(53) and

$$\|(\int_0^\cdot \overbrace{\Gamma_s}^{f(Y)} dX_s, \overbrace{\Gamma'}^{\overbrace{Df(Y)Y'}^{\in L(V\otimes V, W)}})\|_{X,2\alpha} \le \|\Gamma\|_\alpha + \|\Gamma'\|_\infty \|\mathbb{X}\|_{2\alpha}$$

$$+ C\left(\|X\|_\alpha \|R^\Gamma\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha}\|\Gamma'\|_\alpha\right) \qquad \text{(Th. 3.8 (Gubinelli))}$$

$$\le \|\Gamma\|_\alpha + C\left(|\Gamma_0'| + \|(\Gamma,\Gamma')\|_{X,2\alpha}\right)\left(\underbrace{\|X\|_\alpha}_{=\sup_{s\neq t}\left(\frac{|X_{s,t}|}{|s-t|^\alpha |s-t|^{\beta-\alpha}}|s-t|^{\beta-\alpha}\right)} + \|\mathbb{X}\|_{2\alpha}\right)$$

$$\le \|\Gamma\|_\alpha + C\left(|\Gamma_0'| + \|(\Gamma,\Gamma')\|_{X,2\alpha}\right)T^{\beta-\alpha} \qquad (++)$$

<u>Note</u>: inspection of the proofs of L. 4.1 and Th. 3.8 shows that C in $(+),(++)$ can be chosen uniformly w.r.t. $T\in(0,1]$.

1. <u>Invariance</u>:

$$\overset{(+),(++)}{\Longrightarrow} \quad \|M_T(\overbrace{(Y,Y')}^{\in \mathcal{B}_T})\|_{X,2\alpha} \overset{def}{=} \|(\int_0^\cdot \Gamma_s dX_s, \Gamma)\|_{X,2\alpha}$$

$$\overset{(++)}{\le} \underbrace{\|\overbrace{\Gamma}^{=f(Y)}\|_\alpha}_{\le \|f\|_{C_b^1}\|Y\|_\alpha} + C\left(\underbrace{|\Gamma_0'|}_{=|Df(Y_0)Y_0'| \le \|f\|^2_{C_b^1}} + \|(\Gamma,\Gamma')\|_{X,2\alpha}\right)T^{\beta-\alpha}$$

$$\overset{(+)}{\le} \|f\|_{C_b^1}\|Y\|_\alpha + C\Big(\|f\|^2_{C_b^1} + CM\|f\|_{C_b^2}\big(\underbrace{|Y_0'| + \|(Y,Y')\|_{X,2\alpha}}_{\le 1+\|f\|_\infty = M}\big)\Big)T^{\beta-\alpha} \qquad (+++)$$

On the other hand,

$$|Y_{s,t}| \le |Y'|_\infty |X_{s,t}| + \|R^Y\|_{2\alpha}|t-s|^{2\alpha} \overset{T\le 1}{\le} (|Y_0'| + \underbrace{\|Y'\|_\alpha}_{\le \|(Y,Y')\|_{X,2\alpha}})\|X\|_\beta |t-s|^\beta$$

$$+ \underbrace{\|R^Y\|_{2\alpha}}_{\substack{\le \|(Y,Y')\|_{X,2\alpha} \\ \le 1}}|t-s|^{2\alpha}$$

$$\overset{|t-s|^\alpha \le T^\alpha \overset{2\alpha>\beta}{\le} T^{\beta-\alpha}}{\Longrightarrow} \quad \|Y\|_\alpha \le \left(|Y_0'| + \|(Y,Y')\|_{X,2\alpha}\right)\|X\|_\beta T^{\beta-\alpha} \qquad (\diamond)$$

$$+ \|R^Y\|_{2\alpha}T^{\beta-\alpha} \le \left((\|f\|_\infty + 1)\|X\|_\beta + 1\right)T^{\beta-\alpha} \qquad (\diamond\diamond)$$

$$\overset{(+++)}{\Longrightarrow} \quad \|M_T((Y,Y'))\|_{X,2\alpha} \le \|f\|_{C_b^1}(\|f\|_\infty + 1)T^{\beta-\alpha}$$

$$+ CM\left(\|f\|^2_{C_b^1} + \|f\|_{C_b^2}(\|f\|_\infty + 1)\right)T^{\beta-\alpha}$$

Choose $T = T_0 = T_0(\alpha,\beta,\mathbb{X},f) \le 1$ s.t.

(54) $\|\mathcal{M}_{T_0}((Y,Y'))\|_{X,2\alpha} \leq 1 \implies$ invariance
for small $T = T_0$

<u>2. Contraction:</u>

Define $\Delta_s = f(Y_s) - f(\tilde{Y}_s)$

$$\implies \|\mathcal{M}_T((Y,Y')) - \mathcal{M}_T((\tilde{Y},\tilde{Y}'))\|_{X,2\alpha}$$

$$\stackrel{\text{def}}{=} \|(\int_0^\cdot \Delta_s \, dX_s, \Delta)\|_{X,2\alpha}$$

$$\underset{\text{in } (++)}{\overset{\Pi = \Delta}{\leq}} \|\Delta\|_\alpha + C(\underbrace{|\Delta_0'|}_{=0} + \|(\Delta,\Delta')\|_{X,2\alpha}) T^{\beta-\alpha}$$

$$\overset{\text{mean value th.}}{\leq} C\|f\|_{C_b^2} \|Y - \tilde{Y}\|_\alpha + C\|(\Delta,\Delta')\|_{X,2\alpha} T^{\beta-\alpha}$$

$\longrightarrow$ <u>sufficient</u> to show that

$$\|Y - \tilde{Y}\|_\alpha \leq C T^{\beta-\alpha} \|(Y-\tilde{Y}, Y'-\tilde{Y}')\|_{X,2\alpha} \qquad (A)$$

and

$$\|(\Delta,\Delta')\|_{X,2\alpha} \leq C \|(Y-\tilde{Y}, Y'-\tilde{Y}')\|_{X,2\alpha} \qquad (B)$$

<u>(A)</u>: Apply $(\Diamond)$ to $Y - \tilde{Y}$ and we get that

$$\|Y-\tilde{Y}\|_\alpha \leq (\overbrace{|Y_0' - \tilde{Y}_0'|}^{=0} + \|(Y-\tilde{Y}, Y'-\tilde{Y}')\|_{X,2\alpha})\|X\|_\beta T^{\beta-\alpha}$$
$$+ \|RY - R\tilde{Y}\|_{2\alpha} T^{\beta-\alpha}$$
$$\leq C T^{\beta-\alpha} \|(Y-\tilde{Y}, Y'-\tilde{Y}')\|_{X,2\alpha} \implies (A)$$

<u>(B)</u>: $\Delta_s' \overset{L.4.1}{=} f(Y)_s' - f(\tilde{Y})_s' = Df(Y_s)Y_s' - Df(\tilde{Y}_s)\tilde{Y}_s'$

$$= (Df(Y_s) - Df(\tilde{Y}_s))Y_s' + Df(\tilde{Y}_s)(Y_s' - \tilde{Y}_s')$$

$$\overset{\text{meanvalue th.}}{=} \int_0^1 D^2 f(\theta Y_s + (1-\theta)\tilde{Y}_s)[Y_s - \tilde{Y}_s, Y_s'[\cdot]] \, d\theta$$

$$+ Df(\tilde{Y}_s)(Y_s' - \tilde{Y}_s')$$

$\implies$

(55) $\Delta'_{s,t} \overset{\in L(V,W)}{=}$

$$\int_0^1 D^2 f(\theta Y_t + (1-\theta)\tilde{Y}_t)[Y_t - \tilde{Y}_t, Y'_t[\cdot]] - D^2 f(\theta Y_s + (1-\theta)\tilde{Y}_s)[Y_s - \tilde{Y}_s, Y'_s[\cdot]] d\theta$$

$$+ Df(\tilde{Y}_t)(Y'_t - \tilde{Y}'_t) - Df(\tilde{Y}_s)(Y'_s - \tilde{Y}'_s)$$

$$\overset{\text{linearity}}{=} \int_0^1 D^2 f(\theta Y_t + (1-\theta)\tilde{Y}_t)[\underbrace{(Y-\tilde{Y})_{s,t}}_{\in W}, \underbrace{(Y')_{s,t}[\cdot]}_{\in L(V,W)}] d\theta \leftarrow J_1$$

$$+ \int_0^1 D^2 f(\theta Y_t + (1-\theta)\tilde{Y}_t)[(Y-\tilde{Y})_{s,t}, Y'_s[\cdot]] d\theta \leftarrow J_2$$

$$+ \int_0^1 D^2 f(\theta Y_t + (1-\theta)\tilde{Y}_t)[(Y_s - \tilde{Y}_s), Y'_{s,t}[\cdot]] d\theta \leftarrow J_3$$

$$+ \int_0^1 (D^2 f(\theta Y_t + (1-\theta)\tilde{Y}_t) - D^2 f(\theta Y_s + (1-\theta)\tilde{Y}_s))[Y_s - \tilde{Y}_s, Y'_s[\cdot]] d\theta \leftarrow J_4$$

$$+ Df(\tilde{Y}_t)((Y' - \tilde{Y}')_{s,t}) + (Df(\tilde{Y}_t) - Df(\tilde{Y}_s))((Y' - \tilde{Y}')_s) \leftarrow J_5$$

$$= J_1 + \dots + J_5$$

We see that

$$|J_4| \underset{\text{th.}}{\overset{\text{mean value}}{\leq}} \|D^3 f\|_\infty (|Y_{s,t}| + |\tilde{Y}_{s,t}|)\, |Y_s - \tilde{Y}_s| \cdot |Y'_s|$$

$$\Longrightarrow \|J_4\|_\alpha \leq \|D^3 f\|_\infty (\underbrace{\|Y\|_\alpha}_{\overset{(**)}{\leq} ((\|f\|_\infty+1)\|X\|_\beta+1)T^{\beta-\alpha}} + \|\tilde{Y}\|_\alpha)(\underbrace{\|Y'\|_\alpha}_{\leq 1} + \overset{\leq \|f\|_\infty}{|Y'_0|})\, \underbrace{\|Y-\tilde{Y}\|_\alpha}_{\overset{(A)}{\leq} CT^{\beta-\alpha}\|(Y-\tilde{Y}, Y'-\tilde{Y}')\|_{X,2\alpha}}$$

$$\leq C \|(Y-\tilde{Y}, Y'-\tilde{Y}')\|_{X,2\alpha}$$

Similarly, we also get such estimates for J_1, J_2, J_3, J_5

$$\Longrightarrow \|\Delta'\|_\alpha \leq C \|(Y-\tilde{Y}, Y'-\tilde{Y}')\|_{X,2\alpha}$$

Using the Taylor formula of the type

$$g(x_0+h) - g(x_0) - Dg(x_0)h = \int_0^1 (1-\theta) D^2 g(x_0+\theta h)[h,h] d\theta,$$

we also find in the same way that

$$\|R^\Delta\|_{2\alpha} \leq C \|(Y-\tilde{Y}, Y'-\tilde{Y}')\|_{X,2\alpha} \qquad \Longrightarrow (B)$$

56 $\Longrightarrow \exists\, T_0 \in (0,1]$ s.t.

$$\mathcal{M}_{T_0} : B_{T_0} \longrightarrow B_{T_0}$$

and $\|\mathcal{M}_{T_0}((Y,Y')) - \mathcal{M}_{T_0}((\tilde{Y},\tilde{Y}'))\|_{X,2\alpha} \le \frac{1}{2} \|(Y-\tilde{Y}, Y'-\tilde{Y}')\|_{X,2\alpha}$

$\xRightarrow{\text{fixed point th.}}$ $\exists$ unique $(Y,Y') \in \mathcal{D}^{2\alpha}_{X'}$ s.t.

$$\mathcal{M}_{T_0}((Y,Y')) = (Y,Y')$$

$\Longrightarrow$ unique solution on $[0,T_0]$ with $Y_0 = \xi$

Similarly, we get a unique solution on $[T_0, 2T_0]$ with initial value Y_{T_0}

$\vdots$

$\Longrightarrow$ unique solution on $[0,1]$ $\Longrightarrow$ proof.

Rem. 4.4: The diff. eq.

$$Y_t = \xi + \int_0^t f_0(Y_s)\, ds + \int_0^t f(Y_s)\, dX_s$$

is a special case of Th. 4.3 for $\hat{Y}_t = Y_t \in W$, $\hat{X}_t = (X_t, t) \in \hat{V} := V \times \mathbb{R}$ and $\hat{f} \in C^3_b(W, L(\hat{V}, W))$ given by $\hat{f}(y)\hat{h} = f(y)\underbrace{h}_{\in V} + f_0(y)\underbrace{h_0}_{\in \mathbb{R}}$, $\hat{h} = (h, h_0)$ with $f_0 \in C^3_b(W,W)$

$\xrightarrow[\substack{X^{d+1}_t := t \\ X_t = (X^1_t, \dots, X^d_t)}]{V = \mathbb{R}^d}$ $\hat{\mathbb{X}}_{s,t} = (\hat{\mathbb{X}}^{ij}_{s,t})_{1 \le i,j \le d+1} = \left(\int_s^t X^i_r\, dX^j_r\right)_{1 \le i,j \le d+1}$

$$= \begin{pmatrix} \underbrace{\mathbb{X}_{s,t}}_{\in \mathbb{R}^{d\times d}} & \underbrace{\int_s^t X_r\, dr}_{\in \mathbb{R}^d} \\ \int_s^t r\, dX^1_r \cdots & \int_s^t r\, dr \end{pmatrix}$$

$\xrightarrow[\text{after (4)}]{\text{inequality}}$: $(Z, Z') \overset{\text{def.}}{=} (f(Y), f(Y)') = (f(Y), Df(Y)Y') \in \mathcal{D}^{2\alpha}_{X'}$

$\longrightarrow (Y, Y') = (Y, f(Y)) \in \mathcal{D}^{2\alpha}_{X'}$

(57) 4.4 Continuity of the Itô-Lyons map

$\longrightarrow$ solutions $Y = Y(X, \mathbb{X})$ to RDE's are <u>continuous</u> w.r.t. $(X, \mathbb{X}) \in \mathscr{C}^\beta$

$\longrightarrow$ consequence of the stability of rough path integration in the sense of Th. 3.10 and the stability of controlled rough paths under regular transformations!

<u>Prop. 4.5</u> (Stability w.r.t. compositions with regular functions)
Let $\mathbf{X} = (X, \mathbb{X}), \hat{\mathbf{X}} = (\hat{X}, \hat{\mathbb{X}}) \in \mathscr{C}^\alpha$, $(Y, Y') \in \mathscr{D}^{2\alpha}_X$, $(\hat{Y}, \hat{Y}') \in \mathscr{D}^{2\alpha}_{\hat{X}}$ with $\|(Y, Y')\|_{X, 2\alpha}, \|(\hat{Y}, \hat{Y}')\|_{\hat{X}, 2\alpha} \leq M \in [1, \infty)$.
Further, let $g \in C^3_b(W, \bar{W})$ and set

$$(Z, Z') = (g(Y), \underbrace{Dg(Y)Y'}_{\overset{def}{=} f(Y)' \in L(V, \bar{W})}) \in \mathscr{D}^{2\alpha}_X,$$

$$(\hat{Z}, \hat{Z}') = (g(\hat{Y}), Dg(\hat{Y})\hat{Y}') \in \mathscr{D}^{2\alpha}_{\hat{X}}.$$

Then

$$d_{X, \hat{X}, 2\alpha}((Z, Z'), (\hat{Z}, \hat{Z}')) \leq C_M \big(g_\alpha(\mathbf{X}, \hat{\mathbf{X}}) + |Y_0 - \hat{Y}_0| + |Y'_0 - \hat{Y}'_0| + d_{X, \hat{X}, 2\alpha}((Y, Y'), (\hat{Y}, \hat{Y}')) \big)$$

and

$$\overset{def}{=} |X_0 - \hat{X}_0| + \|X - \hat{X}\|_\alpha + \|\mathbb{X} - \hat{\mathbb{X}}\|_{2\alpha}$$

$$\|Z - \hat{Z}\|_\alpha \leq C_M \big(g_\alpha(\mathbf{X}, \hat{\mathbf{X}}) + |Y_0 - \hat{Y}_0| + |Y'_0 - \hat{Y}'_0| + d_{X, \hat{X}, 2\alpha}((Y, Y'), (\hat{Y}, \hat{Y}')) \big)$$

$$\overset{def}{=} \|Y' - \hat{Y}'\|_\alpha + \|R^Y - R^{\hat{Y}}\|_{2\alpha}$$

for some $C_M = C(M, T, \alpha, g)$.

<u>Proof</u>: The proof is based on Taylor's formula and is similar to that of L. 4.1 (see also parts of the proof of Th. 4.3).

$\longrightarrow$ <u>Th. 4.6</u> (Continuity of the Itô-Lyons map)
Suppose that $f \in C^3_b(W, L(V, W))$ and consider the unique solution $(Y, Y') = (Y, f(Y)) \in \mathscr{D}^{2\alpha}_X$ to

$$Y_t = \xi + \int_0^t f(Y_s)\, dX_s$$

with $\alpha < \beta$

(58) and similarly, let $(\hat{Y}, \hat{Y}') = (\hat{Y}, f(\hat{Y}))$ be the RDE solution w.r.t. $\hat{\mathbf{X}}$ and initial value $\tilde{\xi} \in W$.
Assume that

$$|||\mathbb{X}|||_\beta, \ |||\hat{\mathbb{X}}|||_\beta \le M < \infty, \qquad (*)$$

Then

$$d_{X,\hat{X},2\alpha}((Y,Y'),(\hat{Y},\hat{Y}')) \le C_M (|\xi - \tilde{\xi}| + \varsigma_\beta(\mathbb{X},\hat{\mathbb{X}}))$$

$\|Y'-\hat{Y}'\|_\alpha + \|R^Y - R^{\hat{Y}}\|_{2\alpha}$

and $\|Y - \hat{Y}\|_\alpha \le C_M (|\xi - \tilde{\xi}| + \varsigma_\beta(\mathbb{X},\hat{\mathbb{X}}))$,

where $C_M = C(M, \alpha, \beta, f)$.

In particular, the Itô-Lyons

$$\Psi : \mathscr{C}^\beta \longrightarrow C^\alpha ; \quad (X, \mathbb{X}) \mapsto Y = Y(X, \mathbb{X})$$

is continuous.

<u>Proof</u>: We prove Th. 4.6 for some small $T \in [0,1)$:

Define $(Z, Z') = (\xi + \int_0^\cdot f(Y_s)\, d\mathbf{X}_s, f(Y_\cdot)) \in \mathscr{D}_X^{2\alpha}$

and similarly $(\hat{Z}, \hat{Z}')$

Since (Y, Y') is a fixed point of the map $\mathcal{M}_T : \mathcal{B}_T \to \mathcal{B}_T$ (see the proof of Th. 4.3), we have that

$$(Y, f(Y)) = (Y, Y') = (Z, Z') = (Z, f(Y))$$

and similarly for $\hat{Z}$.

Set $(\Pi, \Pi') = (f(Y), f(Y)')$ and similarly for $\hat{\Pi}$

Th. 3.10,
$\Longrightarrow$
(*) in connection with the estimates in the proof of Th. 4.3

$$d_{X,\hat{X},2\alpha}((Y, f(Y)), (\hat{Y}, f(\hat{Y}))) =$$

$$d_{X,\hat{X},2\alpha}((Z,Z'),(\hat{Z},\hat{Z}')) \lesssim \varsigma_\alpha(\mathbb{X},\hat{\mathbb{X}}) + |\xi - \tilde{\xi}| + d_{X,\hat{X},2\alpha}((\Pi,\Pi'),(\hat{\Pi},\hat{\Pi}'))$$

$$\underset{T \le 1}{\overset{\alpha<\beta,}{\le}} \varsigma_\beta(\mathbb{X},\hat{\mathbb{X}}) + |\xi - \tilde{\xi}| + d_{X,\hat{X},2\beta}((\Pi,\Pi'),(\hat{\Pi},\hat{\Pi}'))\, T^{\beta-\alpha} \quad (+)$$

(59) Further, it follows from Prop. 4.5 that

$$d_{X,\widehat{X},2\beta}((\Pi,\Pi'),(\widehat{\Pi},\widehat{\Pi}')) \lesssim \varrho_\beta(\mathbb{X},\widehat{\mathbb{X}}) + |\xi-\widehat{\xi}| + d_{X,\widehat{X},2\beta}((Y,f(Y)),(\widehat{Y},f(\widehat{Y}))) \quad (++)$$

The mean value th. gives

$$\|f(Y)-f(\widehat{Y})\|_\beta \leq 2\|Df\|_\infty \underbrace{\|Y-\widehat{Y}\|_\infty}_{\leq |\xi-\widehat{\xi}|+\|Y-\widehat{Y}\|_\beta}$$

On the other hand, we know that

$$(Y-\widehat{Y})_{s,t} = (Y'_s-\widehat{Y}'_s)X_{s,t} + \widehat{Y}'_s(X_{s,t}-\widehat{X}_{s,t}) + R^Y_{s,t} - R^{\widehat{Y}}_{s,t}$$

$$\overset{2\alpha>\beta,\ T\leq 1}{\Longrightarrow} \|Y-\widehat{Y}\|_\beta \leq \underbrace{\|Y'-\widehat{Y}'\|_\infty}_{\leq |\xi-\widehat{\xi}|+\|Y'-\widehat{Y}'\|_\alpha}\|X\|_\beta + \underbrace{\|\widehat{Y}'\|_\infty}_{\leq \|f\|_\infty+\|\widehat{Y}'\|_\alpha}\|X-\widehat{X}\|_\beta + \|R^Y-R^{\widehat{Y}}\|_{2\alpha}$$

$$\lesssim \varrho_\beta(\mathbb{X},\widehat{\mathbb{X}}) + |\xi-\widehat{\xi}| + d_{X,\widehat{X},2\alpha}((Y,Y'),(\widehat{Y},\widehat{Y}'))$$

$$\Longrightarrow \|f(Y)-f(\widehat{Y})\|_\beta \lesssim \varrho_\beta(\mathbb{X},\widehat{\mathbb{X}}) + |\xi-\widehat{\xi}| + d_{X,\widehat{X},2\alpha}((Y,\underbrace{Y'}_{=f(Y)}),(\widehat{Y},\underbrace{\widehat{Y}'}_{=f(\widehat{Y})}))$$

Using Taylor's formula, we similarly get such an estimate for $\|R^Y-R^{\widehat{Y}}\|_\beta$.

$$\Longrightarrow d_{X,\widehat{X},2\beta}((Y,f(Y)),(\widehat{Y},f(\widehat{Y}))) \lesssim \varrho_\beta(\mathbb{X},\widehat{\mathbb{X}}) + |\xi-\widehat{\xi}| + d_{X,\widehat{X},2\alpha}((Y,f(Y)),(\widehat{Y},f(\widehat{Y})))$$

$$\overset{(++)}{\Longrightarrow} (++) \lesssim \varrho_\beta(\mathbb{X},\widehat{\mathbb{X}}) + |\xi-\widehat{\xi}| + d_{X,\widehat{X},2\alpha}((Y,f(Y)),(\widehat{Y},f(\widehat{Y})))$$

$$\overset{(+)}{\Longrightarrow} d_{X,\widehat{X},2\alpha}((Y,f(Y)),(\widehat{Y},f(\widehat{Y}))) \leq C\big(\varrho_\beta(\mathbb{X},\widehat{\mathbb{X}}) + |\xi-\widehat{\xi}| + d_{X,\widehat{X},2\alpha}((Y,f(Y)),(\widehat{Y},f(\widehat{Y})))T^{\beta-\alpha}\big)$$

for $C = C(\alpha,\beta,f,M)$

(choose $T\leq 1$ s.t. $CT^{\beta-\alpha}\leq \frac{1}{2}$ (and as in the proof of Th. 4.3)

$$\Longrightarrow d_{X,\widehat{X},2\alpha}((Y,f(Y)),(\widehat{Y},f(\widehat{Y}))) \leq 2C\big(\varrho_\beta(\mathbb{X},\widehat{\mathbb{X}}) + |\xi-\widehat{\xi}|\big)$$

$\Longrightarrow$ 1. and 2. inequality $\Longrightarrow$ proof.

Rem.: The estimates in Th. 4.6 also hold, when β is replaced by α

5. Applications: Stochastic differential equations

Let $X_t = B_t$ a d-dimensional Brownian motion and $\mathbb{X}_{s,t} = \mathbb{B}_{s,t} = \left(\int_s^t B^i_{s,r}\, dB^j_r\right)_{1\le i,j\le d}$.

Section 3.4 $\Rightarrow$

$$\mathbf{X} = (B, \mathbb{B}) \in \mathscr{C}^\alpha([0,T], \mathbb{R}^d) \text{ with prob. } 1$$

for any $\alpha \in (\frac{1}{3}, \frac{1}{2})$.

$$\mathbb{B}^{Strat}_{s,t} \underset{\text{Rem. 2.8}}{\overset{\text{def.}}{=}} \mathbb{B}_{s,t} + \frac{1}{2}(t-s)\cdot \mathbb{I}_d$$

$\mathbb{I}_d$: unit matrix in $\mathbb{R}^{d\times d}$

$$\Rightarrow \quad \hat{\mathbf{X}} = (B, \mathbb{B}^{Strat}) \in \mathscr{C}^\alpha_g([0,T], \mathbb{R}^d) \subseteq \mathscr{C}^\alpha([0,T], \mathbb{R}^d),$$

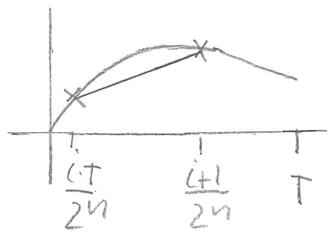
with prob. 1, whenever $\alpha \in (\frac{1}{3}, \frac{1}{2})$.

Consider now the dyadic piecewise-linear approximations B^n to B on $[0,T]$ given by

$$B^n_t = B_{\frac{iT}{2^n}} + 2^n\left(t - \frac{iT}{2^n}\right)\left(B_{\frac{(i+1)T}{2^n}} - B_{\frac{iT}{2^n}}\right),$$

linear interpolation

if $t \in [\frac{iT}{2^n}, \frac{(i+1)T}{2^n}]$ on $[0,T]$



$\longrightarrow$ **Prop. 5.1** (Approx. of $(B, \mathbb{B}^{Strat})$)

$$\left(B^n, \int_s^t B^n_{s,r} \otimes dB^n_r\right) \xrightarrow[n\to\infty]{} (B, \mathbb{B}^{Strat}) \text{ in } \mathscr{C}^\alpha_g$$

with prob. 1, whenever $\alpha \in (\frac{1}{3}, \frac{1}{2})$.

Rem.: $B^n_t = (B^{n,1}_t, \dots, B^{n,d}_t) \Rightarrow \int_s^t B^n_{s,r} \otimes dB^n_r = \left(\int_s^t B^{n,i}_{s,r}\, dB^{n,j}_s\right)_{1\le i,j\le d}$

$$= \left(\int_s^t B^{n,i}_{s,r}\, \dot{B}^{n,j}_r\, dr\right)_{1\le i,j\le d}$$

Proof: Consider $d=1$, $T=1$:

We see that

Gauss bracket (e.g. $[0.9]=0, \dots$)

$$B^n_t = B_{\frac{[2^n t]}{2^n}} + 2^n\left(t - \frac{[2^n t]}{2^n}\right)\left(B_{\frac{[2^n t]+1}{2^n}} - B_{\frac{[2^n t]}{2^n}}\right)$$

for $t<1$

(61) $\frac{[2^n t]}{2^n} \xrightarrow[n\to\infty]{} t \Longrightarrow \forall \omega, t : B^n_t(\omega) \xrightarrow[n\to\infty]{} B_t(\omega)$

B cont.

On the other hand:

$\forall \omega, t$: $\int_s^t B^n_{s,r}(\omega)\, dB^n_r(\omega) \overset{\text{int. by parts}}{=} \frac{1}{2}\left(B^n_t(\omega) - B^n_s(\omega)\right)^2$

$\xrightarrow[n\to\infty]{} \frac{1}{2}\left(B_t(\omega) - B_s(\omega)\right)^2$

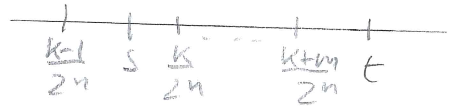
However, $\frac{1}{2}(B_t - B_s)^2 \overset{\text{Itô-formula (2.19)}}{=} \int_s^t B_{s,r}\, dB_r + \frac{1}{2}(t-s) = \mathbb{B}^{Strat}_{s,t}$

$\Longrightarrow \exists\, \Omega^*$ with $P(\Omega^*) = 1$ $\forall \omega \in \Omega^*, s,t$:

$\int_s^t B_{s,r}(\omega)\, dB^n_r(\omega) \xrightarrow[n\to\infty]{} \mathbb{B}^{Strat}_{s,t}(\omega)$

Consider e.g. the case

$\frac{k-1}{2^n} \le s < \frac{k}{2^n} < \dots < \frac{k+m}{2^n} < t \le \frac{k+m+1}{2^n}$, $m \geq 1$



$\Longrightarrow t - s \geq \frac{m}{2^n}$

$$\overset{\alpha<\frac{1}{2}}{\Longrightarrow} \frac{|B^n_s - B^n_t|}{|t-s|^\alpha} \le \frac{|B^n_s - B_{\frac{k}{2^n}}|}{|t-s|^\alpha} + \overbrace{\frac{|B_{\frac{k}{2^n}} - B_{\frac{k+m}{2^n}}|}{\underbrace{|t-s|^\alpha}_{\geq \left(\frac{m}{2^n}\right)^\alpha}}}^{\le \|B\|_\alpha} + \frac{|B_{\frac{k+m}{2^n}} - B^n_t|}{|t-s|^\alpha}$$

$$B^n_s - B_{\frac{k}{2^n}} = B_{\frac{k-1}{2^n}} - B_{\frac{k}{2^n}} + 2^n\left(s - \frac{k-1}{2^n}\right)\left(B_{\frac{k}{2^n}} - B_{\frac{k-1}{2^n}}\right)$$

$$\Longrightarrow \frac{|B^n_s - B_{\frac{k}{2^n}}|}{|t-s|^\alpha} \le \frac{|B_{\frac{k-1}{2^n}} - B_{\frac{k}{2^n}}|}{\underbrace{|t-s|^\alpha}_{\geq \left(\frac{1}{2^n}\right)^\alpha}} + \underbrace{2^n\left(s - \frac{k-1}{2^n}\right)}_{\le 1} \frac{|B_{\frac{k}{2^n}} - B_{\frac{k-1}{2^n}}|}{\underbrace{|t-s|^\alpha}_{\geq \left(\frac{1}{2^n}\right)^\alpha}}$$

$$\le 2\|B\|_\alpha \quad \Longrightarrow \quad \frac{|B^n_s - B^n_t|}{|t-s|^\alpha} \le 5\|B\|_\alpha$$

$\Longrightarrow$ $\|B^n\|_\alpha \le C \cdot \|B\|_\alpha$ for all n $\overset{\text{int. by parts}}{\Longrightarrow}$ $\|\int_s^t B^n_{s,r}\, dB^n_r\|_{2\alpha} \le C\|B\|^2_\alpha$

version of L.2.14 for pointwise convergence, $\tilde{\alpha} < \alpha$ $\Longrightarrow$ $\|B^n - B\|_{\tilde{\alpha}} \xrightarrow[n\to\infty]{} 0$, $\|\int_s^t B^n_{s,r}\, dB^n_r - \mathbb{B}^{Strat}\|_{2\tilde{\alpha}} \xrightarrow[n\to\infty]{} 0$

with prob. 1 $\Longrightarrow$ proof.

Rem. Prop. 5.1 can be also obtained by martingale convergence theorems or by a version of L.3.14 (i.e. L.6.1) and Borel-Cantelli.

In the sequel denote by $\int^t Z_s(\omega)\, dB^{Itô}_s(\omega)$ the rough path int. w.r.t. $(B, \mathbb{B})$ and by $\int^t Z_s(\omega)\, dB^{Strat}_s(\omega)$ the one w.r.t. $(B, \mathbb{B}^{Strat})$ with prob. 1

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Th. 5.2 (Pathwise unique solutions of SDE's)

Assume that $f \in C_b^3(\underset{\underset{W}{\|}}{\mathbb{R}^m}, L(\underset{\underset{V}{\|}}{\mathbb{R}^d}, \mathbb{R}^m))$, $f_0 \in C_b^3(\mathbb{R}^m, \mathbb{R}^m)$, $\xi \in \mathbb{R}^m$, $\alpha \in (\frac{1}{3}, \frac{1}{2})$. Then

(i) There ex. a Ω^* with $P(\Omega^*)=1$ s.t. for all $\omega \in \Omega^*$ there is a unique RDE solution $(Y(\omega), f(Y(\omega))) \in \mathcal{D}^{2\alpha}_{B(\omega)}$ to

$$Y_t(\omega) = \xi + \int_0^t f_0(Y_s(\omega))\,ds + \int_0^t f(Y_s(\omega))\,dB_s^{\text{Itô}}(\omega)$$

Moreover, $Y_t(\cdot)$, $t \geq 0$ is the (strong) solution to the corresponding Itô-SDE.

(ii) Similarly, one obtains from the RDE solutions of

$$Y_t(\omega) = \xi + \int_0^t f_0(Y_s(\omega))\,ds + \int_0^t f(Y_s(\omega))\,dB_s^{\text{Strat}}(\omega)$$

the (strong) solution of the Stratonovich SDE

$$Y_t = \xi + \int_0^t f_0(Y_s)\,ds + \int_0^t f(Y_s) \circ dB_s \, ,$$

where $\int_0^t Z_s \circ dB_s := \int_0^t Z_s\, dB_s + \frac{1}{2}[Z,B]_t$

is the Stratonovich integral and where $[Z,B]$ is the quadratic covariation of (semi-martingales) Z and B:

$$[Z,B]_t = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} Z_{u,v} B_{u,v} \, ,$$

if the limit exists in probability.

Rem: $\int_0^t f(Y_s) \circ dB_s = \int_0^t f(Y_s)\, dB_s + \int_0^t h(Y_s)\, ds$,

where $h_i(x) = \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^d f_{jk}(x) \frac{\partial}{\partial x_j} f_{ik}(x)$, $i=1,\dots,m$,

$f(x) = (f_{ij}(x))$.

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<u>Proof</u>: Denote by $\mathcal{C}_g^{0,\alpha}([0,T],\mathbb{R}^d) \subset \mathcal{C}_g^{\alpha}([0,T],\mathbb{R}^d)$ the closure of the set of all piecewise smooth paths in $\mathcal{C}_g^{\alpha}$ (w.r.t. ρ_α).

$\longrightarrow$ $\mathcal{C}_g^{0,\alpha}$ separable subspace of $\mathcal{C}_g^{\alpha}$ (*)

We know from Prop. 5.1 that

$$(B^n, \int_s^t B^n_{s,r} \otimes dB^n_r) \xrightarrow[n\to\infty]{} (B, \mathbb{B}^{Strat})$$

in $\mathcal{C}_g^{\alpha}([0,T],\mathbb{R}^d)$ with prob. 1

$\overset{(*)}{\Longrightarrow}$ $(\omega \longmapsto (B(\omega), \mathbb{B}^{Strat}(\omega)))\big|_{[0,t]} : \hat{\Omega} \longrightarrow \mathcal{C}_g^{0,\alpha}([0,t],\mathbb{R}^d)$ indep. of $t \in [0,T]$

on $[0,t]$ is measurable, where $\hat{\Omega} \subseteq \Omega$ meas. with $P(\hat{\Omega}) = 1$

$\xrightarrow{\text{Th. 4.6}}$ $Y_\cdot(\omega) = \Psi_t((B, \mathbb{B}^{Strat})|_{[0,t]})$ for a cont. map

$\Psi_t : \mathcal{C}^{\alpha}([0,t],\mathbb{R}^d) \longrightarrow \mathcal{C}^{\alpha}([0,t],\mathbb{R}^m)$

Since $(B, \mathbb{B}^{Strat})$ is $\hat{\Omega} \cap \mathcal{F}_t$-measurable on $[0,t]$, the r.v. given by $\hat{Y}_t(\omega) = \begin{cases} Y_t(\omega), & \omega \in \hat{\Omega} \\ 0, & \omega \in \Omega \setminus \hat{\Omega} \end{cases}$

is $\mathcal{F}_t$-adapted.

On the other hand Prop. 3.15 yields a $\Omega^* \subseteq \hat{\Omega}$ with $P(\Omega^*) = 1$ s.t. for all $\omega \in \Omega^*$, t:

Itô-integral

$$\left(\int_0^t \hat{Y}_s \, dB_s\right)(\omega) = \left(\int_0^t Y_s \, dB_s\right)(\omega) = \int_0^t Y_s(\omega) \, d\mathbf{B}_s^{Itô}(\omega)$$

$\overset{\text{w.l.o.g.}}{\underset{\Omega^* = \hat{\Omega}}{\Longrightarrow}}$ $\hat{Y}$ (or Y) is the unique (strong) solution to the Itô-SDE $\Longrightarrow$ (i) (and similarly (ii))

$\Longrightarrow$ proof.

$\xrightarrow[\text{Th. 4.6}]{\text{Prop. 5.1, Th. 5.2}}$, <u>approximation scheme</u> for solutions of Stratonovich-SDE's by means of <u>random ODE's</u>:

Th. 5.3 (Wong-Zakai, Clark, Stroock-Varadhan)
Retain the conditions of Th. 5.2. Let B^n, $n \geq 1$ be the dyadic piecewise-linear approximation to B on $[0,T]$ (see Prop. 5.1). Further, consider the solutions Y^n to the random ODE's

$$Y_t^n = \xi + \int_0^t f_0(Y_s^n)\,ds + \int_0^t f(Y_s^n)\,dB_s^n, \quad 0 \leq t \leq T$$

and the solution Y to the Stratonovich SDE

$$Y_t = \xi + \int_0^t f_0(Y_s)\,ds + \int_0^t f(Y_s)\circ dB_s, \quad 0 \leq t \leq T.$$

Then

$$\| Y^n - Y \|_{\alpha,[0,T]} \xrightarrow[n\to\infty]{} 0$$

with prob. 1

Rem.: conv. rate: $(2^n)^{\frac{1}{2}-(\alpha-\varepsilon)}$ for all $\varepsilon \in (0,\alpha)$

Let μ be a measure on $\mathcal{B}(X)$ (smallest σ-alg. containing all open sets of X)

$\longrightarrow$ support of μ: $\operatorname{supp}\mu = \{x \in X : \mathcal{U}$ open neighbourhood of $x \Longrightarrow \mu(\mathcal{U}) > 0\}$

Define $\mathcal{H} = W_0^{1,2} = \{ f : [0,T] \to \mathbb{R}^d : f(0) = 0,$ f absolutely cont. with $\int_0^T |\dot f(s)|^2 ds < \infty \}$ and let $S^{(2)}(h) = (h, \int_s^t h_{s,r} \otimes dh_r) \in \mathcal{C}_g^{0,\alpha}$, $h \in \mathcal{H}$

We need the following auxiliary result:

Lemma 5.4: For all $\delta > 0$

$$\lim_{\varepsilon \to 0} P\left(\rho_{\alpha,[0,T]}\left((B, \mathbb{B}^{\text{Strat}}), S^{(2)}(h)\right) < \delta \;\middle|\; \underbrace{\|B - h\|_{\infty,[0,T]} < \varepsilon}_{} \right)$$

$= 1$

has strictly positive probability since the Wiener measure is a strictly positive measure

Proof: See Friz, Lyons, Stroock (2006).

Th. 5.5 (Support theorem of Stroock-Varadhan)
Retain the conditions of Th. 5.2 and consider the unique solution Y to the SDE

$$Y_t = \xi + \int_0^t f_0(Y_s)\,ds + \int_0^t f(Y_s)\circ dB_s \;,\; 0\le t\le T.$$

Further, for $h\in\mathcal{H}$ let y^h be the unique solution to the ODE:

$$y^h_t = \xi + \int_0^t f_0(y^h_s)\,ds + \int_0^t f(y^h_s)\,\dot h_s\,ds \;,\; 0\le t\le T$$

Then

(i) for all $\delta>0$

$$\lim_{\varepsilon\to 0} P(\|Y-y^h\|_{\alpha,[0,T]}<\delta \mid \|B-h\|_{\infty,[0,T]}<\varepsilon) = 1$$

(ii) the support of the measure μ on $C^\alpha([0,T],\mathbb{R}^m)$ given by $\mu(A) = P(Y\in A)$ is the closure of $\{y^h : h\in\mathcal{H}\}$ w.r.t. $\|\cdot\|_\alpha$.

Proof: (i) Because of Th. 5.2 we can choose Y to be the corresponding RDE solution

$$\xrightarrow[\text{modified Th. 4.6 for } \alpha=\beta]{\text{slightly}} \quad \|Y-y^h\|_\alpha \le C_M\, \rho_\alpha((B,\mathbb{B}^{Strat}), S^{(2)}(h)),$$

where $C_M = C(M,f,\alpha,T)$ can be chosen to be

$L := C(\rho_\alpha((B,\mathbb{B}^{Strat}), S^{(2)}(h)) + |||h|||_\alpha)$ for an increasing function C

$$\Rightarrow \; P(\|Y-y^h\|_\alpha \ge\delta \mid \|B-h\|_\infty\le\varepsilon) \le P(L\cdot\rho_\alpha((B,\mathbb{B}^{Strat}), S^{(2)}(h))\ge\delta \mid \|B-h\|_\infty<\varepsilon)$$

$$= P(\{L\cdot\rho_\alpha((B,\mathbb{B}^{Strat}), S^{(2)}(h))\ge\delta\}\cap\{\rho_\alpha((B,\mathbb{B}^{Strat}), S^{(2)}(h))<1\} \mid \|B-h\|_\infty<\varepsilon)$$

$$+ P(\{L\cdot\rho_\alpha((B,\mathbb{B}^{Strat}), S^{(2)}(h))\ge\delta\}\cap\{\rho_\alpha((B,\mathbb{B}^{Strat}), S^{(2)}(h))\ge 1\} \mid \|B-h\|_\infty<\varepsilon)$$

$$\le P(C(1+|||h|||_\alpha)\,\rho_\alpha((B,\mathbb{B}^{Strat}), S^{(2)}(h))\ge\delta \mid \|B-h\|_\infty<\varepsilon) +$$

$$P(\rho_\alpha((B,\mathbb{B}^{Strat}), S^{(2)}(h))\ge 1 \mid \|B-h\|_\infty<\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{L.5.4} 0 \quad\Rightarrow (i)$$

(ii) Suppose there is a $y^h\notin \operatorname{supp}\mu \Rightarrow$

$\exists\,\delta>0$: $\underbrace{\mu(B_\delta(y^h))}_{=P(\|Y-y^h\|_\alpha<\delta)} = 0$ ball with radius δ at y^h

However,

$$P(\|Y-y^h\|_\alpha<\delta) \geq P(\|Y-y^h\|_\alpha<\delta \text{ and } \|B-h\|_\infty<\varepsilon)$$

$$\overset{(i)}{\geq} \frac{1}{2} \underbrace{P(\|B-h\|_\infty<\varepsilon)}_{>0 \text{ for all } \varepsilon<\varepsilon_0(\delta)} \quad \text{for all } \varepsilon<\varepsilon_0(\delta)$$

$\Rightarrow$ ↯ $\Rightarrow$ $y^h \in \operatorname{supp}\mu$ $\Rightarrow$ closure of $\{y^h : h\in\mathcal{H}\}$ contained in $\operatorname{supp}\mu$ (since $\operatorname{supp}\mu$ is closed)

Suppose there ex. a $y\in\operatorname{supp}\mu$ s.t. y is not in the closure of $\{y^h : h\in\mathcal{H}\}$

$\Rightarrow$ $\exists\, \delta>0$ s.t. $B_\delta(y)$ is disjoint with that closure

On the other hand, we know from Th. 5.3 that

$$\|\underset{\in\{y^h : h\in\mathcal{H}\}}{Y^n}-Y\|_\alpha \xrightarrow[n\to\infty]{} 0 \quad \text{with prob. 1}$$

$\Rightarrow$ Y^n converges to Y in distribution

$$\Leftrightarrow \quad \varliminf_{n\to\infty} P(Y^n\in U) \geq P(Y\in U)=\mu(U)$$

for all open U

$$\overset{U=B_\delta(y)}{\Longrightarrow} \quad \varliminf_{n} \underbrace{P(Y^n\in B_\delta(y))}_{=0} \geq P(Y\in B_\delta(y))=\mu(B_\delta(y))$$

$=0$, since $Y^n\in\{y^h : h\in\mathcal{H}\}$ (since $B^n\in\mathcal{H}$ with prob. 1)

$\Rightarrow$ $\mu(B_\delta(y))=0$ $\Rightarrow$ $y\notin\operatorname{supp}\mu$ $\Rightarrow$ ↯

$\Rightarrow$ (ii) $\Rightarrow$ proof.

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6. Applications: Stochastic partial differential equations

We want to study rough (random) partial differential equations of the (formal) type

$$-du = L(u)\,dt + \sum_{i=1}^{d} \Pi_i(u)\, dW_t^i \,, \quad u(T,\cdot) = g \,, \qquad (6.1)$$

see (*) in the proof of Th. 5.2

for $W = (W^1, \dots, W^d) \in \mathcal{C}_g^{0,\alpha}([0,T], \mathbb{R}^d) \subseteq \mathcal{C}^{\alpha}([0,T], \mathbb{R}^d)$,

where $u : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and

$$L(u) \overset{\text{def}}{=} \frac{1}{2} \sum_{i,j=1}^{n} (\sigma(x)\underbrace{\sigma^T}_{\text{transposed}}(x))_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u \,,$$

$$\Pi_i(u) \overset{\text{def}}{=} \sum_{j=1}^{n} \beta_{ij}(x) \frac{\partial u}{\partial x_j} + \gamma_i(x) u$$

for functions $\sigma : \mathbb{R}^n \longrightarrow L(\mathbb{R}^m, \mathbb{R}^n)$, $\beta : \mathbb{R}^n \longrightarrow L(\mathbb{R}^d, \mathbb{R}^n)$, $c : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\gamma_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, \dots, d$, $g : \mathbb{R}^n \longrightarrow \mathbb{R}$

Suppose that $\sigma, \beta, c, g, \gamma_i \in C_b^3$, $i = 1, \dots, d$

Let $W \in C^1([0,T], \mathbb{R}^d)$. Then

$$-\frac{\partial u}{\partial t} = L(u) + \sum_{i=1}^{d} \Pi_i(u) \dot{W}_t^i \,, \quad u(T,\cdot) = g \qquad (6.2)$$

If $u \in C^{1,2}([0,T] \times \mathbb{R}^n)$ and bounded it follows from the Feynman-Kac formula that the solution to the Cauchy problem (6.2) has the representation

$$u(s,x) = E\Big[g(X_T^{s,x}) \exp\Big(\int_s^T c(X_t^{s,x})\, dt + \underbrace{\int_s^T \gamma(X_t^{s,x}) \dot{W}_t\, dt}_{= \int_s^T \gamma(X_t^{s,x})\, dW_t} \Big) \Big] \qquad (6.3)$$

$\gamma = (\gamma_1, \dots, \gamma_d)$

$$=: S(W; g) \,,$$

where $X^{s,x}$ is the unique (strong) solution to the Itô-SDE

$$X_t^{s,x} = x + \int_s^t \sigma(X_u^{s,x})\, dB_u + \int_s^t b(X_u^{s,x})\, du + \int_s^t \beta(X_u^{s,x}) \dot{W}_u\, du \,,$$

B: Brownian motion in $\mathbb{R}^m$

$s \le t \le T$

<u>Rem. 6.1</u>: If in addition to the above cond. there ex. $\delta>0$ s.t. <u>uniform ellipticity</u> holds, that is

$$\sum_{i,j=1}^{n} (\sigma(x)\sigma^T(x))_{ij}\xi_i\xi_j \geq \delta\|\xi\|^2$$

for all $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$, $x\in\mathbb{R}^n$, then there is a bounded $C^{1,2}$-solution u to (6.2).

We need the following useful result:

<u>Lemma 6.1</u> (Kolmogorov criterion for rough path distances)

Let $q\geq 2$, $\beta>\frac{1}{q}$ and let $X_t(\omega), \hat{X}_t(\omega)\in V$, $\mathbb{X}_{s,t}(\omega), \hat{\mathbb{X}}_{s,t}(\omega)\in V\otimes V$, $s,t\in[0,T]$ be processes s.t.

$$E[|X_{s,t}|^q]^{1/q}\leq C|t-s|^\beta, \quad E[|\mathbb{X}_{s,t}|^{q/2}]^{2/q}\leq C|t-s|^{2\beta},$$

for all s,t and the same estimates for $(\hat{X},\hat{\mathbb{X}})$ w.r.t. to the same constant C. Set

$$\Delta X=\hat{X}-X \quad \text{and} \quad \Delta\mathbb{X}=\hat{\mathbb{X}}-\mathbb{X}$$

and assume that

$$E[|\Delta X_{s,t}|^q]^{1/q}\leq C\cdot\varepsilon|t-s|^\beta, \quad E[|\Delta\mathbb{X}_{s,t}|^{q/2}]^{2/q}\leq C\cdot\varepsilon|t-s|^{2\beta}$$

for some $\varepsilon>0$ and all s,t. Then there ex. $M=M(C)$ (increasing in C) s.t. for $\alpha\in(0,\beta-\frac{1}{q})$:

$$E[\|\Delta X\|_\alpha^q]^{1/q}\leq M\cdot\varepsilon, \quad E[\|\Delta\mathbb{X}\|_{2\alpha}^{q/2}]^{2/q}\leq M\cdot\varepsilon.$$

In particular, if $\alpha\in(\frac{1}{3},\beta-\frac{1}{q})$, we have that $|||\hat{\mathbb{X}}|||_\alpha, |||\mathbb{X}|||_\alpha\in L^q$ and

$$E[\rho_\alpha(\hat{\mathbb{X}},\mathbb{X})^q]^{1/q}\leq M\cdot\varepsilon$$

<u>Proof</u>: Similar to L. 3.14.

In the sequel, denote by $BC(\mathbb{R}^m)$ the space of bounded, cont. functions $f:\mathbb{R}^n\to\mathbb{R}$ w.r.t. the semi-norms $\|f\|_n:=\sup_{x\in B_n(0)}|f(x)|$, $n\geq 1$, $B_n(0):=\{x:\|x\|\leq n\}$ (i.e. the topology of locally unif. convergence)

Th. 6.2 (Continuity of $S(\mathbf{W};g)$)

Let $\alpha \in (\frac{1}{3}, \frac{1}{2})$, $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}_g^{0,\alpha}([0,T], \mathbb{R}^d)$.
Choose $W^\varepsilon \in C^1([0,T], \mathbb{R}^d)$ s.t.

$$(W^\varepsilon, \mathbb{W}^\varepsilon) = \Big(W^\varepsilon, \int_s^t W^\varepsilon_{s,r} \otimes dW^\varepsilon_r\Big) \xrightarrow[\varepsilon \searrow 0]{} \mathbf{W}$$

w.r.t. ϱ_α. Then there is $u \in BC([0,T] \times \mathbb{R}^n)$ (depending on $\mathbf{W}$, but not on $W^\varepsilon, \varepsilon > 0$) s.t.

$$u^\varepsilon = S(W^\varepsilon; g) \xrightarrow[\varepsilon \searrow 0]{} u =: S(\mathbf{W}; g)$$

in $BC([0,T] \times \mathbb{R}^n)$. Further,

$$S : \mathcal{C}_g^{0,\alpha}([0,T], \mathbb{R}^d) \longrightarrow BC([0,T] \times \mathbb{R}^n);\ \mathbf{W} \longmapsto S(\mathbf{W}; g)$$

is continuous.

Proof: w.l.o.g. $b = 0$.

1. We first want to give a meaning to the (formal) SDE

$$X_t^{s,x} = x + \int_s^t \sigma(X_u^{s,x})\, dB_u + \int_s^t \beta(X_u^{s,x})\, dW_u \qquad (6.4)$$

(hybrid Itô-rough diff. equation)

for $(W, \mathbb{W}) \in \mathcal{C}_g^{0,\alpha}$ by means of rough path integration.

For this purpose we consider the (random) RDE

$$X_t^{s,x} = x + \int_s^t f(X_u^{s,x})\, dZ_u, \text{ where} \qquad (6.5)$$

$f = (\sigma, \beta)$, $Z = (B, W)^T \in \mathbb{R}^{m+d}$

$f \in C_b^3(\mathbb{R}^n, L(\mathbb{R}^{d+m}, \mathbb{R}^n))$

$$\mathbb{Z}_{s,t} = \begin{pmatrix} \mathbb{B}^{Itô}_{s,t}(\omega) & \int_s^t W_{s,r} \otimes dB_r(\omega) \\ \int_s^t B_{s,r}(\omega) \otimes dW_r & \mathbb{W}_{s,t} \end{pmatrix} \qquad (6.6)$$

If $W \in C^1$, $\mathbb{Z}$ in (6.6) is well-def.

w.l.o.g. let $m = d = 1$ for a moment

int. by parts $\Rightarrow$

$$\int_s^t B_{s,r}\, dW_r = W_{s,t} \cdot B_{s,t} - \underbrace{\int_s^t W_{s,r}\, dB_r}_{\text{Itô-integral}} \qquad (+)$$

Further,

$$\leq \|W\|_\alpha^2 |s-r|^{2\alpha}$$

$$E[|\int_s^t W_{s,r} \, dB_r|^2] \underset{\text{isometry}}{\overset{\text{Itô-}}{=}} E[\int_s^t |W_{s,r}|^2 \, dr]$$

$$\lesssim \|W\|_\alpha^2 |s-t|^{2\alpha+1}$$

$$\overset{(+)}{\Longrightarrow} \quad E[|\int_s^t B_{s,r} \, dW_r|^2] \lesssim |s-t|^{2\alpha+1}$$

$\overset{\text{L.3.14}}{\Longrightarrow}$ $Z^W := (Z, \mathbb{Z}) \in \mathcal{C}^{\alpha'}$ with prob. 1 for any $\alpha' \in (\frac{1}{3}, \alpha)$

Using the same reasoning in connection with L.6.1 for $\varepsilon = \rho_\alpha(W, \widetilde{W})$ we also obtain for all q

$$E[\rho_{\alpha'}(Z^W, Z^{\widetilde{W}})^q]^{1/q} \lesssim \rho_\alpha(W, \widetilde{W}) \qquad (++)$$

for all $W, \widetilde{W} \in C^1$

$\overset{\text{C.S.-sequence}}{\Longrightarrow}$ for $W \in \mathcal{C}_g^{0,\alpha}$ there exists a proc. Z^W with $Z^W(\omega) \in \mathcal{C}_g^{0,\alpha'}$ with probab. 1 s.t.

$$E[\rho_{\alpha'}(Z^{W_\varepsilon}, Z^W)^q]^{1/q} \xrightarrow[\varepsilon \downarrow 0]{} 0$$

$\overset{\text{Th.4.3}}{\longrightarrow}$ there ex. for all $\omega \in \Omega^*$ with $P(\Omega^*) = 1$ a unique solution $X^{s,x}(\omega)$ to (6.5) w.r.t. $Z := Z^W$

On the other hand, if $W \in C^1$ one gets (by using e.g. the Young inequality (right after (4))) that

$$X_t^{s,x} = x + \int_s^t f(X_u^{s,x}(\omega)) \, dZ_u(\omega) =$$

$$x + \underbrace{\int_s^t \sigma(X_u^{s,x}(\omega)) \, dB_u(\omega)}_{\text{rough int. w.r.t } Y = \sigma(X^{s,x}(\omega)),\ Y' = D\sigma(X^{s,x})\sigma(X^{s,x})} + \int_s^t \beta(X_u^{s,x}(\omega)) \dot{W}_u \, du$$

$$\overset{\text{Th.5.2}}{=} \quad x + \underbrace{\int_s^t \sigma(X_u^{s,x}) \, dB_u}_{\text{Itô-integral}} + \int_s^t \beta(X_u^{s,x}) \dot{W}_u \, du \quad \leftarrow \text{Itô-SDE}$$

Further, it also follows from the continuity of the the Itô-Lyons map and (++) that $X^{s,x,\varepsilon}$ for Z^{W_ε} converges in probability in $C^{\tilde{\alpha}}$ for $\tilde{\alpha} < \alpha'$

(71) $\longrightarrow$ it makes sense to define (adapted) solutions to (6.4) for $W \in \mathcal{C}_g^{0,\alpha}$ by solutions to (6.5) for $Z = Z^W$

2. Let $X^{s,x}$ be the solution to (6.4)

$\overset{\text{Th. 4.3}}{\Longrightarrow}$ $(X^{s,x}(\omega), (X^{s,x})'(\omega)) \in \mathcal{D}^{2\alpha'}_{Z(\omega)}$ with prob. 1,

where $(X^{s,x})'(\omega) = f(X^{s,x}(\omega))$, $f = (\sigma, \beta)$

$\overset{\text{L. 4.1}}{\Longrightarrow}$ $\int_s^T \gamma(X_u^{s,x})\,dW_u \overset{\text{def}}{=} \int_s^T (0, \gamma(X_u^{s,x}))\,dZ_u(\omega)$

well-def. with prob. 1

$\Longrightarrow$ $g(X_T^{s,x}) \exp\left(\int_s^T c(X_u^{s,x})\,du + \int_s^T \gamma(X_u^{s,x})\,dW_u\right)$

well-def. as random variable

Consider now the case $c = \gamma = 0$ (for simplicity)

Set $u(s,x) = E[g(X_T^{s,x})]$, $u^{\varepsilon_m}(s,x) = E[g(X_T^{s,x,\varepsilon_m})]$

for $\varepsilon_m \searrow 0$, $m \to \infty$

Then $\|u^{\varepsilon_m} - u\|_n \overset{\text{def}}{=} \sup_{\substack{x \in B_n(0),\\ s \in [0,T]}} |E[g(X_T^{s,x,\varepsilon_m}) - g(X_T^{s,x})]|$

$\leq E[\underbrace{\sup_{\substack{x \in B_n(0),\\ s \in [0,T]}} |g(X_T^{s,x,\varepsilon_m}) - g(X_T^{s,x})|}_{=: \rho_m \leq 2\|g\|_\infty}]$

On the other hand, $\swarrow$ measurable

$\rho_m \overset{\text{mean value th.}}{\leq} \|Dg\|_\infty \sup_{\substack{x \in B_n(0),\\ s \in [0,T]}} \|X^{s,x,\varepsilon_m} - X^{s,x}\|_{\tilde\alpha}$

$\overset{\substack{\text{Th. 4.6},\\ \tilde\alpha < \alpha'}}{\leq}$ $\|Dg\|_\infty C_{M(m,\varepsilon)} \rho_{\alpha'}(Z^{W_{\varepsilon_m}}, Z^W)$ (C_M cont. increasing in M)

But $\rho_{\alpha'}(Z^{\varepsilon_{m_k}}, Z^W) \xrightarrow[k \to \infty]{} 0$ with prob. 1 for a subsequence (m_k) (due to (++))

$\overset{\substack{\text{dominated}\\ \text{conv.}}}{\Longrightarrow}$ $\|u^{\varepsilon_{m_k}} - u\|_n \xrightarrow[k \to \infty]{} 0$

By contradiction we can also argue that

$\forall n$: $\|u^{\varepsilon_m} - u\|_n \xrightarrow[m \to \infty]{} 0$ $\Longrightarrow$ proof.

<u>Rem</u>: The case $\gamma \neq 0$ can be covered by using a generalized Fernique theorem (see Friz, Hairer)