$$\int_0^T Y_r \, d\bar{X}_r = \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y_s \bar{X}_{s,t} + Y'_s \bar{\mathbb{X}}_{s,t}$$

$$= \underbrace{\lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}}_{\overset{\text{Th. 3.4}}{=} \int_0^T Y_r \, dX_r} + \underbrace{\lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y'_s (f(t)-f(s))}_{\overset{\text{Young integral, (4) in Sect. 1}}{=} \int_0^T \underset{C^\alpha}{\underset{\cap}{Y'_r}} \, d\underset{C^{2\alpha}}{\underset{\cap}{f(r)}}}$$

$$\Rightarrow \alpha + 2\alpha > 1$$

<u>Rem. 3.7</u>

(i) $\mathcal{D}^{2\alpha}_X$ becomes a Banach space with norm $\|\cdot\|_{X,2\alpha}$ given by

$$\|(Y,Y')\|_{X,2\alpha} := |Y_0| + |Y'_0| + \|Y'\|_\alpha + \|R^Y\|_{2\alpha} \qquad (3.16)$$

(ii) The <u>Gubinelli derivative Y'</u> is <u>in general not unique</u>!
If e.g. $Y, X \in C^{2\alpha}$ then one can choose any $Y' \in C$ to obtain (3.15).
<u>However</u>, if e.g. $V = \mathbb{R}$ and X is <u>rough at $s \in [0,T)$</u>, i.e.

$$\frac{|X_{s,t_n}|}{|t_n - s|^{2\alpha}} \xrightarrow[n\to\infty]{} \infty \qquad (3.17)$$

for some sequence $t_n \downarrow s$, then Y'_s is uniquely determined and one has $\|R^Y\|_{2\alpha} < \infty$, <u>since</u>:

$$Y_{s,t_n} = Y'_s X_{s,t_n} + R^Y_{s,t_n}, \quad n \geq 1$$

$$\Rightarrow \quad Y'_s = \frac{Y_{s,t_n}}{\underset{\text{for } n \geq n_0}{X_{s,t_n} \neq 0}} - \underbrace{\frac{R^Y_{s,t_n}}{|t_n - s|^{2\alpha}}}_{\text{bounded}} \frac{|t_n - s|^{2\alpha}}{X_{s,t_n}} \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow \quad Y'_s = \lim_{n\to\infty} \frac{Y_{s,t_n}}{X_{s,t_n}}$$

<u>Th. 3.8</u> (Gubinelli)

Let $\bar{W} = L(V,W)$ and $\mathbb{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha([0,T], V)$, $\alpha \in (\frac{1}{3}, \frac{1}{2}]$.
Further, let $(Y, Y') \in \mathcal{D}^{2\alpha}_X([0,T], \bar{W})$. Then

(i)
$$\int_0^T Y_r \, dX_r := \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y X_{s,t} + Y'_s \mathbb{X}_{s,t} \qquad (3.18)$$

exists

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$$\left|\int_s^t Y_r\,dX_r - Y_s X_{s,t} - Y'_s \mathbb{X}_{s,t}\right| \tag{3.19}$$

$$\le C_{T,\alpha}\big(\underbrace{\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha}\|Y'\|_\alpha}_{=:L_\alpha}\big)|t-s|^{3\alpha}$$

(ii) The map $\mathcal{J}: \mathcal{D}^{2\alpha}_{X}([0,T],\overline{W}) \to \mathcal{D}^{2\alpha}_{X}([0,T],W)$ given by

$$\mathcal{J}((Y,Y')) := \Big(\int_0^\cdot Y_r\,dX_r,\ Y\Big) \tag{3.20}$$

Gubinelli derivative of $\int_0^\cdot Y_r\,dX_r$

is a bounded linear map with bound

$$\|\mathcal{J}((Y,Y'))\|_{X,2\alpha} \le \|Y\|_\alpha + \|Y'\|_\infty \|\mathbb{X}\|_{2\alpha} + C_{T,\alpha}\big(\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha}\|Y'\|_\alpha\big) \tag{3.21}$$

<u>Proof</u>: (i) We can proceed exactly as in the proof of Th. 3.4 with $\Xi_{s,t} := Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}$ and obtain (3.18) and (3.19).

(ii) $\|\mathcal{J}((Y,Y')\|_{X,2\alpha} = \|(\underbrace{\int Y_r\,dX_r}_{=:Z}, \underbrace{Y}_{=:Z'})\|_{X,2\alpha}$

w.l.o.g. $Y_0 = 0$

$= \underbrace{\|Z'\|_\alpha}_{=\|Y\|_\alpha} + \|R^Z\|_{2\alpha}$ where $R^Z_{s,t} = Z_{s,t} - Y X_{s,t}$ (Ex 3.6(ii))

$\overset{(3.19),}{\underset{|a|-|b|\le|a-b|}{\Longrightarrow}}$ $\|\mathcal{J}((Y,Y'))\|_{X,2\alpha} = \|Y\|_\alpha + \sup_{s\ne t}\dfrac{|Z_{s,t} - Y X_{s,t}|}{|s-t|^{2\alpha}}$

$$\le \|Y\|_\alpha + \sup_{s\ne t}\frac{|Y'_s|\,|\mathbb{X}_{s,t}|}{|s-t|^{2\alpha}} + C_{T,\alpha} L_\alpha \sup_{s\ne t}\frac{|t-s|^{3\alpha}}{|t-s|^{2\alpha}}$$

$$\le \|Y\|_\alpha + \sup_s |Y'_s| \cdot \|\mathbb{X}\|_{2\alpha} + T^\alpha C_{T,\alpha} L_\alpha$$

$\Rightarrow$ (3.21)

The boundedness of $\mathcal{J}$ will be discussed in connection with the stability of rough path integration later on.

$\Rightarrow$ proof.

Rem. 3.9 (Integration against controlled paths Z)

Let $(X,\mathbb{X}) \in \mathscr{C}^\alpha([0,T],V)$ and let $(Y,Y') \in \mathscr{D}^{2\alpha}_X([0,T],L(\bar V,W))$, $(Z,Z') \in \mathscr{D}^{2\alpha}_X([0,T],\bar V)$.

Then using the Sewing Lemma (L.3.3) we may define

$$\int_0^T Y_u\,dZ_u = \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} \Gamma_{s,t}, \qquad (3.22)$$

where
$$\Gamma_{s,t} := Y_u Z_{u,v} + \tilde Y'_u[\tilde Z'_u[\mathbb{X}_{u,v}]].$$

Here by the lifting theorem (Th. 2.2) $\tilde Z'_u, \tilde Y'_u$ are the unique maps s.t.

$$\tilde Z'_u(w\otimes v) = w \otimes \overbrace{Z'_u}^{L(V,\bar V)}(v),$$
$$\tilde Y'_u(w\otimes v) = Y'_u[w,v] \qquad Y'_u \in L(V,L(\bar V,W))$$

for all $w,v\in V$

$\mathbb{X}_{u,v} = (\mathbb{X}^{ij})_{1\le i,j\le d}$, $\sum_{i,j=1}^d \mathbb{X}^{ij}_{u,v}\, e_i\otimes e_j$

$\xrightarrow[W=\mathbb{R}^m]{V=\mathbb{R}^d,\ \bar V=\mathbb{R}^d}$
$$\tilde Y'_u[\tilde Z'_u[\mathbb{X}_{u,v}]] = \sum_{i,j=1}^d \tilde Y'_u[\underbrace{\tilde Z'_u[e_i\otimes e_j]}_{=e_i\otimes Z'_u(e_j)}]\,\mathbb{X}^{ij}_{u,v}$$

$$Z'_u(e_j) = \sum_{\ell=1}^d (Z'_u)^{\ell j} e_\ell$$

$$= \sum_{i,j,\ell=1}^d Y'_u[e_i,e_\ell](Z'_u)^{\ell j}\,\mathbb{X}^{ij}_{u,v}$$

$$Z'_u = \left((Z'_u)^{\ell j}\right)_{1\le \ell,j\le d}$$

Using Chen's relation just as in the proof of Th. 3.4 we see that

$$\delta\Gamma_{s,u,t} = -R^Z_{s,u} X_{u,t} - (\tilde Y'\tilde Z')_{s,u}\mathbb{X}_{u,t}$$
$$= (\tilde Y'_u \tilde Z'_u - \tilde Y'_s \tilde Z'_s)\,\mathbb{X}_{u,t}$$

$\xrightarrow{\beta=3\alpha}$ $\|\delta\Gamma\|_\beta < \infty$

$\xRightarrow[\text{Lemma}]{\text{Sewing}}$
$$\left|\int_s^t Y_r\,dZ_r - Y_s Z_{s,t} - \tilde Y'_s \tilde Z'_s \mathbb{X}_{s,t}\right|$$
$$\le C\left(\|X\|_\alpha \|R^Z\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha}\|\tilde Y'\tilde Z'\|_\alpha\right)|t-s|^{3\alpha} \qquad (3.23)$$

$\xrightarrow[\text{for all } v,\ \bar V = V]{Z=X,\ Z'(v)=v}$ original def. (3.19).

3.3. Stability of rough path integration

→ rough path integration is stable in the sense that

$$\int_0^t Y_r \, d\mathbf{X}_r \quad \text{and} \quad \int_0^t \tilde{Y}_r \, d\tilde{\mathbf{X}}_r$$

are "close" if the input data

$\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$, $(Y, Y') \in \mathcal{D}_X^{2\alpha}$ and

$\tilde{\mathbf{X}} = (\tilde{X}, \tilde{\mathbb{X}}) \in \mathscr{C}^\alpha$, $(\tilde{Y}, \tilde{Y}') \in \mathcal{D}_{\tilde{X}}^{2\alpha}$ are "close".

Closeness between $(Y, Y') \in \mathcal{D}_X^{2\alpha} \neq \mathcal{D}_{\tilde{X}}^{2\alpha} \ni (\tilde{Y}, \tilde{Y}')$ can be measured by means of the distance between

$$(Y', R^Y) \text{ and } (\tilde{Y}', R^{\tilde{Y}}) \text{ in } C^\alpha \times C^{2\alpha}$$

given by $\|Y' - \tilde{Y}'\|_\alpha + \|R^Y - R^{\tilde{Y}}\|_{2\alpha}$

since we can identify each $(Y, Y') \in \mathcal{D}_X^{2\alpha}$ by the one-to-one map j def. as

$$j((Y, Y')) = (Y', R^Y) \in C^\alpha \times C^{2\alpha}$$

because: If $(Y', R^Y) = (\tilde{Y}', R^{\tilde{Y}})$ then

$$Y_{s,t} = Y_s' X_{s,t} + R^Y_{s,t} = \tilde{Y}_s' X_{s,t} + R^{\tilde{Y}}_{s,t} = \tilde{Y}_{s,t}$$

$\Rightarrow \quad Y_{s,t} = \tilde{Y}_{s,t}$ (and $Y = \tilde{Y}$ if $Y_0 = \tilde{Y}_0$)

→ "distance" between (Y, Y') and $(\tilde{Y}, \tilde{Y}')$ def. as

$$d_{X, \tilde{X}, 2\alpha}((Y, Y'), (\tilde{Y}, \tilde{Y}')) = \|Y' - \tilde{Y}'\|_\alpha + \|R^Y - R^{\tilde{Y}}\|_{2\alpha} \quad (3.24)$$

→ **Th. 3.10** (Stability of rough path integration)

Let $\mathbf{X} = (X, \mathbb{X})$, $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{\mathbb{X}}) \in \mathscr{C}^\alpha$, $(Y, Y') \in \mathcal{D}_X^{2\alpha}$, $(\tilde{Y}, \tilde{Y}') \in \mathcal{D}_{\tilde{X}}^{2\alpha}$ such that

$$|Y_0'| + \|(Y, Y')\|_{X, 2\alpha} \leq M, \quad \varrho_\alpha(0, \mathbf{X}) \overset{\text{def.}}{=} \|X\|_\alpha + \|\mathbb{X}\|_{2\alpha} \leq M \quad (+)$$

and for $\tilde{\mathbf{X}}$, $(\tilde{Y}, \tilde{Y}')$ in the same way for the constant M.

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Let $\qquad$ Gubinelli deriv. of $\int Y dX$

$$(Z,Z') := \Big(\int_0^\cdot Y\,d\mathbf{X},\, Y\Big) \in \mathcal{D}^{2\alpha}_{X}\,,\quad (\tilde Z,\tilde Z') := \Big(\int_0^\cdot \tilde Y\,d\tilde{\mathbf{X}},\, \tilde Y\Big) \in \mathcal{D}^{2\alpha}_{\tilde X}$$

Then

$$d_{X,\tilde X,2\alpha}((Z,Z'),(\tilde Z,\tilde Z')) \le C\big(\rho_\alpha(\mathbf{X},\tilde{\mathbf{X}}) + |Y_0'-\tilde Y_0'| + d_{X,\tilde X,2\alpha}((Y,Y'),(\tilde Y,\tilde Y'))\big) \tag{3.25}$$

and

$$\|Z-\tilde Z\|_\alpha \le C\big(\rho_\alpha(\mathbf{X},\tilde{\mathbf{X}}) + |Y_0-\tilde Y_0| + |Y_0'-\tilde Y_0'| + d_{X,\tilde X,2\alpha}((Y,Y'),(\tilde Y,\tilde Y'))\big) \tag{3.26}$$

for $C = C(M,T,\alpha)$

Proof: We have that

$$d_{X,\tilde X,2\alpha}((Z,Z'),(\tilde Z,\tilde Z')) \overset{def.}{=} \|Z'-\tilde Z'\|_\alpha + \|R^Z - R^{\tilde Z}\|_{2\alpha}$$
$$\overset{def.}{=} \|Y-\tilde Y\|_\alpha + \|R^Z - R^{\tilde Z}\|_{2\alpha}$$

1. $\|Y-\tilde Y\|_\alpha$:

$$|Y_{s,t}-\tilde Y_{s,t}| = \Big|\underbrace{(\overbrace{Y'_{0,s}+Y'_0}^{=Y'_s})X_{s,t} + R^Y_{s,t}}_{=Y_{s,t}} - \underbrace{(\overbrace{\tilde Y'_{0,s}+\tilde Y'_0}^{=\tilde Y'_s})\tilde X_{s,t} - R^{\tilde Y}_{s,t}}_{=\tilde Y_{s,t}}\Big|$$

$$= \Big|\underbrace{(Y'_{0,s}+Y'_0)X_{s,t} - (\tilde Y'_{0,s}+\tilde Y'_0)X_{s,t}}_{=((Y'_{0,s}-\tilde Y'_{0,s})+(Y'_0-\tilde Y'_0))X_{s,t}} + \underbrace{(\tilde Y'_{0,s}+\tilde Y'_0)X_{s,t} - (\tilde Y'_{0,s}+\tilde Y'_0)\tilde X_{s,t}}_{=(\tilde Y'_{0,s}+\tilde Y'_0)(X_{s,t}-\tilde X_{s,t})} + R^Y_{s,t} - R^{\tilde Y}_{s,t}\Big|$$

$$\overset{\Delta\text{-ineq.}}{\le} \Big(\underbrace{|Y'_{0,s}-\tilde Y'_{0,s}|}_{\le \underbrace{\|Y'-\tilde Y'\|_\alpha}_{\le d_{X,\tilde X,2\alpha}((Z,\tilde Z),(Z',\tilde Z'))}\underbrace{|s|^\alpha}_{\le T^\alpha}} + |Y_0'-\tilde Y_0'|\Big)\underbrace{|X_{s,t}|}_{\le \underbrace{\|X\|_\alpha}_{\le M}|s-t|^\alpha} + \Big(\underbrace{|Y'_{0,s}|}_{\le\underbrace{\|Y'\|_\alpha}_{\le M}|s|^\alpha} + \underbrace{|\tilde Y'_0|}_{\le M}\Big)\underbrace{|X_{s,t}-\tilde X_{s,t}|}_{\le \underbrace{\|X-\tilde X\|_\alpha}_{\le \rho_\alpha(\mathbf{X},\tilde{\mathbf{X}})}|s-t|^\alpha} + \underbrace{|R^Y_{s,t}-R^{\tilde Y}_{s,t}|}_{\le \|R^Y-R^{\tilde Y}\|_{2\alpha}\cdot|s-t|^{2\alpha}}$$

$$\le C(M,T,\alpha)\big(\rho_\alpha(\mathbf{X},\tilde{\mathbf{X}}) + |Y_0'-\tilde Y_0'| + d_{X,\tilde X,2\alpha}((Z,\tilde Z),(Z',\tilde Z'))\big)$$

2. $\|R^Z - R^{\tilde Z}\|_{2\alpha}$:

Recall from the proof of Th. 3.4 (Gubinelli) in connection with the Sewing Lemma that

$$\delta\Gamma_{s,t} = -R^Y_{s,u}X_{u,t} - Y'_{s,u}\mathbb{X}_{u,t}\,,\quad \delta\tilde\Gamma_{s,t} = -R^{\tilde Y}_{s,u}\tilde X_{u,t} - \tilde Y'_{s,u}\tilde{\mathbb{X}}_{u,t} \tag{++}$$

for $\Gamma_{s,t} := Y_s X_{s,t} + Y'_s\mathbb{X}_{s,t}$ and $\tilde\Gamma_{s,t} := \tilde Y_s\tilde X_{s,t} + \tilde Y'_s\tilde{\mathbb{X}}_{s,t}$

$$\longrightarrow\quad R^Z_{s,t} \overset{def.}{=} \int_s^t Y_r\,d\mathbf{X}_r - Y_s X_{s,t} = \underbrace{\int_s^t Y_r\,d\mathbf{X}_r}_{=(\mathcal{I}\Gamma)_{s,t}} - \Gamma_{s,t} + Y'_s\mathbb{X}_{s,t}$$

Similarly,

$$R^{\tilde{Y}}_{s,t} = (\mathbb{J}\tilde{\Gamma})_{s,t} - \tilde{\Gamma}_{s,t} + \tilde{Y}'_s \tilde{\mathbb{X}}_{s,t} \in \mathcal{C}_2^{\alpha,\beta} \leftarrow \beta = 3\alpha$$

$$\Rightarrow |R^Z_{s,t} - R^{\tilde{Z}}_{s,t}| \leq |\mathbb{J}(\Gamma-\tilde{\Gamma}))_{s,t} - (\Gamma-\tilde{\Gamma})_{s,t}| + |\underbrace{Y'_s\mathbb{X}_{s,t} - \tilde{Y}'_s\tilde{\mathbb{X}}_{s,t}}_{(\underbrace{Y'_s-\tilde{Y}'_s}_{=Y'_{0,s}-\tilde{Y}'_{0,s}+Y'_0-\tilde{Y}'_0})\mathbb{X}_{s,t} + \underbrace{\tilde{Y}'_s}_{=\tilde{Y}'_{0,s}+\tilde{Y}'_0}(\mathbb{X}_{s,t}-\tilde{\mathbb{X}}_{s,t})}|$$

$$\overset{(\ast\ast\ast)\text{ in the proof of L. 3.3}}{\leq} 2^{3\alpha}\zeta(3\alpha)\|\underbrace{\delta(\Gamma-\tilde{\Gamma})}_{=\delta\Gamma-\delta\tilde{\Gamma}}\|_{3\alpha}|t-s|^{3\alpha} + (\|Y'-\tilde{Y}'\|_\alpha|s|^\alpha + |Y'_0-\tilde{Y}'_0|)\cdot$$

$$\cdot \underbrace{\|\mathbb{X}\|_{2\alpha}}_{\overset{(+)}{\leq} M}|s-t|^{2\alpha} + (\underbrace{\|\tilde{Y}'\|_\alpha}_{\overset{(+)}{\leq} M}|s|^\alpha + |\tilde{Y}'_0|)\|\mathbb{X}-\tilde{\mathbb{X}}\|_{2\alpha}|t-s|^{2\alpha}$$

$$\overset{(++)}{\leq} C(M,T,\alpha)\,(\varrho_\alpha(\mathbb{X},\tilde{\mathbb{X}}) + |Y'_0-\tilde{Y}'_0| + d_{X,\tilde{X},2\alpha}((Y,Y'),(\tilde{Y},\tilde{Y}')))$$

$\Rightarrow$ (3.25).

Exactly the same calculations in 1. for $\|Y-\tilde{Y}\|_\alpha$ give the last estimate (3.26). $\Rightarrow$ proof.

3.4 Rough path and Itô integration

Consider a complete prob. space $(\Omega,\mathcal{A},P)$.

Assume a <u>Brownian motion</u> B on $(\Omega,\mathcal{A},P)$, that is a stoch. proc. $B_t\in\mathbb{R}^d$, $t\geq 0$ which satisfies

(i) $B_0 = 0$ with prob. 1

(ii) B has <u>indep. increments</u>:

$$P(B_{t_1}-B_{t_0}\in A_1,\dots,B_{t_n}-B_{t_{n-1}}\in A_n) = \prod_{i=1}^n P(B_{t_i}-B_{i-1}\in A_i)$$

for all $0=t_0<\dots<t_n$ and (Borel-)meas. $A_i\subseteq\mathbb{R}^d$

(iii) B has <u>stationary Gaussian increm.</u>:

$$P(B_t-B_s\in A) = P(B_{t-s}\in A),\quad t\geq s,\ A\subseteq\mathbb{R}^d$$

with

$$P(B_t\in A) = \frac{1}{(2\pi t)^{d/2}}\int_A \exp(-\tfrac{1}{2t}|x|^2)\,dx,\quad t\geq 0,\ A\subseteq\mathbb{R}^d$$

<u>Rem. 3.11</u> (i) B is a <u>self-similar proc.</u>:

$$(B_{c\cdot t})_{t\geq 0} \overset{d}{=} (\sqrt{c}\,B_t)_{t\geq 0}\quad\text{for all } c>0$$

(ii) B <u>martingale</u> w.r.t. the filtration $\mathcal{F}_t := \sigma(\mathcal{F}^B_t\cup\mathcal{N})$, where σ = smallest σ-algebra containing $\mathcal{F}^B_t$ and $\mathcal{N}$

(41) $\mathcal{F}_t^B = \sigma(B_s, 0 \le s \le t)$ and $\mathcal{N} = \{N \in \mathcal{A} : P(N) = 0\}$,

that is $E[B_t | \mathcal{F}_s] = B_s$, $t \ge s$

<u>Def. 3.12</u> (Itô-integral)

For continuous $\mathcal{F}_t$-meas. (i.e. adapted) proc. $Y_t, t \ge 0$ the <u>Itô-integral</u> of Y w.r.t. B is defined as

$$\int_0^T Y_r \, dB_r = \lim_{n \to \infty} \sum_{[u,v] \in \mathcal{P}_n} Y_u B_{u,v} \quad \text{in probability}$$

for all $\mathcal{P}_n, n \ge 1$ with $|\mathcal{P}_n| \xrightarrow[n \to \infty]{} 0$

<u>Recall</u>: $X_n \to X$ <u>in prob.</u> $\iff$ $\forall$ subsequences (n_k) $\exists$ subseq. (m_k) of (n_k) s.t. $X_{m_k} \xrightarrow[k \to \infty]{} X$ with prob. 1

<u>Rem. 3.13</u>: (i) <u>Itô-isometry</u>: $E[\int_0^T |Y_s|^2 ds] < \infty \implies$

$$\| \int_0^T Y_r \, dB_r \|^2_{L^2(P)} = E[|\int_0^T Y_r \, dB_r|^2]$$

$$= E[\int_0^T |Y_r|^2 dr] = \|Y\|^2_{L^2(P \times dt)}$$

(ii) There ex. a <u>modification</u> $M_t, t \ge 0$ of $\int_0^t Y_s \, dB_s$, i.e. $P(M_t = \int_0^t Y_s \, dB_s) = 1$ for all t, under the assumption of (i) s.t. $M_t, t \ge 0$ is a continuous martingale w.r.t. $\mathcal{F}_t$, i.e.

$E[M_t | \mathcal{F}_s] = M_s$, $s \ge t$

<u>Lemma 3.14</u> (Kolmogorov's continuity criterion for rough paths)

Let $q \ge 2$, $\beta > \frac{1}{q}$. Assume processes $X_t(\omega) \in V$ and $\mathbb{X}_{s,t}(\omega) \in V \otimes V$, $s,t \in [0,T]$ satisfying

$$E[|X_{s,t}|^q]^{1/q} \le C |t-s|^\beta, \quad E[|\mathbb{X}_{s,t}|^{q/2}]^{2/q} \le C |t-s|^{2\beta}$$

for all s,t. Then for all $\alpha \in [0, \beta - \frac{1}{q})$ there is a modification of $(X, \mathbb{X})$ (which we for convenience denote by $(X, \mathbb{X})$) and r.v.'s $K_\alpha \in L^q(P)$, $\mathbb{K}_\alpha \in L^{q/2}(P)$, $\Omega^* \in \mathcal{A}$ with $P(\Omega^*) = 1$ s.t.

$$|X_{s,t}(\omega)| \le K_\alpha(\omega) |t-s|^\alpha, \quad |\mathbb{X}_{s,t}(\omega)|_\alpha \le \mathbb{K}_\alpha(\omega) |t-s|^{2\alpha}$$

for all s,t

Hence, if $(X(\omega), \mathbb{X}(\omega))$ satisfies Chen's relation and $\beta - \frac{1}{q} > \frac{1}{3}$ then $(X(\omega), \mathbb{X}(\omega)) \in \mathcal{C}^\alpha$ for all $\alpha \in (\frac{1}{3}, \beta - \frac{1}{q})$

42 **Proof:** Similar to the classical continuity criterion

$\xrightarrow{\beta=\frac{1}{2}}$ $E[|B_{s,t}|^q]^{1/q} \underset{\text{Rem. 3.11}}{\overset{\text{stationarity, self-similarity,}}{=}} E[|s-t|^{q/2}|B_1|^q]^{1/q}$

$= C|s-t|^{\frac{1}{2}}$ for all $q \geq 2$

$\xRightarrow{\text{L.3.14}}$ B has a α-Hölder cont. modification for all $\alpha \in (\frac{1}{3}, \frac{1}{2})$

On the other hand, we have for $B_t = (B_t^1, \dots, B_t^d)^T$ and $q=4$

$$E[|\int_s^t (B_r^i - B_s^i)\, dB_r^j|^2]^{1/2} \underset{\text{Rem. 3.13,}}{\overset{\text{Itô-isometry,}}{=}} E[\int_s^t |B_r^i - B_s^i|^2 dr]^{1/2}$$

$$\overset{\text{Fubini}}{=} \Big(\int_s^t \underbrace{E[|B_r^i - B_s^i|^2]}_{\overset{\text{stationarity, self-similarity}}{=} |r-s| E[|B_1^i|^2]} dr\Big)^{1/2} = C|t-s|^{2\beta}$$

$\xRightarrow{\text{L.3.14}}$
$$\mathbb{B}_{s,t} = \mathbb{B}_{s,t}^{\text{Itô}} \overset{\text{def.}}{=} \int_s^t B_{s,r} \otimes dB_r = \Big(\int_s^t B_{s,r}^i\, dB_r^j\Big)_{1 \leq i,j \leq d} \quad (3.27)$$

has a modification s.t. $\|\mathbb{B}(\omega)\|_{2\alpha} < \infty$ for all $\alpha \in (0, \frac{1}{4})$ on some Ω^* with $P(\Omega^*) = 1$

Rem.: One shows that the latter also holds for all $\alpha \in (0, \frac{1}{2})$

$\xRightarrow{\text{Rem. 2.5}}$ $\mathbf{X} = (B(\omega), \mathbb{B}(\omega)) \in \mathscr{C}^\alpha$, $\alpha \in (\frac{1}{3}, \frac{1}{2})$ on some Ω^* with $P(\Omega^*) = 1$

$\longrightarrow$ **Prop. 3.15:** Assume that $(Y(\omega), Y'(\omega)) \in \mathscr{D}_{B(\omega)}^{2\alpha}$ on Ω^* with $P(\Omega^*) = 1$ and that Y_t, Y_t' are $\mathcal{F}_t$-meas. for all t. Let $\hat{\Omega} \subseteq \Omega^*$ with $P(\hat{\Omega}) = 1$ s.t.

$$\int_0^T Y_r\, dB_r^{\text{Itô}} \overset{\text{Def 3.12}}{=} \lim_{n\to\infty} \sum_{[u,v]\in \mathcal{P}_n} Y_u(\omega) B_{u,v}(\omega) \quad (3.28)$$

on $\hat{\Omega}$ for some $\mathcal{P}_n$ with $|\mathcal{P}_n| \xrightarrow[n\to\infty]{} 0$.

Then
$$\int_0^T Y_r(\omega)\, d\mathbf{B}_r(\omega) = \int_0^T Y_r\, dB_r^{\text{Itô}}$$

for all $\omega \in \Omega^+$ with $P(\Omega^+) = 1$, where $\Omega^+ \subseteq \hat{\Omega}$

Proof: W.l.o.g. assume that there is a $M > 0$ s.t.

$$\sup_{\substack{\omega \in \hat{\Omega} \\ s \in [0,T]}} |Y_s'(\omega)| \leq M \quad (*)$$

(43) where $\bar{\Omega} \subseteq \hat{\Omega}$ with $P(\bar{\Omega})=1$.

We know that

$$\int_0^T Y_r(\omega)\,d\mathbb{B}_r = \lim_{n\to\infty} \sum_{[u,v]\in\mathcal{P}_n} Y_u B_{u,v}^{(\omega)} + Y_u' \mathbb{B}_{u,v}^{(\omega)}$$

for all $\omega \in \Omega^* \supseteq \bar{\Omega}$, where $|\mathcal{P}_n| \xrightarrow[n\to\infty]{} 0$

$$\overset{(3.28)}{\Longrightarrow} \lim_{n\to\infty} \sum_{[u,v]\in\mathcal{P}_n} Y_u' \mathbb{B}_{u,v}^{(\omega)} \text{ exists} \qquad (+)$$

and equals

$$\int_0^T Y_r\,dB_r^{\text{Itô}} - \int_0^T Y_r(\omega)\,d\mathbb{B}_r^{(\omega)}$$

for all $\omega \in \bar{\Omega}$.

We want to show that (+) is zero on some $\Omega^+ \subseteq \bar{\Omega}$ with $P(\Omega^+)=1$:

Choose a partition $\mathcal{P} = \{0=\tau_0 < \tau_1 < \ldots < \tau_N = T\}$.

Then

$$E\Big[\Big|\sum_{[u,v]\in\mathcal{P}} Y_u' \mathbb{B}_{u,v}\Big|^2\Big] = E\Big[\Big|\sum_{k=0}^{N-1} Y_{\tau_k}' \mathbb{B}_{\tau_k,\tau_{k+1}}\Big|^2\Big]$$

$$= \sum_{k_1,k_2=0}^{N-1} \underbrace{E[\langle Y_{\tau_{k_1}}' \mathbb{B}_{\tau_{k_1},\tau_{k_1+1}}, Y_{\tau_{k_2}}' \mathbb{B}_{\tau_{k_2},\tau_{k_2+1}} \rangle]}_{=:\,\Delta_{k_1 k_2}} \qquad (++)$$

inner product on $\mathbb{R}^m$

Claim: $\Delta_{k_1 k_2} = 0$, if $k_1 \neq k_2$

w.l.o.g. assume that $m=d=1$ and $k_2 > k_1$. Then, using Itô's isometry (Rem. 3.13(i)) and Def. 3.12 one finds that

$$\mathbb{B}_{\tau_{k_2},\tau_{k_2+1}} = \lim_{|\mathcal{P}^*|\to 0} \sum_{[u,v]\in\mathcal{P}^*} B_{\tau_{k_2},u} B_{u,v}$$

def of $\mathbb{B}_m$

in $L^2(P)$.

indep. r.v.'s

$$\overset{(*)}{\Longrightarrow} \Delta_{k_1 k_2} = \lim_{|\mathcal{P}^*|\to 0} \sum_{[u,v]\in\mathcal{P}^*} E[Y_{\tau_{k_1}}' \mathbb{B}_{\tau_{k_1},\tau_{k_1+1}} \cdot Y_{\tau_{k_2}}' B_{\tau_{k_2},u} \cdot B_{u,v}]$$

$$= \lim_{|\mathcal{P}^*|\to 0} \sum_{[u,v]\in\mathcal{P}^*} E[Y_{\tau_{k_1}}' \cdot \mathbb{B}_{\tau_{k_1},\tau_{k_1+1}} \cdot Y_{\tau_{k_2}}' B_{\tau_{k_2},u}] \cdot \underbrace{E[B_{u,v}]}_{=0}$$

$= 0 \Longrightarrow$ claim

$$\Longrightarrow (++) = \sum_{k=0}^{N-1} E[\underbrace{|Y_{\tau_k}' \mathbb{B}_{\tau_k,\tau_{k+1}}|^2}_{\leq |Y_{\tau_k}'|^2 |\mathbb{B}_{\tau_k,\tau_{k+1}}|^2}] \leq M^2 \sum_{k=0}^{N-1} \underbrace{E[|\mathbb{B}_{\tau_k,\tau_{k+1}}|^2]}_{\text{Itô-isom.}}$$

$$\int_{\tau_k}^{\tau_{k+1}} \underbrace{E[|B_r - B_{\tau_k}|^2]}_{\substack{\text{self-similarity,}\\ \text{stationarity}\\ (r-\tau_k)E[|B_1|^2]}} dr$$

$$= M^2 \sum_{k=0}^{N-1} \frac{1}{2} |\tau_{k+1} - \tau_k|^2 \leq C \cdot |\mathcal{P}| \xrightarrow[|\mathcal{P}|\to 0]{} 0$$

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$\Rightarrow$ there ex. $\Omega^+ \subseteq \overline{\Omega}$ s.t.

$$\int_0^T Y_r \, dB_r^{\text{Itô}} - \int_0^T Y_r(\omega) \, dB_r(\omega) = 0$$

$\Rightarrow$ proof.

4. Solutions to rough differential equations

→ objective: Study of rough path differential equations (RDE's) of the form

$$dY_t = f(Y_t)\, dX_t, \quad Y_0 = \xi \in W, \qquad (4.1)$$

where $X : [0,T] \to V$ is the driving input signal, $Y : [0,T] \to W$ the output signal and $f : W \to L(V,W)$ the driving vectorfield for Banach spaces V, W.

4.1 Composition of regular functions with controlled rough paths

Let $g : W \to \overline{W}$ be "nice" (e.g. $g \in C_b^2$).

→ classical chain rule suggests:

If $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0,T], W)$ then

$$(g(Y), g(Y)') \in \mathcal{D}_X^{2\alpha}([0,T], \overline{W}), \qquad (4.2)$$

where
$$g(Y)_t = g(Y_t), \quad g(Y)'_t = Dg(Y_t) Y'_t \in L(V, \overline{W})$$

→ Lemma 4.1: Assume that $g \in C_b^2$ (i.e. space of twice cont. diff. functions with bounded derivatives) and that $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0,T], W)$ for some $X \in C^\alpha$ with $\|(Y, Y')\|_{X,2\alpha} \le M$ for $M \ge 1$. Then $(g(Y), g(Y)') \in \mathcal{D}_X^{2\alpha}([0,T], \overline{W})$ with $g(Y), g(Y)'$ as in (4.2). Furthermore,

$$\|g(Y), g(Y)'\|_{X,2\alpha} \le C_{\alpha,T}\, M \|g\|_{C_b^2} \cdot (1 + \|X\|_\alpha)^2 \, \|(Y, Y')\|_{X,2\alpha}$$

where

$$\|g\|_{C_b^2} := \|g\|_\infty + \|Dg\|_\infty + \|D^2 g\|_\infty$$

and $\|F\|_\infty := \sup_x |F(x)|$ for functions F

(46) **Proof**: We first show that $(\varphi(Y),\varphi(Y)') = (\varphi(Y_\cdot), D\varphi(Y_\cdot)Y'_\cdot) \in \mathcal{D}^{2\alpha}_{X}$:

Using the mean value theorem we see that

$|\varphi(Y_t)-\varphi(Y_s)| = |D\varphi(Y_s+\xi Y_{s,t})Y_{s,t}|$ for some $\xi\in(0,1)$.

$$\Rightarrow \quad |\varphi(Y_t)-\varphi(Y_s)| \le \|D\varphi\|_\infty |Y_{s,t}|$$

$$\Rightarrow \quad \|\varphi(Y_\cdot)\|_\alpha = \sup_{s\neq t}\frac{|\varphi(Y_t)-\varphi(Y_s)|}{|t-s|^\alpha} \le \|D\varphi\|_\infty \underbrace{\sup_{s\neq t}\frac{|Y_{s,t}|}{|t-s|^\alpha}}_{=\|Y_\cdot\|_\alpha} < \infty$$

Similarly, we have that

$$|D\varphi(Y_t)Y'_t - D\varphi(Y_s)Y'_s| = |(D\varphi(Y_t)-D\varphi(Y_s))Y'_t + D\varphi(Y_s)(Y'_t-Y'_s)|$$

$$\le \|D^2\varphi\|_\infty |Y_{t,s}|\cdot\sup_t|Y'_t| + \|D\varphi\|_\infty |Y'_{t,s}|$$

$$\Rightarrow \quad \|\varphi(Y)'\|_\alpha \le \|D^2\varphi\|_\infty \|Y_\cdot\|_\alpha \|Y'_\cdot\|_\infty + \|D\varphi\|_\infty \|Y'_\cdot\|_\alpha < \infty \qquad (+)$$

$$\Rightarrow \quad \varphi(Y),\ \varphi(Y)' \in C^\alpha$$

Further,

$$R^{\varphi(Y)}_{s,t} \overset{\text{def.}}{=} \varphi(Y_t)-\varphi(Y_s) - D\varphi(Y_s)\overbrace{Y'_sX_{s,t}}^{=Y_{s,t}-R^Y_{s,t}}$$

$$= \underbrace{\varphi(Y_t)-\varphi(Y_s)-D\varphi(Y_s)Y_{s,t}}_{\overset{\text{Taylor expansion}}{=}\ \frac{1}{2}D^2\varphi(Y_s+\xi Y_{s,t})[Y_{s,t},Y_{s,t}] \text{ for some } \xi\in(0,1)} + D\varphi(Y_s)R^Y_{s,t}$$

$$\Rightarrow \quad \|R^{\varphi(Y)}\|_\alpha \le \frac{1}{2}\|D^2\varphi\|_\infty \|Y_\cdot\|_\alpha^2 + \|D\varphi\|_\infty \|R^Y\|_{2\alpha} < \infty \qquad (++)$$

$$\Rightarrow \quad (\varphi(Y_\cdot),\varphi(Y_\cdot)') \in \mathcal{D}^{2\alpha}_X$$

$$\Rightarrow \quad \|(\varphi(Y_\cdot),\varphi(Y_\cdot)')\|_{X,2\alpha} \underset{\substack{\text{w.l.o.g. } \varphi(Y_0)=0,\\ \varphi(Y)'_0=0}}{\overset{\text{def}}{=}} \|\varphi(Y_\cdot)'\|_\alpha + \|R^{\varphi(Y)}\|_{2\alpha}$$

$$\overset{(+),(++)}{\le} \|D^2\varphi\|_\infty \|Y_\cdot\|_\alpha \|Y'_\cdot\|_\infty + \|D\varphi\|_\infty \|Y'_\cdot\|_\alpha$$

$$+ \frac{1}{2}\|D^2\varphi\|_\infty \|Y_\cdot\|_\alpha^2 + \|D\varphi\|_\infty \|R^Y\|_{2\alpha}$$

$$\le \|\varphi\|_{C^2_b}\big(\|Y_\cdot\|_\alpha \|Y'_\cdot\|_\infty + \|Y'_\cdot\|_\alpha + \|Y_\cdot\|_\alpha^2 + \|R^Y\|_{2\alpha}\big) \qquad (*)$$

$$\|Y_\cdot\|_\alpha \le \|Y'_\cdot\|_\infty \|X\|_\alpha + \|R^Y\|_\alpha \le C(1+\|X\|_\alpha)\|(Y,Y')\|_{X,2\alpha}$$

$$\Rightarrow (*) \le C_{\alpha,T}\|\varphi\|_{C^2_b}(1+\|X\|_\alpha)^2(1+\|(Y,Y')\|_{X,2\alpha})\|(Y,Y')\|_{X,2\alpha}$$

$\Rightarrow$ proof.

4.2 A priori estimates with respect RDE's

Prop. 4.2: Suppose that $\xi \in W$, $f \in C_b^2(W, L(V,W))$ and $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^\alpha$ for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. Let $(Y, Y') = (Y, f(Y)) \in \mathcal{D}_X^{2\alpha}$ be a solution to the RDE

$$Y_t = \xi + \int_0^t f(Y_s)\,d\mathbf{X}_s .$$

Then

$$\|Y\|_\alpha \le C_\alpha \left[\left(\|f\|_{C_b^2} |||\mathbf{X}|||_\alpha\right) \vee \left(\|f\|_{C_b^2} |||\mathbf{X}|||_\alpha\right)^{1/\alpha} \right],$$

where $|||\mathbf{X}||_\alpha \overset{\text{def.}}{=} \|X\|_\alpha + \|\mathbb{X}\|_{2\alpha}^{1/2}$ and $a \vee b \overset{\text{def.}}{=} \max(a,b)$

Rem.: We know from L. 4.1 that $(f(Y_\cdot), f(Y_\cdot)') = (f(Y_\cdot), Df(Y_\cdot)Y_\cdot') \in \mathcal{D}_X^{2\alpha} \longrightarrow \int_0^t f(Y_s)\,d\mathbf{X}_s$ in Prop. 4.2 well-def. in the sense of Th. 3.8 (Gubinelli)

Proof: Denote by $\|\cdot\|_{\alpha,J}$ the Hölder seminorm w.r.t. an interval $J \subseteq [0,T]$ ($\Longrightarrow \|X\|_{\alpha,J} \le \|X\|_{\alpha,[0,T]} =: \|X\|_\alpha$)

Let $J = [s,t]$.

$$\overset{(3.19)\text{ in Th. } 3.8}{\Longrightarrow} \quad |R^Y_{s,t}| \overset{\text{def.}}{=} |Y_{s,t} - f(Y_s)X_{s,t}|$$

$$\overset{\Delta\text{-ineq.}}{\le} \Big| \underbrace{\int_s^t f(Y_u)\,dX_u}_{= Y_{s,t}} - f(Y_s)X_{s,t} - \underbrace{\overset{\in L(W,L(V,W))}{Df(Y_s)}\,\overset{\in L(V,W)}{f(Y_s)}}_{\text{lifting theorem} \ \in L(V\otimes V, W)} \mathbb{X}_{s,t} \Big|$$

$$+ |Df(Y_s)f(Y_s)\mathbb{X}_{s,t}| \lesssim \left(\|X\|_{\alpha,J} \|R^{f(Y)}\|_{2\alpha,J} + \|\mathbb{X}\|_{2\alpha,J} \|f(Y_\cdot)\|_{\alpha,J}\right) \cdot |t-s|^{3\alpha} + \|\mathbb{X}\|_{2\alpha,J} \cdot |t-s|^{2\alpha} \qquad (4.3)$$

Further, define

$$\|Z\|_{\alpha,h} = \sup_{\substack{J \text{ s.t.} \\ |J| \le h}} \|Z\|_{\alpha,J}$$

(length of J $\longrightarrow |J| \le h$)

($\Longrightarrow \|Z\|_{\alpha,J} \le \|Z\|_{\alpha,h}$ for all J with $|J| \le h$)

(48) $\overset{(4.3)}{\Longrightarrow}$ $$\|R^Y\|_{2\alpha,h} \lesssim \|\mathbb{X}\|_{2\alpha,h} + \left(\|X\|_{\alpha,h}\|R^{f(Y)}\|_{2\alpha,h} + \|\mathbb{X}\|_{2\alpha,h}\|f(Y)\|_{\alpha,h}\right)h^\alpha \qquad (*)$$

On the other hand, we have that

$$R^{f(Y)}_{s,t} \overset{\text{def}}{=} f(Y_t)-f(Y_s)-Df(Y_s)\underbrace{Y'_s X_{s,t}}_{=Y_{s,t}-R^Y_{s,t}}$$

$$= f(Y_t)-\overset{\in L(V,W)}{f(Y_s)}-\overbrace{Df(Y_s)}^{\in L(V,W)}Y_{s,t}+Df(Y_s)R^Y_{s,t}$$

$\overset{(\#)\text{ in}}{\Longrightarrow}$ the proof of L.4.1 $$\|R^{f(Y)}\|_{2\alpha,h} \le \frac{1}{2}\|D^2f\|_\infty \|Y\|^2_{\alpha,h} + \|Df\|_\infty \|R^Y\|_{2\alpha,h}$$

$$\lesssim \|Y\|^2_{\alpha,h} + \|R^Y\|_{2\alpha,h}$$

Further, the mean value theorem also yields

$$\|f(Y)\|_{\alpha,h} \lesssim \|Y\|_{\alpha,h}$$

$\overset{(*)}{\Longrightarrow}$ $$\|R^Y\|_{2\alpha,h} \le c_1\|\mathbb{X}\|_{2\alpha,h} + c_1\|X\|_{\alpha,h}h^\alpha\|Y\|^2_{\alpha,h} + c_1\|X\|_{\alpha,h}h^\alpha\|R^Y\|_{2\alpha,h} + c_1\|\mathbb{X}\|_{2\alpha,h}h^\alpha\|Y\|_{\alpha,h}, \qquad (**)$$

where c_1 is indep. of $\mathbb{X}$ and Y.

Now choose h s.t.

$$c_1\|X\|_\alpha h^\alpha,\ c_1\|\mathbb{X}\|^{1/2}_{2\alpha,h}h^\alpha \le \frac{1}{2} \qquad (+)$$

$\overset{(**)}{\Longrightarrow}$ $$\|R^Y\|_{2\alpha,h} \le c_1\|\mathbb{X}\|_{2\alpha,h} + \frac{1}{2}\|Y\|^2_{\alpha,h} + \frac{1}{2}\|R^Y\|_{2\alpha,h} + \frac{1}{2}\|\mathbb{X}\|^{1/2}_{2\alpha,h}\|Y\|_{\alpha,h}$$

$\Longrightarrow$ $$\|R^Y\|_{2\alpha,h} \le 2c_1\|\mathbb{X}\|_{2\alpha,h} + \|Y\|^2_{\alpha,h} + \underbrace{\|\mathbb{X}\|^{1/2}_{2\alpha,h}\|Y\|_{\alpha,h}}_{\le \frac{1}{2}(\|\mathbb{X}\|_{2\alpha,h}+\|Y\|^2_{\alpha,h})}$$

$$\le c_2\|\mathbb{X}\|_{2\alpha,h} + 2\|Y\|^2_{\alpha,h} \quad \text{with } c_2 = (2c_1+1) \qquad (***)$$

We also have that

$$\|Y\|_{\alpha,h} \lesssim \|X\|_\alpha + \|R^Y\|_{2\alpha,h}\cdot h^\alpha$$

since $Y_{s,t} = \underbrace{f(Y_s)}_{=Y'_s}X_{s,t} + R^Y_{s,t}$ and $f \in C^2_b$

$\overset{(***)}{\Longrightarrow}$ $$\|Y\|_{\alpha,h} \le c_3\|X\|_\alpha + c_3\|\mathbb{X}\|_{2\alpha,h}h^\alpha + c_3\|Y\|^2_{\alpha,h}h^\alpha$$

(49) $\overset{(+)}{\leq} \; c_3 \|X\|_\alpha + c_4 \|X\|_{2\alpha,h}^{1/2} + \underbrace{c_3 \|Y\|_{\alpha,h} h^\alpha}_{=:\,\Psi_h} \cdot \|Y\|_{\alpha,h}$

$\overset{\cdot c_3 h^\alpha}{\Longrightarrow}$

$$\Psi_h \leq \lambda_h + \Psi_h^2 \qquad (++)$$

for $\lambda_h := c_5 \|\|X\|\|_\alpha h^\alpha$

Suppose $\lambda_{h_0} < \frac{1}{4}$ for some $h_0 > 0$

$\overset{(++)}{\Longrightarrow}$ $\forall \; h \leq h_0$ we have <u>either</u>

$$\Psi_h \geq \Psi_+ := \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_h} \geq \frac{1}{2} \qquad (i)$$

<u>or</u>

$$\Psi_h \leq \Psi_- := \frac{1}{2} - \sqrt{\frac{1}{4} - \lambda_h} \overset{\text{mean}}{\underset{\text{value th.}}{=}} \int_0^1 \frac{1}{\sqrt{1 - 4\theta\lambda_h}} \, d\theta \, \lambda_h \qquad (ii)$$

Now choose $h_0 > 0$ by means of λ_h s.t. $\forall h \leq h_0$

$$\int_0^1 \frac{1}{\sqrt{1-4\theta\lambda_h}} \, d\theta \, \lambda_h < \frac{1}{7}$$

$\Longrightarrow$ $\Psi_h < \frac{1}{7}$ in the case of (ii)

<u>claim</u>: $\forall h \in (0, h_0] : \Psi_h < \frac{1}{7}$

otherwise: Define

$$\gamma = \inf \{ h \in (0, h_0] : \Psi_h \geq \tfrac{1}{2} \} < \infty$$

$\gamma > 0$, since $\Psi_h = c_3 \underbrace{\|Y\|_{\alpha,h}}_{\leq \|Y\|_\alpha} h^\alpha \underset{h \downarrow 0}{\Longrightarrow} 0$

<u>However</u>, we have that

$$\Psi_h \leq 3 \lim_{\ell \uparrow h} \Psi_\ell \quad ,$$

since

$$\|Y\|_{\alpha,h} \leq 3 \overset{\leq \lim_{\ell \uparrow h} \|Y\|_{\alpha,\ell}}{\|Y\|_{\alpha,h/3}}$$

and similarly

$$\lim_{\ell \downarrow h} \Psi_\ell \leq 3 \Psi_h$$

1. case: $\Psi_\gamma \geq \frac{1}{2}$

$\Longrightarrow$ $\Psi_\gamma \leq 3 \lim_{\ell \uparrow \gamma} \Psi_\ell \leq 3 \cdot \frac{1}{7} < \frac{1}{2}$ $\Longrightarrow$ ↯

2. case: $\Psi_\gamma < \frac{1}{7}$ $\Longrightarrow$ $\frac{1}{2} \leq \lim_n \Psi_{\ell_n} \leq 3\Psi_\gamma < \frac{3}{7}$

for some $\ell_n \downarrow \gamma$ $\Longrightarrow$ ↯

(50) $\Longrightarrow$ claim with h_0 depending on $\mathbb{X}$ (and not Y)

$\overset{(+)}{\Longrightarrow}$ $\psi_h \leq 2\lambda_h \quad \forall h \in (0, h_0]$ $\Longrightarrow$

$$\|Y\|_{\alpha,h} \leq C_6 \, |||\mathbb{X}|||_\alpha \qquad (+++)$$

$\forall \, h \in (0, h_0]$

(Claim: Let $\alpha \in [0,1)$, $h>0$, $M>0$. Suppose that

$$\|Z\|_{\alpha,h} \leq M.$$

Then

$$\|Z\|_\alpha \leq M\left(1 \vee 2h^{-(1-\alpha)}\right)$$

$\longrightarrow$ proof left as exercise

$$\xRightarrow[h=h_0]{(+++)} \quad \|Y\|_\alpha \leq C_6 \, |||\mathbb{X}|||_\alpha \left(1 \vee 2h_0^{-(1-\alpha)}\right) \qquad (\lozenge)$$

We have that $\underset{\overset{\|}{C_5 |||\mathbb{X}|||_\alpha h_0^\alpha}}{\lambda_{h_0}} < \frac{1}{4}$

$\Longrightarrow \quad h_0 = C \cdot |||\mathbb{X}|||^{-\frac{1}{\alpha}}$

$$\xRightarrow{\lambda^{\alpha}(a^{\alpha} \vee b^{\alpha}) = \lambda a \vee \lambda b} \quad \|Y\|_\alpha \leq C\left(|||\mathbb{X}|||_\alpha \vee |||\mathbb{X}|||^{\frac{1}{\alpha}}\right).$$

$\Longrightarrow$ proof.

4.3 Rough differential equations

$\longrightarrow$ One of our main results:

<u>Th. 4.3</u> (Existence and uniqueness of RDE's)
Let $\xi \in W$, $f \in C_b^3(W, L(V,W))$ and $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^\beta([0,\infty), V)$ for $\beta \in (\frac{1}{3}, \frac{1}{2})$. Then there exists a unique $(Y, Y') \in \mathcal{D}_X^{2\beta}([0,1], W)$ s.t.

$$Y_t = \xi + \int_0^t f(Y_s)\, d\mathbf{X}_s, \quad 0 \leq t \leq 1.$$

Moreover, $Y' = f(Y)$.

<u>Proof</u>: <u>idea</u>: <u>fixed point argument</u> in $\mathcal{D}_X^{2\alpha}$ in connection with L.4.1 and Th.3.8 (Gubinelli):
Let $\alpha \in (\frac{1}{3}, \beta)$ and $(Y, Y') \in \mathcal{D}_X^{2\alpha}$ $(\Rightarrow \mathbf{X} \in \mathcal{C}^\alpha \supseteq \mathcal{C}^\beta)$.

$\longrightarrow$ fixed point map:

$$\mathcal{M}_T((Y,Y')) \overset{\text{def}}{=} \Big(\xi + \int_0^{\cdot} \Gamma_s\, d\mathbf{X}_s, \underbrace{\Gamma}_{\text{Gubinelli derivative of } \xi + \int_0^{\cdot} \Gamma_s d\mathbf{X}_s}\Big) \in \mathcal{D}_X^{2\alpha},$$

where $(\Gamma, \Gamma') \overset{\text{def}}{=} (f(Y), f(Y)') \overset{\text{def}}{=} (f(Y), Df(Y)Y') \overset{L.4.1}{\in} \mathcal{D}_X^{2\alpha}$

on a closed subspace $B_T \subset \mathcal{D}_X^{2\alpha}$ w.r.t. a small interval $[0,T] \subseteq [0,1]$.

$\longrightarrow$ fixed point argument yields: $(Y,Y') \in \mathcal{D}_X^{2\beta}$, <u>since</u>:

$$|Y_{s,t}| \leq \sup_s |Y'_s|\,|X_{s,t}| + \|R^Y\|_{2\alpha} |t-s|^{2\alpha > \frac{2}{3}} \Rightarrow Y \in C^\beta$$

$\Rightarrow Y' \in C^\beta$ (since $Y' = f(Y)$).

On the other hand,

$$|R^Y_{s,t}| \overset{\text{def}}{=} |Y_{s,t} - Y'_s X_{s,t}| = \Big|\int_s^t (f(Y_r) - f(Y_s))\, d\mathbf{X}_r\Big|$$

$$\underset{\Delta\text{-ineq}}{\overset{\text{Th 3.8.}}{\leq}} \sup_s |Y'_s|\,|\mathbb{X}_{s,t}| + O(|t-s|^{3\alpha})$$

$\Rightarrow \|R^Y\|_{2\beta} < \infty \quad \Rightarrow (Y,Y') \in \mathcal{D}_X^{2\beta}$

Now choose $B_T \subseteq \mathcal{D}_X^{2\alpha}$ to be the closed unit ball with centre

$$\Big(\xi + f(\xi) X_{0,\cdot}, \underbrace{f(\xi)}_{\text{Gubinelli derivative}}\Big) \in \mathcal{D}_X^{2\alpha}$$

(52) Since $\mathcal{M}_T((Y,Y'))_0 = (\xi, f(\xi))$, we define

$$B_T = \widehat{B}_T \cap \{(Y,Y') \in \mathcal{D}_X^{2\alpha}([0,T], W) : Y_0 = \xi, Y_0' = f(\xi)\}$$

($\Rightarrow$ closed set in $\mathcal{D}_X^{2\alpha}$)

$\longrightarrow$ B_T set of all $(Y,Y') \in \mathcal{D}_X^{2\alpha}$ with $Y_0 = \xi$, $Y_0' = f(\xi)$ s.t.

$$\|(\xi + f(\xi)X_{0,\cdot} - Y, f(\xi) - Y')\|_{X,2\alpha} \overset{\text{def.}}{=} \quad (*)$$

$$|\xi + \underbrace{f(\xi)X_{0,0}}_{=0} - \overset{\xi}{Y_0}| + |\underbrace{f(\xi) - \overset{=f(\xi)}{Y_0'}}_{=0}| + \|f(\xi) - Y'\|_\alpha + \|R^{\xi + f(\xi)X_{0,\cdot} - Y}\|_{2\alpha}$$

≤ 1

From now on, denote by $\|\cdot\|_{X,2\alpha}$ the seminorm

$\|(Y,Y')\|_{X,2\alpha} = \|Y'\|_\alpha + \|R^Y\|_{2\alpha}$. (for simplicity)

$\xrightarrow{\Delta\text{-ineq.}}$

$$\Big| \underbrace{\|(\xi + f(\xi)X_{0,\cdot}, f(\xi))\|_{X,2\alpha}}_{=0} - \|(Y,Y')\|_{X,2\alpha} \Big|$$

$$\leq \|(\xi + f(\xi)X_{0,\cdot} - Y, f(\xi) - Y')\|_{X,2\alpha} \leq \underbrace{\|(\xi + f(\xi)X_{0,\cdot}, f(\xi))\|_{X,2\alpha}}_{= \|f(\xi)\|_\alpha + \|0\|_{2\alpha} = 0} + \|(Y,Y')\|_{X,2\alpha}$$

$\Rightarrow$ $(*) = \|(Y,Y')\|_{X,2\alpha}$

$\Rightarrow$ $B_T = \{(Y,Y') \in \mathcal{D}_X^{2\alpha} : Y_0 = \xi, Y_0' = f(\xi), \|(Y,Y')\|_{X,2\alpha} \leq 1\}$ $(**)$

$\longrightarrow$ We want to show that

(i) <u>Invariance</u>, i.e. $\mathcal{M}_T : B_T \longrightarrow B_T$ for small T

and (ii) <u>Contraction</u>, i.e.

$$\|\mathcal{M}_T((Y,Y')) - \mathcal{M}_T((\tilde{Y},\tilde{Y}'))\|_{X,2\alpha} \leq K \|(Y,Y') - (\tilde{Y},\tilde{Y}')\|_{X,2\alpha}$$

for all $(Y,Y'), (\tilde{Y},\tilde{Y}') \in B_T$ and some $K < 1$

To this end, we need the following estimates

$$\|\Gamma, \Gamma'\|_{X,2\alpha} \overset{\text{def}}{=} \|(f(Y), f(Y)')\|_{X,2\alpha}$$

$$\leq C M \|f\|_{C_b^2} (|Y_0'| + \|(Y,Y')\|_{X,2\alpha}), \quad \text{(Lemma 4.1)} \quad (+)$$

where $M := \|f\|_\infty + 1 \overset{(**)}{\geq} |Y_0'| + \|(Y,Y')\|_{X,2\alpha}$