Rough Path Theory

1. Introduction

Basic literature:

1. Friz, P., Hairer, M.: A Course on Rough Paths. Springer (2014).
2. Lyons, T., Caruana, M., Lévy, T.: Differential Equations Driven by Rough Paths. LNM, Springer (2006).
3. Friz, P., Victoir, N.: Multidimensional Stochastic Processes as Rough Paths. Cambridge Univ. Press (2010).

The *motivation* of rough path theory initiated by T. Lyons in the mid-nineties comes from *controlled differential equations* of the form

$$\dot{Y}_t = f(Y_t)\dot{X}_t, \quad Y_0 = \xi, \qquad (1)$$

where $\dot{X}$ is the derivative of a given function X (driving signal), ξ the initial condition, f (smooth) function and Y the unknown.

More generally, if X is of finite 1-variation (bounded variation), i.e. if

$$\sup_{\mathcal{P}} \sum |X_{t_{i+1}} - X_{t_i}|^p < \infty \qquad (2)$$

for $p=1$, where the supremum is taken over all partitions $\mathcal{P}$ of $[0,T]$, then we can recast (1) in integral form w.r.t. the Lebesgue-Stieltjes integral as

$$Y_t = \xi + \int_0^t f(Y_s)\,dX_s \qquad (3)$$

However, in applications to physics, chemistry or finance (e.g. Y_t stockprice at time t, X: "market noise") the control or driving signal may not be smooth, but "rough" in the sense of Hölder continuity with exponent $0<\alpha<1$, that is

$$X \in C^{\alpha}([0,T];\mathbb{R}^d) := \Big\{ g \in C([0,T];\mathbb{R}^d) : \sup_{s \neq t} \frac{|g(s)-g(t)|}{|s-t|^{\alpha}} =: |||g|||_{\alpha} < \infty \Big\}$$

In this case, one needs to make sense of the integral term in (3) or more generally of the integral

$$\int_0^t Z_s \, dX_s$$

for $X \in C^{\alpha}([0,T];\mathbb{R}^d)$.

It turns out that this integral can be defined as a Young integral:

Theorem (Young 1936):

(i) Let $Z \in C^{\beta}([0,T];\mathbb{R})$, $X \in C^{\alpha}([0,T];\mathbb{R})$ with $0<\alpha,\beta<1$ and $\alpha+\beta>1$. Then

$$\int_0^T Z_s \, dX_s := \lim_{|\mathfrak{P}| \to 0} \sum_{[s,t] \in \mathfrak{P}} Z_s (X_t - X_s) \qquad (4)$$

converges, where $|\mathfrak{P}|$ is the mesh of the partition $\mathfrak{P}$ of $[0,T]$.

The resulting bilinear map $(X,Z) \mapsto \int_0^T Z_s \, dX_s$ is continuous, that is

$$\Big| \int_0^T Z_s \, dX_s \Big| \leq C_{\alpha+\beta}(|Z_0|) \cdot \|Z\|_{\beta} \cdot \|X\|_{\alpha}$$

(ii) If $\alpha+\beta \leq 1$, then there exist $Z \in C^{\beta}([0,T];\mathbb{R})$, $X \in C^{\alpha}([0,T],\mathbb{R})$ such that (4) does not converge (i.e. different sequences of partitions yield different limits or none at all)

If $X \in C^{\alpha}([0,T];\mathbb{R}^m)$ with $\alpha > \frac{1}{2}$ and $f \in C_b^2(\mathbb{R}^d;\mathbb{R}^{d\times m})$ (space of twice cont. diff. $g: \mathbb{R}^d \to \mathbb{R}^{d\times m}$ with bounded part. derivatives), one shows by using the Picard iteration scheme that there is a unique solution Y to (3) or equivalently

$$dY_t = f(Y_t)\,dX_t, \quad Y_0 = \xi \in \mathbb{R}^d$$

③ Although the concept of the Young integral is useful, it is still too <u>restrictive</u> in many applications: E.g. in math. finance $\mathcal{X}$ could be a path of a Brownian motion $(t \mapsto B_t(\omega))$, i.e. a continuous process, starting in zero with independent and stationary increments on some probability space $(\Omega, \mathcal{A}, P)$. Here

$$\int_0^t Z_s(\omega)\, dB_s(\omega) \qquad (5)$$

may represent the value of a portfolio of stocks at time t w.r.t. a (admissible) hedging strategy process $Z_s(\omega)$, $0 \le s \le T$, $\omega \in \Omega$.

It is known that

$$(t \mapsto B_t(\omega)) \in C^{\frac{1}{2}-\varepsilon}([0,T]) \quad P\text{-a.e.}$$

for all $\varepsilon \in (0, \frac{1}{2})$.

<u>However</u>, paths of Z are in general not in $C^{\beta}([0,T])$

⟶ definition of (5) in the sense of <u>Young</u> may <u>fail</u>

⟶ solution: <u>Itô-integral</u> (K. Itô 1944):

$$\int_0^T Z_s \, dB_s = \lim_{|B| \to 0} \sum_{[u,v]} Z_u (B_v - B_u) \qquad (6)$$

in probability for processes Z (with right continuous paths having existing left limits), which are <u>adapted</u> (i.e. Z_s $\sigma((B_u)_{0 \le u \le s})$-measurable for all s)

This concept relies on the martingale property of B and can be even extended to semi-martingales $\mathcal{X}$:

$$\mathcal{X}_t = \mathcal{X}_0 + \underset{\substack{\uparrow \\ \text{bounded} \\ \text{variation} \\ \text{process}}}{A_t} + \underset{\substack{\uparrow \\ \text{(local)} \\ \text{martingale}}}{M_t}$$

→ Itô integration quite general integration concept

→ *problem*: may not be general enough in applications:

E.g. recent empirical observations in finance suggest the stoch. modeling of volatilities of stock prices by means of integral processes w.r.t. a *fractional Brownian motion* $X = B^H$ for *Hurst parameters* $H < \frac{1}{2}$, which is *not* a semi-martingale

Definition (fract. Bm)

Let $0 < H < 1$. The *fractional Brownian motion* with *Hurst parameter* H is a continuous Gaussian process B^H on $[0,T]$ with $B_0^H = 0$, $E[B_t^H] = 0$ and

$$E[B_t^H \cdot B_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

→ (i) B^H for $H = \frac{1}{2}$ standard Brownian motion

(ii) paths of B^H are α-Hölder-continuous for all $\alpha < H$ with probability 1

(iii) B^H is not a semi-martingale for $H \neq \frac{1}{2}$

→ concept of Itô-integration fails

→ *crucial objective* in rough path theory:

Definition of a *rough path integral*

$$\int_0^T Z_s \, dX_s$$

for "rough" driving signals X in $\mathbb{R}^d$ (i.e. $X \in C^\alpha$ for $\alpha \leq \frac{1}{2}$ as e.g. $X = B^H$ for $H \leq \frac{1}{2}$) and "rough" integrands $Z \, (\in C^\alpha, \alpha \leq \frac{1}{2})$

→ integration concept, which is *path by path* in contrast to Itô integration and which is *not* within the framework of Young integration

⑤ What is this type of integration theory good for ?

→ It allows e.g. for the study of the following problems :

1. Pathwise (i.e. deterministic) study of stochastic differential equations (SDE's) : To explain this consider the Itô-SDE

$$Y_t = \xi + \int_0^t f(Y_s)\, dB_s \;\; (\leftarrow \text{Itô-integral}), \quad \xi \in \mathbb{R}^d \qquad (7)$$

If f is Lipschitz continuous, Picard iteration gives a unique adapted solution $Y = \Phi(B) = \Phi_{f,\xi}(B)$

→ map Φ (Itô-map) defined on the path space is measurable, but not continuous in general

→ Example (discont. Φ) :

$$\Phi(B) = \begin{pmatrix} B^{(i)} \\ \int_0^{\cdot} B^{(i)}(t)\, dB^{(j)}(t) \end{pmatrix} \qquad (8) \quad (B^{(i)}, B^{(j)} \text{ 1-dim Bm's})$$

as the solution to the 2-dim. SDE

$$Y_t^{(1)} = B_t^{(i)}$$

$$Y_t^{(2)} = \int_0^t Y_s^{(1)}\, dB_s^{(j)}$$

See e.g. T. Lyons 1991

However the integrals in ⑧ contain all information to obtain the following continuity result !

Rough path principle :

$$\begin{tikzcd} B \arrow[rr, "\Phi"] \arrow[d, "\mathcal{L}"] & & Y \\ \hat{X} \arrow[rru, "\Psi"] & & \end{tikzcd}$$

where

$$\hat{X} \text{ "=" } \left(B, \left(\int_0^{\cdot} B^{(i)}(t)\, dB^{(j)}(t)\right)_{i,j}\right) \quad (\text{"enhanced rough path"})$$

and where

Ψ is the Itô-Lyons map, which is continuous (in the rough path topology)

$\longrightarrow$ this striking fact allows a path by path analysis of SDE's

2. Deterministic study of (non-linear) stochastic partial differential equations (SPDE's):

E.g. study of the Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 - C + \xi \leftarrow \text{space-time white noise} \qquad (9)$$

(as a model in e.g. the field of interacting particle systems) in the framework of a unifying theory of rough paths ("regularity structures")

3. Stochastic regularization: To explain this, consider e.g. the deterministic transport equation:

$$\frac{\partial u}{\partial t} + (b\cdot\overset{\text{gradient}}{\nabla})u = 0 \quad , u|_{t=0} = u_0 \qquad (10)$$

for a given vector field $b:[0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, an initial condition $u_0:\mathbb{R}^d\to\mathbb{R}$ and the unknown scalar field $u:[0,T]\times\mathbb{R}^d\to\mathbb{R}$

If b is not a Lipschitz function, then uniqueness or even existence of solutions to (10) may fail!

In order to restore well-posedness of (10) in the sense of uniqueness, existence and regularity of solutions one may "regularize" (10) by adding a noise ξ_t to it:

$$d_t u + (b\cdot\nabla)u\,dt + d\xi_t = 0 \quad , u|_{t=0} = u_0 , \qquad (11)$$

where e.g.

$$d\xi_t = \sum_{i=1}^d \partial_{x_i} u \circ dB_t^{(i),H} \Longleftrightarrow \xi_t = \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} u \circ dB_s^{(i),H}$$

for indep. fract. 1-dim. Bm's $B^{(i),H}$, $i=1,\dots,d$ with Hurst parameter $H \in (0,\frac{1}{2}]$

Here $\circ dB_s^{(i),H}$ stands for a Stratonovich type of integration w.r.t. $B^{(i),H}$ (see later)

If e.g. $H=\frac{1}{2}$, b is bounded and measurable, u_0 smooth (or in C_b^1) then (11) has a unique Sobolev differentiable (weak) solution process to (11)!

! Related equations to (11) are e.g. stochastic continuity equations with even C^k-solutions under mild conditions on the coefficients (see later)

4. Generalization and simplification of proofs of central results in stochastic analysis!

E.g. Hörmander's theorem in the case of fBm B^H, $H \in (\frac{1}{4},\frac{1}{2}]$:

Consider the SDE

$$dX_t = V_0(X_t)\,dt + \sum_{i=1}^{m} V_i(X_t) \overset{\text{Stratonovich integration}}{\circ} dB_t^{(i),H}, \quad X_0 = y \in \mathbb{R}^m,$$

where $B_t^H = (B_t^{(1),H}, \dots, B_t^{(m),H})^T$ ← transposed, $V_0, V_i \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$ (space of inf. diff. funct. with bounded partial derivatives of all orders)

Define

$$[V,W](x) := \overset{\text{derivative of } V}{D}V(x)\cdot W(x) - DW(x)\cdot V(x) \qquad \text{(Lie bracket)}$$

for cont. diff. $V,W: \mathbb{R}^d \to \mathbb{R}^d$

Assume Hörmander's condition: For all $x \in \mathbb{R}^d$ the vectors $V_{j_0}(x)$, $[V_{j_0},V_{j_1}](x)$, $[[V_{j_0},V_{j_1}],V_{j_2}](x), \dots$, $0 \le j_0, j_1, \dots, j_m \le m$ span $\mathbb{R}^d$.

Then X_t has a probability density w.r.t. the Lebesgue meas. for all $t>0$

If $H=\frac{1}{2}$ (i.e. the case of Bm) then this density is smooth

→ can be used to construct smooth fundamental solutions to the Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \mathcal{A}u(t,x) , & t>0 , x \in \mathbb{R}^d \\ u(t,\cdot) \longrightarrow f \quad \text{as } t \to 0 , \end{cases}$$

where

$$\mathcal{A}g = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} g(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} g(x)$$

5. Applications to machine learning, statistics and finance

<u>course programme</u>:

We aim at discussing e.g. the following topics

1. Integration of controlled rough paths
2. Doob-Meyer type decomposition for rough paths
3. Itô's formula for rough paths and applications
4. SDE's driven by rough paths
    - (i) Continuity of the Itô-Lyons map
    - (ii) Flows of rough path driven SDE's
    - (iii) Wong-Zakai theorem
    - (iv) Hörmander's theorem
5. SPDE's driven by rough paths
6. Regularity structures with applications (e.g. KPZ equation)

2. The space of rough paths

Motivation: Consider the diff. eq.

$$Y_t = \xi + \int_0^t \underbrace{F(Y_u)}_{=(F_{ij}(Y_u))_{ij}} \, dX_u \qquad (2.1)$$

for smooth F and $X = (X^{(1)}, \dots, X^{(d)})$

$$\Longleftrightarrow \quad Y_t^{(i)} = \xi^{(i)} + \sum_{j=1}^{d} \int_0^t F_{ij}(Y_u) \, dX_u^{(j)} \, , \quad i = 1, \dots, m$$

$\longrightarrow$ first-order Euler approximation:

$$Y_t^{(i)} - Y_s^{(i)} \approx \sum_{j=1}^{d} F_{ij}(Y_s) \int_s^t dX_u^{(j)} \, , \quad i = 1, \dots, m \qquad (2.2)$$

Then Taylor's expansion applied to (2.1) gives a 2-step Euler approximation:

$$Y_t^{(i)} - Y_s^{(i)} = \sum_{j=1}^{d} F_{ij}(Y_s) \int_s^t dX_u^{(j)} + \sum_{j=1}^{d} \int_s^t \left(F_{ij}(Y_u) - F_{ij}(Y_s) \right) dX_u^{(j)}$$

$$\underset{(2.2)}{\overset{\text{Taylor}}{\approx}} \sum_{j=1}^{d} F_{ij}(Y_s) \int_s^t dX_u^{(j)} + \sum_{j,k=1}^{d} \sum_{\ell=1}^{m} \frac{\partial}{\partial x_\ell} F_{ij}(Y_s) F_{\ell k}(Y_s) \int_s^t \int_s^u dX_r^{(k)} \, dX_u^{(j)} \, , \qquad (2.3)$$

$i = 1, \dots, m$ for small $|t-s|$

$\longrightarrow$ the approximation (2.3) suggests that in addition to

$$\left(\int_s^t dX_u^{(j)} \right)_{j=1,\dots,d}$$

also the information contained in

$$\left(\int_s^t\int_s^u dX_r^{(k)}\, dX_u^{(j)}\right)_{k,j=1,\dots,d} \qquad (2.4)$$

or equivalently in tensor notation form

$$\int_s^t (X_u - X_s)\otimes dX_u \qquad (2.5)$$

may play a crucial role in the definition of the integral

$$\int_0^t F(Y_u)\, dX_u$$

w.r.t to rough paths X or more generally of integrals of the form

$$\int^t Z_u\, dX_u$$

for controlled rough paths Z as integrands (see later)

Since rough path theory is also valid for paths X with values in inf. dim. spaces, we shall recall here (in view of applications later on) tensor products of Banach spaces w.r.t. the projective tensor norm:

Def. 2.1 (alg. tensor product)

Let V, W be vector spaces over $\mathbb{R}$. Set $\mathcal{I} = V\times W$

Define the vector space F by

$$F := \{ f : \mathcal{I} \to \mathbb{R} : f(i) \neq 0 \text{ for at most finitely many } i \}$$

Define $\mathbb{1}_i \in F$ by

$$\mathbb{1}_i(j) = \begin{cases} 1, & \text{if } j = i \\ 0 & \text{else} \end{cases}$$

Consider the subspace U of F spanned by the vectors

$$\mathbb{1}_{\mathcal{I}\ni(v,w_1+w_2)} - \mathbb{1}_{(v,w_1)} - \mathbb{1}_{(v,w_2)}$$

$$\mathbb{1}_{(v_1+v_2,w)} - \mathbb{1}_{(v_1,w)} - \mathbb{1}_{(v_2,w)}$$

$$\mathbb{1}_{(\alpha v,w)} - \lambda\, \mathbb{1}_{(v,w)}$$

$$\mathbb{1}_{(v,\lambda w)} - \lambda \, \mathbb{1}_{(v,w)}$$

for $v_i, v \in V$, $w_i, w \in W$, $i=1,2$, $\lambda \in \mathbb{R}$

Then the <u>tensor product of V and W</u> denoted by $V \otimes W$ is defined as

$$V \otimes W = F/U$$

$$(= \{ f+U : f \in F \} , \ (f_1+U)+(f_2+U) := (f_1+f_2)+U$$
$$\lambda(f+U) := (\lambda f)+U \ , \ f+U := \{ f+u : u \in U \})$$

<u>Rem.</u> : (i) $v \in V, w \in W \longrightarrow v \otimes w := \mathbb{1}_{(v,w)} + U$

$\longrightarrow$ $V \otimes W$ (linear in v and w)

(ii) $(e_i)_{i \in J_1}$, $(e'_i)_{i \in J_2}$ bases of V, W, resp.

$\longrightarrow$ $(e_i \otimes e'_j)_{i \in J_1, j \in J_2}$ basis of $V \otimes W$

Consider now a <u>Banach space</u> V with norm $\| \cdot \|_1$, thas is a complete vector space w.r.t. $\| \cdot \|_1$ (i.e. $\| x_n - x_m \|_1 \xrightarrow[n,m \to \infty]{} 0 \implies$ there is a $x \in V$: $\| x_n - x \|_1 \xrightarrow[n \to \infty]{} 0$)

Let W another Banach space w.r.t. to the norm $\| \cdot \|_2$

Consider $Z := V \otimes W$ as in Def. 2.1 and define the norm $\| \cdot \|$ (<u>projective norm</u>) on Z by

$$\| u \| := \inf \Big\{ \sum_{i=1}^{n} \| v_i \|_1 \cdot \| w_i \|_2 \ : \ u = \sum_{i=1}^{n} v_i \otimes w_i \Big\} \qquad (2.6)$$

Denote by $\hat{Z}$ the Banach space given by the completion of Z w.r.t. $\| \cdot \|$ (which we also denote by $V \otimes W$ from now on by convenience)

Then one shows that

$$\| v \otimes w \| = \| v \|_1 \cdot \| w \|_2 \qquad (2.7)$$

for all $v \in V$, $w \in W$.

Let F be another Banach space w.r.t. the norm $\|\cdot\|_3$

Now let $A: V\times W \rightarrow F$ be a <u>bounded bilinear map</u>, i.e.

(i) $A(v_1+v_2, w) = A(v_1,w)+A(v_2,w)$, $A(\lambda v, w) = \lambda A(v,w)$

$\quad\quad\quad\quad\quad ''\quad\quad\quad\quad\quad\quad ''\quad\quad\quad$, $A(v,\lambda w) = \lambda A(v,w)$

(ii) A bounded, i.e.:

$$|A| := \sup_{v\neq 0, w\neq 0} \frac{\|A(v,w)\|_3}{\|v\|_1\cdot\|w\|_2} < \infty \qquad (2.8)$$

Denote by $L(V\times W; F)$ the space of all such maps A

$\longrightarrow$ $Q := L(V\times W, F)$ Banach space with norm $|A|$

Further, we also denote for another Banach space R with norm $\|\cdot\|_R$ by $L(R,F)$ the space of linear maps $B: R\rightarrow F$ which are bounded, i.e.

$$|B|_* := \sup_{r\neq 0} \frac{\|B(r)\|_3}{\|r\|_R} < \infty \qquad (2.9)$$

$\longrightarrow$ $E := L(R,F)$ Banach space with norm $|B|_*$

$\longrightarrow$ <u>Theorem 2.2</u> (Lifting theorem)

Let $A\in L(V\times W, F)$ and $R := V\otimes W$ with projective norm $\|\cdot\|$. Then there exists a <u>unique</u> $B\in L(R,F)$ such that

$$A(v,w) = B(v\otimes w) \quad \text{for all } v\in V, w\in W$$

Furthermore

$$|A| = |B|_* \qquad (2.10)$$

(isometry)

<u>Rem. 2.3</u> So the linear map $\overset{L(V\times W,F)}{\overset{\rotatebox{90}{$\in$}}{A}} \longmapsto \overset{L(R,F)}{\overset{\rotatebox{90}{$\in$}}{B}}$ is one-to-one onto $L(R,F)$ and distance preserving in the sense of the isometry (2.10)

$(\longrightarrow L(V\times W, F) \cong L(V\otimes W, F))$

$\longrightarrow$ any (bounded) bilinear map can be turned into a (bounded) linear map

Example 2.4: $V=\mathbb{R}^d$, $W=\mathbb{R}^m$ with norms

$\|v\|_1 = (v_1^2+\dots+(v_d)^2)^{1/2}$, $v=(v_1,\dots,v_d)$,

$\|w\|_2 = (w_1^2+\dots+w_m^2)^{1/2}$, $w=(w_1,\dots,w_m)$

$\longrightarrow$ $\|v\otimes w\| = \|v\|_1\cdot\|w\|_2$ and

matrix norm

$v\otimes w$ can be identified with

$$v\cdot w^{\overset{\text{transposed}}{T}} = (v_i\cdot w_j)_{1\le i\le d,1\le j\le m} = \begin{pmatrix} v_1w_1 & \cdots & v_1w_m \\ \vdots & & \\ v_dw_1 & \cdots & v_dw_m \end{pmatrix}$$

So $\mathbb{R}^d\otimes\mathbb{R}^m$ can be identified with $\mathbb{R}^{d\times m}$

Consider now $V=\mathbb{R}^d$ and a smooth path $X:[0,T]\to V$ with $X_t=(X_t^{(1)},\dots,X_t^{(d)})$.

Then

$$(X_u-X_s)\otimes(X_t-X_u) \overset{\text{Ex. 2.4}}{=} \left((X_u^{(i)}-X_s^{(i)})\cdot(X_t^{(j)}-X_u^{(j)})\right)_{1\le i,j\le d} \qquad (2.11)$$

Let $s\le u\le t$. Then integration by parts gives

$$(X_u^{(i)}-X_s^{(i)})(X_t^{(j)}-X_u^{(j)}) = \int_0^T \mathbb{1}_{[s,u]}(\ell)\,dX_\ell^{(i)}\cdot\int_0^T \mathbb{1}_{[u,t]}(\ell)\,dX_\ell^{(j)}$$

$$= \int_0^T\int_0^r \mathbb{1}_{[s,u]}(\ell)\,dX_\ell^{(i)}\,\mathbb{1}_{[u,t]}(r)\,dX_r^{(j)}$$

$$+ \underbrace{\int_0^T\int_0^r \mathbb{1}_{[u,t]}(\ell)\,dX_\ell^{(j)}\,\mathbb{1}_{[s,u]}(r)\,dX_r^{(i)}}_{=0}$$

$$= \int_u^t \underbrace{(X_u^{(i)}-X_s^{(i)})}_{=:X_{s,u}^{(i)}}\,dX_r^{(j)} \qquad (2.12)$$

On the other hand, we have that

$$\underbrace{\int_s^t X_{s,r}^{(i)}\,dX_r^{(j)} - \int_s^u X_{s,r}^{(i)}\,dX_r^{(j)}}_{=\int_u^t X_{s,r}^{(i)}\,dX_r^{(j)}} - \int_u^t X_{u,r}^{(i)}\,dX_r^{(j)}$$

$$= \int_u^t (X_{s,r}^{(i)}-X_{u,r}^{(i)})\,dX_r^{(j)} = \int_u^t \overbrace{(X_u^{(i)}-X_s^{(i)})}^{=X_{s,u}^{(i)}}\,dX_r^{(j)} \qquad (2.13)$$

Define the matrices

$$\mathbb{M}_{\ell,r} = \Big(\int_\ell^r X^{(i)}_{\ell,r}\, dX^{(j)}_r\Big)_{1\le i,j\le d} \in \mathbb{R}^{d\times d}$$

$$\xrightarrow[(2.13)]{(2.11),(2.12)} \quad \mathbb{M}_{s,t} - \mathbb{M}_{s,u} - \mathbb{M}_{u,t} = X_{s,u} \otimes X_{u,t}, \qquad (2.14)$$

where

$$X_{\ell,r} := X_r - X_\ell \qquad (2.15)$$

Assume now that a sequence of smooth paths $X^{(n)}$ approximates a rough path X. Then one may expect that

$$X^{(n)}_{s,u} \otimes X^{(n)}_{u,t} \xrightarrow[n\to\infty]{} X_{s,u} \otimes X_{u,t} \qquad (2.16)$$

and

$$\mathbb{M}^{(n)}_{s,t} - \mathbb{M}^{(n)}_{s,u} - \mathbb{M}^{(n)}_{u,t} \xrightarrow[n\to\infty]{} \mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} \qquad (2.17)$$

for a (continuous) function $\mathbb{X} : \underbrace{[0,T]\times[0,T]}_{=[0,T]^2} \longrightarrow \underbrace{\mathbb{R}^{d\times d}}_{=V\otimes V}$

$\xrightarrow{(2.14),(2.16),(2.17)}$ a (Hölder continuous) rough path $X : [0,T] \to V$ should satisfy the following condition (which also reflects the additivity property of integrals):

<u>Chen's relation</u>:

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}, \quad s,u,t \in [0,T] \qquad (2.18)$$

for a continuous function $\mathbb{X} : [0,T]\times[0,T] \to V\otimes V$

In view of (2.16) and (2.17) we denote $\mathbb{X}_{s,t}$ by $\int_s^t X_{s,r} \otimes dX_r$

Rem. 2.5:

(i) $s=u=t$ in (2.18) $\longrightarrow \mathbb{X}_{t,t} = \Theta$

(ii) $\mathbb{X}$ is determined up to an increment of a cont. function $F : [0,T] \to V\otimes V$:
With $\mathbb{X}_{\ell,r}$ also $\mathbb{X}_{\ell,r} + F_r - F_\ell$ satisfies (2.18)
On the other hand:

If $\mathbb{X}, \bar{\mathbb{X}}$ satisfy (2.18), then it follows from (2.18) that

$$G_{s,t} = G_{u,t} + G_{s,u}$$

for $G_{s,t} := \mathbb{X}_{s,t} - \bar{\mathbb{X}}_{s,t}$

$$\xrightarrow{s=0} \quad G_{u,t} = \underbrace{G_{0,t}}_{=: F_t} - G_{0,u}$$

(iii) Using Itô's formula w.r.t. the Bm $B = (B^{(1)}, \dots, B^{(d)})$, one shows (2.18) for $\mathbb{X}_{s,t} = \left(\int_s^t (B_r^{(i)} - B_s^{(i)}) \, dB_r^{(j)} \right)_{1 \le i,j \le d}$

Itô-formula:

$$f(B_t) = f(0) + \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} f(B_s) \, dB_s^{(i)} + \sum_{i=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i^2} f(B_s) \, ds \qquad (2.19)$$

$$\xrightarrow{d=1,\ f(x) = \frac{1}{2}x^2} \quad \int_s^t (B_r - B_s) \, dB_r = \frac{1}{2}(B_t - B_s)^2 - \frac{1}{2}(t-s) = \mathbb{X}_{s,t} \text{ for } d=1 \quad (*)$$

$B \in C^\alpha([0,T];\mathbb{R})$ for all $\alpha < \frac{1}{2}$

$$\xrightarrow{(*)} \quad \sup_{s \neq t} \frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}} < \infty \quad \text{with prob. } 1$$

⟶ suggests the following def. of a rough path space:

Def. 2.6 (Rough path space)
Let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. Then the space of α-Hölder rough paths denoted by $\mathscr{C}^\alpha([0,T],V)$ is the collection of all $Z := (X, \mathbb{X})$ satisfying (2.18) and

$$\|X\|_\alpha := \sup_{s \neq t} \frac{|X_{s,t}|}{|t-s|^\alpha} < \infty, \quad \|\mathbb{X}\|_{2\alpha} := \sup_{s \neq t} \frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}} < \infty$$

Rem 2.7:

(i) $\mathscr{C}^\alpha([0,T],V)$ is not a vector space

(ii) $\mathscr{C}^\alpha([0,T],V)$ becomes a complete metric space with metric ϱ_α given by

distance between Z_1 and Z_2 ⟶ $\varrho_\alpha(Z_1, Z_2) \overset{\text{def.}}{=} |X_0 - Y_0| + \|X - Y\|_\alpha + \|\mathbb{X} - \mathbb{Y}\|_{2\alpha}$

for $Z_1 = (X, \mathbb{X}),\ Z_2 = (Y, \mathbb{Y}) \in \mathscr{C}^\alpha([0,T],V)$

⟶ deficiency of Chen's relation:
It captures the basic property of additivity of

integrals, <u>but not</u> integration by parts as e.g. in the case of smooth paths:

Let $V=\mathbb{R}^d$ and $X=(X^{(1)},\dots,X^{(d)})$ be a smooth path

integration by parts $\longrightarrow$

$$X^{(i)}_{s,t}X^{(j)}_{s,t} = (X^{(i)}_t - X^{(i)}_s)\cdot(X^{(j)}_t - X^{(j)}_s)$$

$$= \int_s^t \underbrace{(X^{(i)}_r - X^{(i)}_s)}_{=X^{(i)}_{s,r}}\, dX^{(j)}_r + \overbrace{\int_s^t \underbrace{(X^{(j)}_r - X^{(j)}_s)}_{=X^{(j)}_{s,r}}\, dX^{(i)}_r}^{\mathbb{X}^{ji}_{s,t}}$$

Now choosing a sequence of smooth $X^{(n)}$ approximating a rough path X (in some reasonable sense) we can argue just as in (2.16) and (2.17) and postulate that

$$\mathbb{X}^{ij}_{s,t} + \mathbb{X}^{ji}_{s,t} = X^{(i)}_{s,t}X^{(j)}_{s,t}, \quad 1\le i,j\le d,\ s,t\in[0,T]$$

or equivalently

$$\mathrm{Sym}(\mathbb{X}_{s,t}) = \frac{1}{2}X_{s,t}\otimes X_{s,t}, \quad s,t\in[0,T] \qquad (2.20)$$

holds for a continuous function $\mathbb{X}:[0,T]\times[0,T]\to V\otimes V$,

$$\mathrm{Sym}(\overset{\in\mathbb{R}^{d\times d}}{A}) \overset{\text{def.}}{=} \frac{1}{2}(A+A^{T}) \leftarrow \text{transposed}$$

<u>Rem. 2.8</u>: Using Itô's formula (2.19) we find that

$$\mathbb{X}_{s,t} \overset{\text{def.}}{=} \mathbb{B}^{\text{strat}}_{s,t} \overset{\text{def}}{=} \left(\int_s^t (B^{(i)}_r - B^{(i)}_s)\circ dB^{(j)}_r\right)_{1\le i,j\le d},$$

satisfies (2.18) and (2.20), where

$$\int_s^t (B^{(i)}_r - B^{(i)}_s)\circ dB^{(j)}_r \overset{\text{def}}{=} \int_s^t (B^{(i)}_r - B^{(i)}_s)\, dB^{(i)}_r + \frac{1}{2}(t-s)\overset{=\begin{cases}1, \text{if } i=j\\ 0 \text{ else}\end{cases}}{\boxed{\delta_{ij}}}$$

is a Stratonovich integral and $B=(B^{(1)},\dots,B^{(d)})$ a Bm

$\longrightarrow$ space of geometric rough paths $\mathscr{C}^\alpha_g \subset \mathscr{C}^\alpha$!

<u>Def. 2.9</u> (Geometric rough paths)
Let $\alpha\in(\frac{1}{3},\frac{1}{2}]$. The space of geometric rough paths, denoted by $\mathscr{C}^\alpha_g$, is defined as the set of all $(X,\mathbb{X})\in\mathscr{C}^\alpha$ satisfying in addition to 2.18) the relation (2.20).

Rem: $\mathscr{C}^\alpha_g \subset \mathscr{C}^\alpha$ closed subspace w.r.t. the metric ϱ_α

→ <u>interpretation of rough paths as paths with values in Lie groups</u>:

Let $V=\mathbb{R}^d$ and $(X,\mathbb{X})\in\mathscr{C}^{\alpha}([0,T],V)$. Then

$$\mathbf{X}_{s,t} := (1, X_{s,t}, \mathbb{X}_{s,t}) \in \sum_{i=0}^{N}\Big(\bigotimes_{j=0}^{i} V\Big) = \mathbb{R}\times V\times(V\otimes V) =$$

$$= \mathbb{R}\times\mathbb{R}^{d}\times\mathbb{R}^{d\times d} =: T^{(2)}(\mathbb{R}^d) = T^{(N)}(V) \qquad (2.21)$$

for $N=2$ $\quad\Big(\bigotimes_{j=0}^{0} V \overset{def}{=} \mathbb{R}\Big)$

→ $T^{(2)}(\mathbb{R}^d)=T^{(N)}(\mathbb{R}^d)$ is a vector space over $\mathbb{R}$

If $g=(\overset{\in\mathbb{R}}{g_0}, \overset{\in\mathbb{R}^d}{g_1}, \overset{\in\mathbb{R}^{d\times d}}{g_2})$, $h=(h_0,h_1,h_2)\in T^{(2)}(\mathbb{R}^d)$, define multiplication $\otimes$ of g and h by

$$g\otimes h \overset{def}{=} \sum_{k=0}^{N}\Big(\sum_{i=0}^{k} g_{k-i}\otimes h_i\Big) \qquad (2.22)$$

$$\overset{N=2}{=} (g_0 h_0,\ h_0\cdot g_1 + g_0\cdot h_1,\ h_0\cdot g_2 + g_1\otimes h_1 + g_0\cdot h_2)$$

Recall: $x\otimes y = x\cdot y^T$, $x,y\in\mathbb{R}^d$

→ map A given by $A(g,h)=g\otimes h$, $g,h\in T^{(2)}(\mathbb{R}^d)$ is bilinear and the multiplication is associative:

$a\otimes(b\otimes c) = (a\otimes b)\otimes c$ for all $a,b,c\in T^{(2)}(\mathbb{R}^d)$

⟹ $T^{(2)}(\mathbb{R}^d)=T^{(N)}(\mathbb{R}^d)$ becomes an <u>associative algebra</u> under $\otimes$, called <u>truncated tensor algera of level $N=2$</u>

Ex. 2.10: $\mathbf{X}_{s,u}\otimes\mathbf{X}_{u,t} \overset{(2.22)}{=} \Big(1, \underbrace{\overset{X_u-X_s}{X_{s,u}}+X_{u,t}}_{=X_{s,t}}, \underbrace{\mathbb{X}_{s,u}+X_{s,u}\otimes X_{u,t}+\mathbb{X}_{u,t}}_{\overset{(2.18)}{=}\ \mathbb{X}_{s,t}}\Big)$

$= \mathbf{X}_{s,t}$

Now define

$$T_a^{(2)}(\mathbb{R}^d) \overset{N=2}{=} T_a^{(N)}(\mathbb{R}^d) := \{(\overset{\mathbb{R}}{g_0}, \overset{\mathbb{R}^d}{g_1}, \overset{\mathbb{R}^{d\times d}}{g_2}) : g_0=a\} \qquad (2.23)$$

In fact for $a=1$, $T_1^{(2)}(\mathbb{R}^d)$ is a <u>group</u> <u>w.r.t. $\otimes$</u> with <u>unit element $(1,0,0)$</u>, since e.g.

⑱ if $g=(1,g_1,g_2)\in T_1^{(2)}(\mathbb{R}^d)$, then the inverse of g is

$$g^{-1} = \sum_{K=0}^{N} (-1)^K \Big(\bigotimes_{i=0}^{K} a\Big) = \big(\overset{\in\mathbb{R}}{1}, -\overset{\in\mathbb{R}^d}{g_1}, \overset{\in\mathbb{R}^{d\times d}}{-g_2+g_1\otimes g_2}\big) \qquad (2.24)$$

for $N=2$, $a=(0,g_1,g_2)$, i.e.

$$g\otimes g^{-1} = (1,g_1,g_2)\otimes(1,-g_1,-g_2+g_1\otimes g_2) = (1,0,0)$$

Further, $T_1^{(2)}(\mathbb{R}^d) = T_1^{(N)}(\mathbb{R}^d)$ is a <u>Lie group</u>, that is a smooth manifold M, which is a group w.r.t. $\cdot$ such that the maps

$$\begin{aligned} &(x,y) \mapsto x\cdot y \\ \text{and}\quad &x \mapsto x^{-1} \end{aligned} \qquad (2.25)$$

are smooth.

$\longrightarrow$ here: $M = T_1^{(2)}(\mathbb{R}^d)$ ($\cong \mathbb{R}^{d+d^2}$) and $\cdot = \otimes$
(take the chart $(T_1^{(2)}(\mathbb{R}^d), \varphi)$ with $\varphi(1,g_1,g_2) = (g_1,g_2) \in \mathbb{R}^{d+d^2}$)

Ex. 2.10 $\longrightarrow$ $\mathbb{X}_{u,t} = \mathbb{X}_{s,u}^{-1}\otimes\mathbb{X}_{s,t} \overset{s=0}{\Longrightarrow} \mathbb{X}_{u,t} = \mathbb{X}_{0,u}^{-1}\otimes\mathbb{X}_{0,t}$

Similarly to the classical we can introduce an <u>exponential function</u>

$$\mathrm{Exp}: T_0^{(N)}(\mathbb{R}^d) \longrightarrow T_1^{(N)}(\mathbb{R}^d) \text{ by} \qquad (2.26)$$

$$\mathrm{Exp}(g) \overset{\text{def.}}{=} \underset{(1,0,0)}{1} + \sum_{K=1}^{N} \frac{1}{K!} g^{\otimes K} \text{ with } g^{\otimes K} \overset{\text{def.}}{=} \bigotimes_{i=1}^{K} g$$

and a <u>logarithmic function</u>

$$\mathrm{Log}: T_1^{(N)}(\mathbb{R}^d) \longrightarrow T_0^{(N)}(\mathbb{R}^d) \text{ by} \qquad (2.27)$$

$$\mathrm{Log}(h) \overset{\text{def.}}{=} \sum_{K=1}^{N} \frac{(-1)^{K+1}}{K} \hat{h}^{\otimes K} \quad \text{with } \hat{h} = (0,h_1,\dots,h_N)$$

for $h = (1,h_1,\dots,h_N)$

for $N=2$

A direct calculation shows that

$$\mathrm{Exp}(\mathrm{Log}(h)) = h \quad\text{and}\quad \mathrm{Log}(\mathrm{Exp}(g)) = g \qquad (2.28)$$

$\Longrightarrow \quad T_1^{(N)}(\mathbb{R}^d) = \mathrm{Exp}[T_0^{(N)}(\mathbb{R}^d)]$

<u>However</u> : $\mathrm{Exp}(a+b) \neq \mathrm{Exp}(a) \otimes \mathrm{Exp}(b)$ in general

But for $N=2$ we find that

$$\mathrm{Exp}(a) \otimes \mathrm{Exp}(b) \overset{\text{def.}}{=} (1+a+a^{\otimes 2}) \otimes (1+b+\frac{b^{\otimes 2}}{2})$$
$$= 1+a+b+\frac{1}{2}(a\otimes b - b\otimes a) + \frac{(a+b)^{\otimes 2}}{2}$$
$$= \mathrm{Exp}(a+b+\frac{1}{2}[a,b]), \qquad (2.29)$$

where

$$[a,b] \overset{\text{def}}{=} a\otimes b - b \otimes a$$

is a <u>Lie bracket</u> on $T_0^{(N)}(\mathbb{R}^d)$

$\longrightarrow$ $T_0^{(N)}(\mathbb{R}^d)$ is a <u>Lie algebra</u>, <u>that is</u> it is (2.30) a vector space U with a bilinear map $[\cdot,\cdot] : U \times U \longrightarrow U$ satisfying the conditions

(i) $[g,h] = -[h,g]$, $h,g \in U$ (anticommutativity law)

(ii) $[g,[h,K]] + [h,[K,g]] + [K,[g,h]] = 0$, $g,h,K \in U$ (Jacobi identity)

<u>In fact</u>, one can generalize (2.29) and gets for $N>2$

<u>Th. 2.11</u> (Campbell-Baker-Hausdorff)

$$\mathrm{Log}(\mathrm{Exp}(a) \otimes \mathrm{Exp}(b)) = b + \int_0^1 \sum_{n=0}^{N} \frac{(-1)^n}{n+1} e^{t\,\mathrm{ad}\,a}(e^{t\,\mathrm{ad}\,b}(a))\, dt,$$

where

$\mathrm{ad}\,a : T_0^{(N)}(\mathbb{R}^d) \longrightarrow T_0^{(N)}(\mathbb{R}^d)$ defined by

$$(\mathrm{ad}\,a)(g) = [a,g] \qquad (2.31)$$

and

$$e^{t\,\mathrm{ad}\,a}(g) := \sum_{n=0}^{N} \frac{t^n}{n!} (\mathrm{ad}\,a)^n (g)$$

$A^n = A \circ \ldots \circ A$ (n times) for a map

$\longrightarrow$ here : $A = \mathrm{ad}\,a$

⑳ Rem.: $(\mathrm{ad}\,a)^n = 0$ for $n > N$ (2.32)

Consider now the vector space $\mathfrak{g}^{(N)}(\mathbb{R}^d) \subset T_0^{(N)}(\mathbb{R}^d)$ spanned by vectors from

$$V,\ [V,V],\ [V,[V,V]],\ \dots,\ \underbrace{[V,[\dots,[V,V]]]}_{N-1 \text{ brackets}},$$

where $V \overset{\text{def}}{=} \{(0,a,0,\dots,0) : a \in \mathbb{R}^d\}$

and $[Z,W] \overset{\text{def}}{=} \{[a,b] : a \in Z,\ b \in W\}$,

for $Z, W \subseteq T_0^{(N)}(\mathbb{R}^d)$

$\longrightarrow$ $\mathfrak{g}^{(N)}(\mathbb{R}^d)$ Lie algebra w.r.t. $[\cdot,\cdot]$, called *free step-N nilpotent Lie algebra*

$$\xrightarrow{(2.22)} \mathfrak{g}^{(N)}(\mathbb{R}^d) = V \oplus \underset{\in \mathbb{R}^{d\times d}}{[V,V]} \oplus \dots \oplus [V,[\dots,[V,V]]] \quad (2.33)$$

Define

$$G^{(N)}(\mathbb{R}^d) = \{\mathrm{Exp}(a) : a \in \mathfrak{g}^{(N)}(\mathbb{R}^d)\} \quad (2.34)$$
$$\subseteq T_1^{(N)}(\mathbb{R}^d)$$

$\longrightarrow$ *group* w.r.t. $\otimes$ because of (2.29) for $N=2$ or because of Th. 2.11 for $N>2$

It is *also* a Lie group

(take the chart w.r.t. $\pi \circ \mathrm{Log}$, where $\pi((0,g_1,\dots,g_N)) := (g_1,\dots,g_N) \in \mathbb{R}^d \times \dots \times \mathbb{R}^{d^N}$)

Let now $(X_{s,t}, \mathbb{X}_{s,t}) \in \mathscr{C}_g^\alpha$ be a geometric rough path

Then for $\mathbf{X}_{s,t} = (1, X_{s,t}, \mathbb{X}_{s,t})$

$$\mathrm{Log}(\mathbf{X}_{s,t}) \overset{(2.27)}{=} (0, X_{s,t}, \mathbb{X}_{s,t}) - \tfrac{1}{2}(0, X_{s,t}, \mathbb{X}_{s,t})^{\otimes 2}$$

$$\overset{(2.22)}{=} (0, X_{s,t}, \mathbb{X}_{s,t}) - (0, 0, \underbrace{\tfrac{1}{2} X_{s,t} \otimes X_{s,t}}_{= \mathrm{Sym}(\mathbb{X}_{s,t})})$$

$$= (0, X_{s,t}, \underbrace{\mathbb{X}_{s,t} - \mathrm{Sym}(\mathbb{X}_{s,t})}_{\in [V,V]})$$

since e.g. $e_i \otimes e_j - \mathrm{Sym}(e_i \otimes e_j) = \frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i)$
$= \frac{1}{2}[e_i, e_j]$ for basis elements e_i

$\Rightarrow$ $\mathrm{Log}(\mathbb{X}_{s,t}) \in g^{(N)}(\mathbb{R}^d)$

$\overset{(2.34)}{\Rightarrow}$ $(1, X_{s,t}, \mathbb{X}_{s,t}) = (1, X^{(1)}_{s,t}, X^{(2)}_{s,t}, \dots, \overset{\in \mathbb{R}^{d^N}}{X^{(N)}_{s,t}})$

$\in G^{(N)}(\mathbb{R}^d)$ for $N=2$ (2.35)

In fact one can introduce a <u>metric</u> on $G^{(N)}(\mathbb{R}^d)$ by means of the <u>Carnot-Caratheodory norm</u> $\|\cdot\|_C$:

Consider the <u>step-N signature</u> of a smooth path γ:

$$S^{(N)}(\gamma)_{s,t} := \left(1, \int_{s<u<t} d\gamma_u, \dots, \int_{s<u_1<\dots<u_N<t} d\gamma_{u_1} \otimes \dots \otimes d\gamma_{u_N}\right)$$

$$\overset{N=2}{=} \left(1, \int_s^t d\gamma_{u_1}, \int_s^t \int_s^{u_2} d\gamma_{u_1} \otimes d\gamma_{u_2}\right)$$

$$= \left(1, \gamma_t - \gamma_s, \int_s^t (\gamma_{u_2} - \gamma_s) \otimes d\gamma_{u_2}\right) \in G^{(N)}(\mathbb{R}^d)$$

⟶ $\|g\|_C \overset{\text{def}}{=} \inf\left\{ \underbrace{\int_0^1 |\dot{\gamma}_t| dt}_{\text{length of } \gamma} : \gamma \in C^1([0,1]; \mathbb{R}^d), S^{(N)}(\gamma)_{0,1} = g \right\}$ (2.36)

⟶ metric d_C on $G^{(N)}(\mathbb{R}^d)$ defined by
$d_C(g, h) = \|g^{-1} \otimes h\|_C$ (2.37)

⟶ <u>properties</u>:

(i) d_C left invariant: $d_C(g \otimes h, g \otimes k) = d_C(h, k)$

(ii) $\|g\|_C = 0 \Leftrightarrow g = (1, 0, \dots, 0)$

(iii) sub-additivity: $\|g \otimes h\|_C \leq \|g\|_C + \|h\|_C$

(iv) symmetry: $\|g\|_C = \|g^{-1}\|_C$

(v) homogeneity: $g = (1, g_1, \dots, g_N)$, $g_\lambda := (1, \lambda g_1, \dots, \lambda^N g_N)$
$\Rightarrow$ $\|g_\lambda\| = |\lambda| \, \|g\|$, $\lambda \in \mathbb{R}$

(vi) $\exists C_1, C_2 > 0$: $C_1(|g_1| + |g_2|^{1/2}) \leq \|g\|_C \leq$
$C_2 \cdot (|g_1| + |g_2|^{1/2})$ for all $g = (1, g_1, g_2) \in G^{(N)}(\mathbb{R}^d)$
for $N=2$

$$\mathbf{X}_{u,t}=\mathbf{X}_{0,u}^{-1}\otimes\overset{=:\,\mathbf{X}_t}{\mathbf{X}_{0,t}} \xRightarrow[(vi)]{} C_1 |||\mathbf{X}|||_{\alpha,[0,T]} \le \sup_{s,t\in[0,T]} \frac{d_c(\mathbf{X}_s,\mathbf{X}_t)}{|t-s|^\alpha} \le$$

$$C_2 |||\mathbf{X}||_{\alpha,[0,T]} \quad , \qquad (2.38)$$

where

$$|||\mathbf{X}||_{\alpha,[0,T]} \overset{def.}{=} \sup_{\substack{s,t\in\\ [0,T]}} \frac{|X_{s,t}|}{|s-t|^\alpha} + \sup_{\substack{s,t\in\\ [0,T]}} \frac{|\mathbb{X}_{s,t}|^{1/2}}{|s-t|^\alpha}$$

for $C_1, C_2 > 0$

$\longrightarrow$ Prop. 2.12 (Characterization of geometric rough paths)

Let $\alpha\in(1/3,\frac{1}{2}]$. Then

1. $(X,\mathbb{X})\in\mathcal{C}_g^\alpha$

$\Longleftrightarrow$ 2. $\mathbf{X}_t=(1, X_{0,t}, \mathbb{X}_{0,t})\in G^{(N)}(\mathbb{R}^d)$ and $\mathbf{X}$ is α Hölder continuous w.r.t. the metric d_c

Prop. 2.13 : Let $\beta\in(\frac{1}{3},\frac{1}{2}]$. Then for all $(X,\mathbb{X})\in\mathcal{C}_g^\beta([0,T];\mathbb{R}^d)$ there exists a sequence of smooth paths $X^n:[0,T]\to\mathbb{R}^d$ such that

$$X_t^n \xrightarrow[n\to\infty]{} X_t \text{ uniformly in } t,$$

$$\mathbb{X}^n_{s,t}=\int_s^t X^n_{s,r}\otimes dX^n_r \xrightarrow[n\to\infty]{} \mathbb{X}_{s,t} \text{ unif. in } s,t$$

with $\sup_n (\|X^n\|_\beta + \|\mathbb{X}^n\|_{2\beta}) \le C < \infty$ (2.39)

Lemma 2.14 : Let $(X^n,\mathbb{X}^n)\in\mathcal{C}^\beta$, $n\ge1$ satisfying (2.39) with $X^n_{s,t}\xrightarrow[n\to\infty]{} X_{s,t}$, $\mathbb{X}^n_{s,t}\xrightarrow[n\to\infty]{}\mathbb{X}_{s,t}$ uniformly in s,t. Then

$$S_\alpha((X^n,\mathbb{X}^n),(X,\mathbb{X}))\xrightarrow[n\to\infty]{} 0$$

for all $\alpha\in(\frac{1}{3},\beta)$.

Proof: We find that

$$|X_{s,t}| = \lim_{n\to\infty} |X^n_{s,t}| \le C|t-s|^\beta ,$$

$$|\mathbb{X}_{s,t}| = \lim_{n\to\infty} |\mathbb{X}^n_{s,t}| \le C|t-s|^{2\beta} \text{ for all } s,t$$

$$\Longrightarrow \quad (X,\mathbb{X}) \in \mathscr{C}^\beta$$

On the otherhand, uniform convergence and (2.39) give for a $\varepsilon_n \longrightarrow 0$ that

$$|X_{s,t} - X^n_{s,t}| \le \varepsilon_n , \quad |\mathbb{X}^n_{s,t} - \mathbb{X}_{s,t}| \le \varepsilon_n$$

and $|X_{s,t} - X^n_{s,t}| \le 2C|t-s|^\beta , \; |\mathbb{X}^n_{s,t} - \mathbb{X}_{s,t}| \le 2C|t-s|^{2\beta}$

Then using

$$\min(a,b) \le a^{1-\theta} b^\theta , \; a,b>0 , \; \theta \in (0,1).$$

for $\theta = \frac{\alpha}{\beta}$ we see that

means: inequality up to a constant

$$|X_{s,t} - X^n_{s,t}| \lesssim \varepsilon_n^{1-\alpha/\beta} |t-s|^\alpha$$

and

$$|\mathbb{X}^n_{s,t} - \mathbb{X}_{s,t}| \lesssim \varepsilon_n^{1-\alpha/\beta} |t-s|^{2\alpha}$$

$$\Longrightarrow \quad \varrho_\alpha((X^n_{s,t}, \mathbb{X}^n_{s,t}),(X_{s,t}, \mathbb{X}_{s,t}))$$

$$= |X^n_0 - X_0| + \underbrace{\sup_{s\neq t} \frac{|X^n_{s,t} - X_{s,t}|}{|t-s|^\alpha}}_{\le \varepsilon_n^{1-\alpha/\beta}} + \underbrace{\sup_{s\neq t} \frac{|\mathbb{X}^n_{s,t} - \mathbb{X}_{s,t}|}{|t-s|^{2\alpha}}}_{\le \varepsilon_n^{1-\alpha/\beta}}$$

$$\xrightarrow[n\to\infty]{} 0$$

$\Longrightarrow$ proof.

3. Rough path integrals

Objective: Definition of integrals of the type

$$\int_0^T Y_s \, dX_s , \qquad (3.1)$$

where $(X, \mathbb{X}) \in \mathcal{C}^\alpha([0,T], V)$.

$\longrightarrow$ natural starting point: integration w.r.t. to integrands given by

$$Y_s = F(X_s)$$

for smooth functions (i.e. 1-forms)

$F : V \to L(V,W)$, $\leftarrow \mathbb{R}^{m \times d}$ if $V = \mathbb{R}^d$, $W = \mathbb{R}^m$

since Y_s "looks like" X_s

$\longrightarrow$ 3.1 Integration of 1-forms

Motivation: $F(X_s) = (F_{ij}(X_s))_{1 \le i \le m, 1 \le j \le d}$ $\leftarrow$ matrix entries, $V = \mathbb{R}^d, W = \mathbb{R}^m$,

$$X_t = (X_t^1, \dots, X_t^d)$$

$[s,t]$ small time interval

$$\longrightarrow \int_s^t F(X_r)\, dX_r = \int_s^t F(X_s)\, dX_r + \int_s^t (F(X_r) - F(X_s))\, dX_r$$

$$\underset{\text{approximation}}{\overset{\text{Taylor}}{\approx}} \underbrace{F(X_s) X_{s,t}}_{\mathbb{R}^m \ni \left(\sum_{j=1}^d F_{ij}(X_s) X^j_{s,t}\right)^T_{1 \le i \le m}} + \int_s^t \underbrace{DF(X_s) X_{s,r}}_{\left(\sum_{l=1}^d \frac{\partial}{\partial x_l} F_{ij}(X_s) \cdot X^l_{s,r}\right)_{1 \le i \le m, 1 \le j \le d} \in \mathbb{R}^{m \times d}} dX_r$$

$$= F(X_s) X_{s,t} + \begin{pmatrix} \sum_{j=1}^d \int_s^t \sum_{l=1}^d \frac{\partial}{\partial x_l} F_{1j}(X_s) X^l_{s,r} \, dX^j_r \\ \vdots \\ \sum_{j=1}^d \int_s^t \sum_{l=1}^d \frac{\partial}{\partial x_l} F_{mj}(X_s) X^l_{s,r} \, dX^j_r \end{pmatrix}$$

$$= F(X_s) X_{s,t} + DF(X_s) \Big(\underbrace{\int_s^t X_{s,r} \otimes dX_r}_{= \mathbb{X}_{s,t} = \left(\int_s^t X^l_{s,r} \, dX^j_r \right)_{1 \le l, j \le d}} \Big), \qquad (3.1)$$

where

$$DF(X_s) : \overset{=\mathbb{R}^{d\times d}}{V \otimes V} \longrightarrow \overset{\mathbb{R}^m}{\overset{\parallel}{W}} \quad \text{is given by}$$

$$DF(X_s)(v \otimes w) = \left(\sum_{l,j=1}^{d} \frac{\partial}{\partial x_l} F_{ij}(X_s)\, v_l \cdot w_j \right)^{T}_{1 \leq i \leq m} \leftarrow \text{transposition}$$

See the Lifting theorem (Th. 2.2) in the case of Banach spaces.

Consider $\{t_i\}_{i=1}^{n}$ with $0 = t_1 < \dots < t_n = T$

→ <u>partition</u> $\mathcal{P} = \{[t_i, t_{i+1}] : i = 1, \dots, n-1\}$ of $[0,T]$,

<u>mesh of $\mathcal{P}$</u> : $|\mathcal{P}| := \max_{i=1}^{n-1} |t_i - t_{i+1}|$

Now by <u>"sewing"</u> together the local approximations in (3.1) on small intervals $[s,t]$ to a global approximation on $[0,T]$, we can expect that

$$\int_0^T F(X_r)\, dX_r = \sum_{[s,t]\in\mathcal{P}} \int_s^t F(X_r)\, dX_r$$

$$\approx \sum_{[s,t]\in\mathcal{P}} F(X_s) X_{s,t} + DF(X_s)(\mathbb{X}_{s,t}), \qquad (3.2)$$

if $|\mathcal{P}|$ is small enough

→ <u>natural question</u> :

Why should (3.2) be a good approximation of the integral ?

→ <u>reason</u> : We could have also applied in our previous reasoning w.r.t. (3.1) the 3rd order Taylor approximation similarly and we would have arrived at

$$\int_0^T F(X_r)\, dX_r \approx \sum_{[s,t]\in\mathcal{P}} F(X_s) X_{s,t} + DF(X_s) \mathbb{X}_{s,t} + \frac{1}{2} D^2 F(X_s) \mathbb{X}^{(3)}_{s,t}, \qquad (3.3)$$

where

$$D^2F(X_s)(h^1\otimes h^2\otimes h^3) = \Big(\sum_{j_1,j_2,j_3=1}^{d} \frac{\partial^2}{\partial x_{j_1}\partial x_{j_2}} F_{i j_3}(X_s)\, h^1_{j_1} h^2_{j_2} h^3_{j_3}\Big)^T_{1\le i\le m}$$

for $h^\ell = (h^\ell_1, \dots, h^\ell_d)_{\ell=1,2,3}$

and

$$\overset{\in\mathbb{R}^{d^3}}{\mathbb{X}^{(3)}_{s,t}} = \int_s^t \Big(\int_s^u X_{s,r}\otimes dX_r\Big)\otimes dX_u$$

$$= \Big(\int_s^t\int_s^u X^{i_1}_{s,r}\, dX^{i_2}_r\, dX^{i_3}_u\Big)_{1\le i_1,i_2,i_3\le d}$$

Since for $\alpha\in(\frac{1}{3},\frac{1}{2}]$

$$|X_{s,t}| \lesssim |s-t|^\alpha \text{ and } |\mathbb{X}_{s,t}| \lesssim |s-t|^{2\alpha}$$

we may expect that

$$|\mathbb{X}^{(3)}_{s,t}| \lesssim |s-t|^{3\alpha}$$

$$\overset{3\alpha>1}{\Longrightarrow} \quad \frac{|\mathbb{X}^{(3)}_{s,t}|}{|s-t|} \xrightarrow[|s-t|\to 0]{} 0 \quad, \text{ that is}$$

$$|\mathbb{X}^{(3)}_{s,t}| = o(|s-t|) \text{ for } |s-t|\to 0$$

$$\overset{F \text{ bounded}}{\underset{\text{derivatives}}{\Longrightarrow}} \quad |D^2F(X_s)\mathbb{X}^{(3)}_{s,t}| = o(|s-t|)$$

$$\Rightarrow \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} |D^2F(X_s)\mathbb{X}^{(3)}_{s,t}| \lesssim \lim_{|\mathcal{P}|\to 0} |\mathcal{P}|^{3\alpha-1} = 0$$

$\Rightarrow$ third term in (3.3) negligible

$\Rightarrow$ We <u>should have</u> (as we will see later on)

$$\int_0^T F(X_s)\,dX_s = \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} F(X_s)X_{s,t} + DF(X_s)\mathbb{X}_{s,t} \qquad (3.4)$$

$\longrightarrow$ definition of the rough path integral, if the limit exists

We need the following auxiliary result:

<u>Lemma 3.1</u>: Let $F: V\to L(V,W)$ be in C^2_b (space of twice cont. diff. functions with bounded derivatives) and let $(X,\mathbb{X})\in\mathcal{C}^\alpha$ for some $\alpha\in(\frac{1}{3},\frac{1}{2}]$. Further,

set $Y_s := F(X_s)$, $Y'_s := DF(X_s)$ and $R^Y_{s,t} := Y_{s,t} - Y'_s X_{s,t}$ (in view of the definition of the Gubinelli derivative Y' in a more general framework later on). Then

$$\|Y\|_\alpha \le \|DF\|_\infty \|X\|_\alpha,$$

$$\|Y'\|_\alpha \le \|D^2F\|_\infty \|X\|_\alpha,$$

$$\|R^Y\|_{2\alpha} \le \frac{1}{2} \|D^2F\|_\infty \|X\|_\alpha^2,$$

where $\|DF\|_\infty = \sup_{x\in V} \|DF(x)\|$, $\|D^2F\|_\infty = \sup_{x \in V} \|D^2F(x)\|$.

Rem. 3.2: $V = \mathbb{R}^d$, $W = \mathbb{R}^m$

(i) $$R^Y_{s,t} = \underset{Y_{s,t}}{\overset{\text{matrix}}{\left(F_{ij}(X_t) - F_{ij}(X_t)\right)_{\substack{1\le i\le m\\ 1\le j\le d}}}} - \underset{Y'X_{s,t}}{\left(\sum_{\ell=1}^{d} \frac{\partial}{\partial x_\ell} F_{ij}(X_s)\cdot X^\ell_{s,t}\right)_{\substack{1\le i\le m\\ 1\le j\le d}}}$$

(ii) $$\|DF(x)\| \overset{\text{equivalent}}{\sim} \left(\sum_{\ell,j=1}^{d} \sum_{i=1}^{m} \left(\frac{\partial}{\partial x_\ell} F_{ij}(x)\right)^2\right)^{1/2},$$

$$\|D^2F(x)\| \sim \left(\sum_{\ell_1,\ell_2,j=1}^{d} \sum_{i=1}^{m} \left(\frac{\partial^2}{\partial x_{\ell_1}\partial x_{\ell_2}} F_{ij}(x)\right)^2\right)^{1/2}$$

Proof of L. 3.1: The mean value theorem gives

$$Y_{s,t} = F(X_t) - F(X_s) = DF(X_s + \xi X_{s,t}) X_{s,t}$$

for some $\xi \in (0,1)$

$$\Rightarrow \quad |Y_{s,t}| = |DF(X_s + \xi X_{s,t}) X_{s,t}| \le \|DF(X_s + \xi X_{s,t})\| \, |X_{s,t}| \le \|DF\|_\infty |X_{s,t}|$$

$$\Rightarrow \quad \|Y\|_\alpha \overset{\text{def.}}{=} \sup_{s \ne t} \frac{|Y_{s,t}|}{|s-t|^\alpha} \le \|DF\|_\infty \overbrace{\sup_{s\ne t} \frac{|X_{s,t}|}{|s-t|^\alpha}}^{\|X\|_\alpha}$$

Similarly, we get the estimate for $\|Y'\|_\alpha$

Using Taylor expansion, we find that

$$R^Y_{s,t} \overset{\text{def}}{=} F(X_t) - F(X_s) - DF(X_s) X_{s,t} = \frac{1}{2} D^2F(X_s + \xi X_{s,t})[X_{s,t}, X_{s,t}]$$

for some $\xi \in (0,1)$

$$\Rightarrow \quad |R^Y_{s,t}| \le \frac{1}{2} \underbrace{\|D^2F(X_s + \xi X_{s,t})\|}_{\le \|D^2F\|_\infty} |X_{s,t}| \, |X_{s,t}|$$

$$\Rightarrow \quad \|R^Y_{s,t}\|_{2\alpha} \le \frac{1}{2} \|D^2F\|_\infty \overset{\|X\|_\alpha^2}{\left(\sup_{s\ne t} \frac{|X_{s,t}|}{|s-t|^\alpha}\right)^2}$$

$\Rightarrow$ proof

Our next aim is to "sew" the local approximations

$$\Gamma_{s,t} = F(X_s)X_{s,t} + DF(X_s)\mathbb{X}_{s,t} \qquad (3.5)$$

in (3.1) together to an integral

$$\mathcal{J}\Gamma = \int_0^T F(X_s)\,dX_s$$

It turns out that this procedure can be also done for more general choices of Γ than in (3.5) as functions in $C_2^{\alpha,\beta}([0,T],W)$ defined as the space of functions Γ on $0 \le s \le t \le T$ with values in W such that

$$\Gamma_{t,t} = 0 \text{ for all } t, \qquad (3.6)$$
$$\|\Gamma\|_{\alpha,\beta} \overset{def.}{=} \|\Gamma\|_\alpha + \|\delta\Gamma\|_\beta < \infty$$

Where $\|\Gamma\|_\alpha = \sup_{s<t} \frac{|\Gamma_{s,t}|}{|t-s|^\alpha}$ and where

$\delta\Gamma_{s,u,t} \overset{def.}{=} \Gamma_{s,t} - \Gamma_{s,u} - \Gamma_{u,t}$, $\|\delta\Gamma\|_\beta \overset{def.}{=} \sup_{s<u<t} \frac{|\delta\Gamma_{s,u,t}|}{|t-s|^\beta}$

→ Lemma 3.3 (Sewing Lemma)

Let $0 < \alpha \le 1 < \beta$. Then there exists a unique bounded linear map $\mathcal{J}: C_2^{\alpha,\beta}([0,T],W) \longrightarrow C^\alpha([0,T],W)$ such that $(\mathcal{J}\Gamma)_0 = 0$ and

$$|(\mathcal{J}\Gamma)_{s,t} - \Gamma_{s,t}| \le C|t-s|^\beta, \qquad (3.7)$$

where $C = C(\beta, \|\delta\Gamma\|_\beta)$ is a constant.

Proof:

1. Uniqueness of $\mathcal{J}$: Suppose $F := \mathcal{J}_1\Gamma \neq \bar{F} := \mathcal{J}_2\Gamma$ with $\mathcal{J}_1, \mathcal{J}_2$ satisfying the above cond.

$\Rightarrow$ $F_0 - \bar{F}_0 = 0$ and $|F_{s,t} - \bar{F}_{s,t}| = |F_{s,t} - \Gamma_{s,t} + \Gamma_{s,t} - \bar{F}_{s,t}|$

$\le |F_{s,t} - \Gamma_{s,t}| + |\Gamma_{s,t} - \bar{F}_{s,t}| \overset{(3.7)}{\le} \tilde{C}|t-s|^\beta$

$\mathcal{P}$ partition of $[0,u]$

$$\Rightarrow |F_t - \bar{F}_t| = \Big|\sum_{[a,b]\in\mathcal{P}} (F_{a,b} - \bar{F}_{a,b})\Big| \le \sum_{[a,b]\in\mathcal{P}} |F_{a,b} - \bar{F}_{a,b}|$$

$$\overset{\beta>1}{\le} \tilde{C} \sum_{[a,b]\in\mathcal{P}} |b-a|^\beta \le \tilde{C}|\mathcal{P}|^{\beta-1} \underset{u=}{} \boxed{\sum_{[a,b]\in\mathcal{P}} |b-a|} \xrightarrow[|\mathcal{P}|\to 0]{} 0$$

$\Rightarrow$ $F_t = \bar{F}_t$ for all t $\Longrightarrow$ ↯ $\Rightarrow$ uniqueness

<u>2. Existence of $\int$</u>: Let $\mathcal{P}$ be a partition of $[s,t]$.

Define

$$\int_{\mathcal{P}} \Gamma = \sum_{[u,v]\in\mathcal{P}} \Gamma_{u,v} \qquad (3.8)$$

We want to show that $\int_{\mathcal{P}}\Gamma$ converges for $|\mathcal{P}| \to 0$

Let r be the number of intervals in $\mathcal{P}$

<u>Claim</u>: If $r \geq 2$, then there ex. a $u \in [s,t]$ with $[u_-,u], [u,u_+] \in \mathcal{P}$ s.t.

$$|u_+ - u_-| \leq \frac{2}{r-1}|t-s| \qquad (*)$$

→ otherwise: there ex. $u_1,\dots,u_{r-1}$ such points s.t.

$$|(u_i)_+ - (u_i)_-| > \frac{2}{r-1}|t-s| \text{ for all } i=1,\dots,r-1$$

$$\Rightarrow \underbrace{\sum_{i=1}^{r-1} \underbrace{|(u_i)_+ - (u_i)_-|}_{\leq |(u_i)_+ - u_i| + |u_i - (u_i)_-|}}_{\leq 2|t-s|} > 2|t-s| \quad \Rightarrow ↯$$

$\Rightarrow$ claim

Choose a u_1 as in $(*)$. Define

$\mathcal{P}_r = \mathcal{P}$ and

i.e. replace $[(u_1)_-,u_1], [u_1,(u_1)_+]$ by $[(u_1)_-,(u_1)_+]$ in $\mathcal{P}$

$$\mathcal{P}_{r-1} = (\mathcal{P} \setminus \{[(u_1)_-,u_1],[u_1,(u_1)_+]\}) \cup \{[(u_1)_-,(u_1)_+]\}$$

$\Rightarrow \mathcal{P}_{r-1}$ with $r-1$ intervals

Choose u_2 as in $(*)$ w.r.t. $\mathcal{P}_{r-1}$

$$\longrightarrow \mathcal{P}_{r-2} := (\mathcal{P}_{r-1} \setminus \{[(u_2)_-,u_2],[u_2,(u_2)_+]\}) \cup \{[(u_2)_-,(u_2)_+]\}$$

$\vdots$

$$\mathcal{P}_1 = \{[s,t]\}$$

$$\Rightarrow \left|\Gamma_{s,t} - \int_{\mathcal{P}}\Gamma\right| = \left|\int_{\mathcal{P}_1}\Gamma - \int_{\mathcal{P}_r}\Gamma\right| \overset{\substack{\text{telescope}\\ \text{sum}}}{=} \left|\sum_{j=0}^{r-2}\left(\int_{\mathcal{P}_{r-j-1}}\Gamma - \int_{\mathcal{P}_{r-j}}\Gamma\right)\right|$$

$$\leq \sum_{j=0}^{r-2} \Bigg|\underbrace{\int_{\mathcal{P}_{r-j-1}}\Gamma - \int_{\mathcal{P}_{r-j}}\Gamma}_{\overset{(3.8)}{=} \Gamma_{(u_{j+1})_-,(u_{j+1})_+} - \Gamma_{(u_{j+1})_-,u_{j+1}} - \Gamma_{u_{j+1},(u_{j+1})_+} \overset{\text{def.}}{=} \delta\Gamma_{(u_{j+1})_-,u_{j+1},(u_{j+1})_+}}\Bigg| = \sum_{j=0}^{r-2} \overbrace{|\delta\Gamma_{(u_{j+1})_-,u_{j+1},(u_{j+1})_+}|}^{\overset{(3.6)}{\leq} \|\delta\Gamma\|_\beta \cdot |(u_{j+1})_+ - (u_{j+1})_-|^\beta} \qquad (**)$$

$$|(u_{j+1})_+ - (u_{j+1})_-|^\beta \overset{(*)}{\leq} 2^\beta \frac{1}{(r-j-1)^\beta} |t-s|^\beta$$

$$\xRightarrow{(**)} \quad |\Gamma_{s,t} - \int_{\mathcal{P}} \Gamma| \leq 2^\beta \|\delta\Gamma\|_\beta |t-s|^\beta \underbrace{\sum_{j=0}^{r-2} \frac{1}{(r-j-1)^\beta}}_{\leq \sum_{j\geq 1} \frac{1}{j^\beta} = \zeta(\beta)} \qquad (***)$$

zeta function

Remains to show that

$$\sup_{|\mathcal{P}| \vee |\mathcal{P}'| < \varepsilon} \left| \int_{\mathcal{P}} \Gamma - \int_{\mathcal{P}'} \Gamma \right| \xrightarrow[\varepsilon \downarrow 0]{} 0 \qquad (+) \qquad (a \vee b = \max(a,b))$$

Since

$$\left| \int_{\mathcal{P}} \Gamma - \int_{\mathcal{P}'} \Gamma \right| \leq \left| \int_{\mathcal{P}} \Gamma - \int_{\mathcal{P} \cup \mathcal{P}'} \Gamma \right| + \left| \int_{\mathcal{P} \cup \mathcal{P}'} \Gamma - \int_{\mathcal{P}'} \Gamma \right|$$

we can assume that $\mathcal{P}'$ is a refinement of $\mathcal{P}$ (i.e. $\mathcal{P} \leq \mathcal{P}'$)

$$\Rightarrow \quad \int_{\mathcal{P}} \Gamma - \int_{\mathcal{P}'} \Gamma = \sum_{[u,v] \in \mathcal{P}} \Big(\Gamma_{u,v} - \sum_{\substack{[u',v'] \in \mathcal{P}' \text{ with} \\ [u',v'] \subseteq [u,v]}} \Gamma_{u',v'} \Big)$$

provides a partition of $[u,v]$

$$\xRightarrow{(***)} \quad \left| \int_{\mathcal{P}} \Gamma - \int_{\mathcal{P}'} \Gamma \right|$$

$$\leq 2^\beta \zeta(\beta) \|\delta\Gamma\|_\beta \underbrace{\sum_{[u,v] \in \mathcal{P}} |v-u|^\beta}_{\leq C \cdot |\mathcal{P}|^{\beta-1} \xrightarrow[|\mathcal{P}| \to 0]{} 0} \qquad \Rightarrow (+)$$

$$\xrightarrow{(+)} \quad (\mathcal{I}\Gamma)_{s,t} \overset{\text{def.}}{=} \lim_{|\mathcal{P}| \to 0} \int_{\mathcal{P}} \Gamma$$

$$\xRightarrow[|\mathcal{P}| \to 0]{(***),} \quad |\Gamma_{s,t} - (\mathcal{I}\Gamma)_{s,t}| \leq \zeta(\beta) 2^\beta \|\delta\Gamma\|_\beta |t-s|^\beta$$

$|a| - |b| \leq |a-b|$

$$\Rightarrow \quad \underbrace{\sup_{s \neq t} \frac{|(\mathcal{I}\Gamma)_{s,t}|}{|t-s|^\alpha}}_{= \|\mathcal{I}\Gamma\|_\alpha} \leq \underbrace{\sup_{s \neq t} \frac{|\Gamma_{s,t}|}{|s-t|^\alpha}}_{= \|\Gamma\|_\alpha} + \zeta(\beta) 2^\beta \|\delta\Gamma\|_\beta \, T^{\beta - \alpha}$$

$$\Rightarrow \quad \|\mathcal{I}\Gamma\|_\alpha \leq C \left(\|\Gamma\|_\alpha + \|\delta\Gamma\|_\beta \right) = C \|\Gamma\|_{\alpha,\beta}$$

$\Rightarrow$ $\mathcal{I}$ bounded linear map $\Rightarrow$ proof

Sewing Lemma $\longrightarrow$ Th. 3.4 (T. Lyons) Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^{\alpha}([0,T], V)$ for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ and let $F : V \to L(V, W)$ be in C^2_b.
Then the rough path integral of $Y_s = F(X_s)$ w.r.t. $\mathbf{X}$ can be defined as the existing limit

$$\int_0^T F(X_s)\,dX_s = \lim_{|\mathcal{P}| \to 0} \sum_{[s,t] \in \mathcal{P}} F(X_s) X_{s,t} + DF(X_s)\mathbb{X}_{s,t} \qquad (3.9)$$

Further, we have that

$$\left| \int_s^t F(X_r)\,dX_r - F(X_s)X_{s,t} - DF(X_s)\mathbb{X}_{s,t} \right|$$

$$\leq C_\alpha \|F\|_{C^2_b} \left(\|X\|^3_\alpha + \|X\|_\alpha \cdot \|\mathbb{X}\|_{2\alpha} \right) |t-s|^{3\alpha}, \qquad (3.10)$$

where $\|F\|_{C^2_b} := \|F\|_\infty + \|DF\|_\infty + \|D^2F\|_\infty$

Moreover

$$\left\| \int_0^{\cdot} F(X_s)\,dX_s \right\|_\alpha \leq C_{\alpha,T} \|F\|_{C^2_b} \left(|||\mathbf{X}|||_\alpha \vee |||\mathbf{X}|||^3_\alpha \right), \quad (3.11)$$

where

$$|||\mathbf{X}|||_\alpha = \|X\|_\alpha + \left(\|\mathbb{X}\|_{2\alpha}\right)^{1/2}.$$

Proof: idea: Sewing Lemma (L 3.3) applied to

$$\Gamma_{s,t} := Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}$$

with $Y_s := F(X_s)$, $Y'_s := DF(X_s)$

We have to check in (3.6) that

$\|\Gamma\|_{\alpha,\beta} \overset{\text{def}}{=} \|\Gamma\|_\alpha + \|\delta\Gamma\|_\beta < \infty$ for a $\beta > 1$,

where $\delta\Gamma_{s,u,t} = \Gamma_{s,t} - \Gamma_{s,u} - \Gamma_{u,t}$

We see that

$$|Y_s X_{s,t}| \leq \|F\|_\infty \overset{\lesssim |t-s|^\alpha}{|X_{s,t}|} \text{ and } |Y'_s \mathbb{X}_{s,t}| \leq \|DF\|_\infty \overset{\lesssim |t-s|^{2\alpha}}{|\mathbb{X}_{s,t}|}$$

$\Longrightarrow \|\Gamma\|_\alpha < \infty$

$$\left(\sum_{\ell=1}^{d} \frac{\partial}{\delta x_\ell} F_{ij}(X_s) X^\ell_{s,u} \right)_{\substack{1 \leq i \leq m,\\ 1 \leq j \leq d}} \in \mathbb{R}^{m \times d} \text{ if } V = \mathbb{R}^d, W = \mathbb{R}^m$$

Set $R^Y_{s,t} = Y_{s,t} - \boxed{Y'_s X_{s,t}}$

$\left(\sum_{\ell=1}^{d}\frac{\partial}{\partial x_\ell}F_{ij}(X_s)\,\mathbb{X}^{\ell j}_{u,t}\right)^T_{1\le i\le m}$ if $V=\mathbb{R}^d, W=\mathbb{R}^m$

$\mathbb{X}_{u,t}=\left(\mathbb{X}^{ij}_{u,t}\right)_{1\le i,j\le d}$

$$\Longrightarrow -R^Y_{s,u}X_{u,t}-\boxed{Y'_{s,u}\mathbb{X}_{u,t}}$$

$$=-Y_{s,u}X_{u,t}+Y'_s[X_{s,u},X_{u,t}]-Y'_{s,u}\mathbb{X}_{u,t}$$

$\left(X^i_{s,u}\cdot X^j_{u,t}\right)_{1\le i,j\le d}$ if $V=\mathbb{R}^d$

$$\overset{\text{Lifting theorem}}{\underset{\text{Th. 2.2}}{=}} -Y_{s,u}X_{u,t}+Y'_s\big(\underbrace{X_{s,u}\otimes X_{u,t}}_{\overset{\text{Chen's}}{\underset{\text{relation}}{=}}\ \mathbb{X}_{s,t}-\mathbb{X}_{s,u}-\mathbb{X}_{u,t}}\big)-Y'_{s,u}\mathbb{X}_{u,t}$$

$$\overset{\text{linearity}}{\underset{\text{of } Y, Y'}{=}} -Y_{s,u}X_{u,t}+Y'_s\mathbb{X}_{s,t}-Y'_s\mathbb{X}_{s,u}-Y'_s\mathbb{X}_{u,t}-(Y'_u-Y'_s)\mathbb{X}_{u,t}$$

$$=\boxed{-Y_{s,u}X_{u,t}}+Y'_s\mathbb{X}_{s,t}-Y'_s\mathbb{X}_{s,u}-Y'_u\mathbb{X}_{u,t}$$

$$\parallel\ Y_sX_{s,t}-Y_sX_{s,u}-Y_uX_{u,t}$$

$$=\Gamma_{s,t}-\Gamma_{s,u}-\Gamma_{u,t}\overset{(3.6)}{=}\delta\Gamma_{s,u,t}\qquad(*)$$

$$|R^Y_{s,u}X_{u,t}|\le\underbrace{|R^Y_{s,u}|}_{\overset{L.3.1}{\lesssim}|s-u|^{2\alpha}}\overbrace{|X_{u,t}|}^{\lesssim|u-t|^{\alpha}}\ \text{and}\ |Y'_{s,u}\mathbb{X}_{u,t}|\le\underbrace{|Y'_{s,u}|}_{\overset{L.3.1}{\lesssim}|s-u|^{\alpha}}\overbrace{|\mathbb{X}_{u,t}|}^{\lesssim|u-t|^{2\alpha}}$$

$$\overset{\beta:=3\alpha>1}{\underset{(*)}{\Longrightarrow}}\quad\|\delta\Gamma\|_\beta=\sup_{s<u<t}\frac{|\delta\Gamma_{s,u,t}|}{|s-t|^{3\alpha}}$$

$$\le\|R^Y\|_{2\alpha}\|X\|_\alpha+\|Y'\|_\alpha\|\mathbb{X}\|_{2\alpha}$$

$$\overset{L.3.1}{\le}\frac{1}{2}\|D^2F\|_\infty\|X\|^3_\alpha+\|D^2F\|_\infty\|X\|_\alpha\|\mathbb{X}\|_{2\alpha}<\infty$$

$$\overset{(***)\text{ in}}{\underset{\text{the proof of } L.3.3,\ \beta=3\alpha}{\Longrightarrow}}\quad\left|\int_s^t Y_r\,dX_r-Y_sX_{s,t}-Y'_s\mathbb{X}_{s,t}\right|\qquad(**)$$

$$\le\zeta(3\alpha)2^{3\alpha}\|F\|_{C^2_b}\left(\|X\|^3_\alpha+\|X\|_\alpha\|\mathbb{X}\|_{2\alpha}\right)|t-s|^{3\alpha}\quad\Longrightarrow(3.10)$$

$$\overset{|a|-|b|\le|a-b|}{\Longrightarrow}\quad\left|\int_s^t Y_r\,dX_r\right|\le\|F\|_\infty\overbrace{|X_{s,t}|}^{\lesssim|||\mathbb{X}|||_\alpha\cdot|s-t|^\alpha}+\|DF\|_\infty\overbrace{|\mathbb{X}_{s,t}|}^{\lesssim|||\mathbb{X}|||^2_\alpha\cdot|s-t|^{2\alpha}}$$

$$+\zeta(3\alpha)2^{3\alpha}\|F\|_{C^2_b}\Big(\underbrace{\|X\|^3_\alpha}_{\le|||\mathbb{X}|||^3_\alpha}+\underbrace{\|X\|_\alpha\|\mathbb{X}\|_{2\alpha}}_{\le|||\mathbb{X}|||^3_\alpha}\Big)|t-s|^{3\alpha}$$

$$|||\mathbb{X}|||^2_\alpha\le|||\mathbb{X}|||_\alpha\vee|||\mathbb{X}|||^3_\alpha$$

$\Longrightarrow(3.11)\ \Longrightarrow$ proof

3.2 Integration of controlled rough paths

Recall that

$$\int_0^T Z_s\, dX_s = \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Z_s X_{s,t} + Z'_s \mathbb{X}_{s,t}$$

for $Z_s = F(X_s)$, $Z'_s = DF(X_s)$

$$\longrightarrow \qquad Z_{s,t} = Z'_s X_{s,t} + R^Z_{s,t}, \qquad (3.12)$$

where $R^Z_{s,t} := Z_{s,t} - Z'_s X_{s,t}$

with $\|Z\|_\alpha, \|Z'\|_\alpha, \|R^Z_{s,t}\|_{2\alpha} < \infty$.

In order to generalize the class of such integrands Z, one could e.g. consider integrands Y which are obtained as a (formal) limit of $Z^n = F_n(X_\cdot)$, that is

$$Z^n \xrightarrow[n\to\infty]{} Y$$

with

$$Z^{\prime,n} = DF_n(X_\cdot) \xrightarrow[n\to\infty]{} Y'$$

$$R^{Z_n} \xrightarrow[n\to\infty]{} R^Y$$

$$\xrightarrow{(3.12)} \qquad Y_{s,t} = Y'_s X_{s,t} + R^Y_{s,t} \qquad (3.13)$$

If we additionally require that

$$\|Y\|_\alpha, \|Y'\|_\alpha, \|R^Y\|_{2\alpha} < \infty$$

then we could choose $\Gamma_{s,t} = Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}$ in the Sewing Lemma (L.3.3) with

$$\|\delta\Gamma\|_\beta \le \|R^Y\|_{2\alpha}\|X\|_\alpha + \|Y'\|_\alpha \|\mathbb{X}\|_{2\alpha} < \infty$$

(see the proof of Th. 3.4)

and define

$$\int_0^T Y_s\, dX_s = \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t} \qquad (3.14)$$

$\longrightarrow$ <u>Y' Gubinelli derivative of Y</u> for the special case $\overline{W} = L(V,W)$ in the sense of

<u>Def. 3.5</u> (Space of controlled rough paths)
Let $V, \overline{W}$ be Banach spaces and let $X \in C^\alpha([0,T],V)$, $Y \in C^\alpha([0,T],\overline{W})$. Then Y is a <u>controlled rough path of X</u> iff there ex. $Y' \in C^\alpha([0,T],\overline{V})$ with $\overline{V} = L(V,\overline{W})$ such that

$$Y_{s,t} = Y'_s X_{s,t} + R^Y_{s,t} \tag{3.15}$$

for a remainder term R^Y with $\|R^Y\|_{2\alpha} < \infty$.
We denote by

$$\mathcal{D}^{2\alpha}_X([0,T],\overline{W})$$

the vector space of all controlled rough paths (Y,Y').
Further, we define the space

$$\mathcal{C}^\alpha \ltimes \mathcal{D}^{2\alpha} = \overset{\text{disjoint union}}{\bigsqcup_{X \in \mathcal{C}^\alpha}} \{X\} \times \mathcal{D}^{2\alpha}_X$$

with base space $\mathcal{C}^\alpha$ and "fibres" $\mathcal{D}^{2\alpha}_X$.

<u>Ex. 3.6</u>: (i) $\overline{W} = L(V,W)$, $Y_s = F(X_s)$, $Y'_s = DF(X_s)$ as in Th. 3.4.

(ii) $\overline{W} = W$, $Y_s = \int_0^s F(X_r)\,dX_r$, $Y'_s = F(X_s)$,

$$R^Y_{s,t} = Y_{s,t} - Y'_s X_{s,t} = \int_s^t F(X_r)\,dX_r - F(X_s) X_{s,t}$$

$$\underset{|a|-|b| \le |a-b|}{\overset{(3.10)}{\Longrightarrow}} \|R^Y\|_{2\alpha} < \infty$$

(iii) $Y = F(X_\cdot)$ controlled by X with Gubinelli derivative $Y' = DF(X_\cdot)$.
Let $(\overline{X}, \overline{\mathbb{X}}) \in \mathcal{C}^\alpha$ with

$$\overline{X}_t = X_t \quad \text{and} \quad \overline{\mathbb{X}}_{s,t} = \overset{\text{satisfies Chen's relation}}{\mathbb{X}_{s,t} + f(t) - f(s)}$$

for some $f \in C^{2\alpha}$

$\Longrightarrow$ $(Y,Y') \in \mathcal{D}^{2\alpha}_{\overline{X}}$, <u>but</u>