimation using kernel based methods

- Clustering we have $\lambda_2(\boldsymbol{s}, \boldsymbol{x})$> 1 when $\|\boldsymbol{x} - \boldsymbol{s}\|$ is small
- Repulsion we have $\lambda_2(\boldsymbol{s}, \boldsymbol{x}) < 1$ when $\|\boldsymbol{x} - \boldsymbol{s}\|$ is small

K-function

- Assume stationary and isotropic point process

$$K(h) = E\left\{\begin{matrix}\text{number of events within} \\ \text{distance h of an abitrary event}\end{matrix}\right\}/\lambda$$

- Related to pair correlation function
- Estimator:

$$\widehat{K}(h) = \frac{1}{m\hat{\lambda}}\sum_{i=1}^{m}\sum_{\substack{j=1 \\ j\neq i}}^{m} I\left(\|s_i - s_j\| \leq h\right)$$

$$\hat{\lambda} = m/|A|$$

Also edge effects se book 4.180

K-function for an Poisson process

- In $\mathbb{R}^d$

$$K(h) = \pi^{\frac{d}{2}} / \, \Gamma\left(1 + \frac{d}{2}\right) h^d$$

- In $\mathbb{R}^2$

$$K(h) = \pi h^2$$

To compare with Poisson distribution use L function which center and normalize K for Poisson distribution

L-function

- In $\mathbb{R}^d$

$$L(h) = \left(\frac{K(h)\Gamma\left(1 + \frac{d}{2}\right)}{\pi^{d/2}} \right)^{1/d} - h$$

- In $\mathbb{R}^2$

$$L(h) = \left(\frac{K(h)}{\pi} \right)^{1/2} - h$$

Some definitions do not subtract h

Clustering / Repulsion

- Clustering
  - More events at short distance than Poisson
  - If Estimated K function is larger than the response from Poisson process
  - If L-function is positive (significantly positive)
- Repulsion
  - Less events at short distance than Poisson
  - If Estimated K function is less than the response from Poisson process
  - If L-function is negative (significantly negative)
- Example Page 213 in book