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Let $W \in C^1([0,T],\mathbb{R}^d)$ and let ~~be~~ u be a bounded $C^{1,2}$ solution to

$$u_s = g + \int_s^T L u_t\,dt + \underbrace{\int_s^T \Gamma u_t\,dW_t}_{=\int_s^T \Gamma u_t \dot W_t\,dt}, \tag{6.7}$$

where $\Gamma := (\Gamma_1, \ldots, \Gamma_d)$

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ (space of ∞-diff. functions on $\mathbb{R}^d$ with compact support) and set

$\langle \varphi, \zeta \rangle = \int_{\mathbb{R}^d} \varphi(x)\zeta(x)\,dx$ for $\varphi, \zeta \in C_0^\infty$

Further, denote by L^* and Γ^* the dual operators of L and Γ, resp. w.r.t. domains in $H = L^2(\mathbb{R}^d)$

$\longrightarrow$ $\langle L u_t(\cdot), \varphi \rangle = \langle u_t(\cdot), L^*\varphi \rangle$ and

$\langle \Gamma u_t(\cdot), \varphi \rangle = \langle u_t(\cdot), \Gamma^*\varphi \rangle$ for $\varphi \in C_0^\infty(\mathbb{R}^d)$

(e.g. if $L = \frac{1}{2}\Delta = \frac{1}{2}\frac{\partial^2}{\partial x^2} \longrightarrow L^* = \frac{1}{2}\Delta$)

$\xrightarrow[\text{applied to (6.7)}]{\text{int. by parts}}$ $\langle u_s, \varphi \rangle = \langle g, \varphi \rangle + \int_s^T \langle L u_t, \varphi \rangle\,dt + \int_s^T \langle \Gamma u_t, \varphi \rangle\,dW_t$

$$= \langle g, \varphi \rangle + \int_s^T \langle u_t, L^*\varphi \rangle\,dt + \int_s^T \langle u_t, \Gamma^*\varphi \rangle\,dW_t \tag{6.8}$$

for $\varphi \in C_0^\infty(\mathbb{R}^d)$

$\longrightarrow$ it makes sense to say u is a (analytically weak) solution to the rough PDE (RPDE) w.r.t. $\mathbf{W} = (W, \mathbb{W}) \in \mathscr{C}_g^{0,\alpha}([0,T],\mathbb{R}^d)$ in the following sense:

<u>Def. 6.3</u> (Analytically weak backward RPDE solution)

A bounded and measurable function $u = u_t(x; \mathbf{W})$ is <u>an analytically weak solution</u> to (6.7), if for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, we have $(Y^\varphi, (Y^\varphi)') \in \mathscr{D}_W^{2\alpha}$ with

$$Y_t^\varphi := \langle u_t, \Gamma^*\varphi \rangle := (\langle u_t, \Gamma_1^*\varphi \rangle, \ldots \langle u_t, \Gamma_d^*\varphi \rangle),$$

$$(Y_t^\varphi)' := -\langle u_t, \Gamma^*\Gamma^*\varphi \rangle = (\langle u_t, \Gamma_i^*\Gamma_j^*\varphi \rangle)_{1 \le i,j \le d}$$

<u>and</u> u satisfies (6.8) for all $0 \le t \le T$, where the rough integral in (6.8) is defined w.r.t. $(Y^\varphi, (Y^\varphi)')$

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Denote by $\Pi_{(f)}(0, y, \mathbf{X})$ the solution Y to (solution starts in zero)

$$Y_t = y + \int_0^t f(Y_s)\, d\mathbf{X}_s$$

for $\mathbf{X} \in \mathscr{C}_g^\alpha([0,T], \mathbb{R}^d)$, $f \in C_b^3$

exercise $\rightarrow$ the inverse $\Pi_{(f)}(0, \cdot, \mathbf{X})_t^{-1}$ of $(y \mapsto \Pi_{(f)}(0, y, \mathbf{X})_t)$ (flow)

exists and is given by

$$\Pi_{(f)}(0, y, \mathbf{X})_t^{-1} = \Pi_{(f)}(0, y, \mathbf{X}_{t-\cdot})_t \qquad (6.9)$$

$\longrightarrow$ consequence of the continuity of the Itô-Lyons map!

<u>Th. 6.4</u> (Stability of flows)

Let $\mathbf{X}, \hat{\mathbf{X}} \in \mathscr{C}_g^\alpha$, $f \in C_b^4$. Then the flows and their inverse flows are in C^2 (w.r.t. y). Further

$$\underbrace{D Y_t}_{\Pi_{(f)}(0,y,\mathbf{X})_t} = \underbrace{\mathbb{I}_d}_{\text{unit matrix}} + \int_0^t Df(Y_s)\, DY_s\, d\mathbf{X}_s,$$

$$\underbrace{D Y_{T-t}^{-1}}_{\Pi_{(f)}(0,y,\mathbf{X})_{T-t}^{-1}} = \mathbb{I}_d + \int_t^T Df(Y_{T-s}^{-1})\, DY_{T-s}^{-1}\, d\mathbf{X}_{T-s}$$

and similarly for $\hat{\mathbf{X}}$. Moreover, if $|||\mathbf{X}|||_\alpha, |||\hat{\mathbf{X}}|||_\alpha \leq M < \infty$, there ex. $C = C(M, f)$, $K = K(M, f)$ s.t.

$$\sup_{\xi \in \mathbb{R}^d} \| D\Pi_{(f)}(0, \xi, \mathbf{X}) - D\Pi_{(f)}(0, \xi, \hat{\mathbf{X}}) \|_{\alpha, [0,T]} \leq C \rho_\alpha(\mathbf{X}, \hat{\mathbf{X}}),$$

$$\sup_{\xi \in \mathbb{R}^d} \| D\Pi_{(f)}(0, \xi, \mathbf{X}) \|_{\alpha, [0,T]} \leq K$$

and similarly for the inverse flows and $\hat{\mathbf{X}}$.

$\longrightarrow$ <u>application</u>: Existence of solutions to (6.8):

For simplicity, c, γ, b are zero, $d = m = 1$.

$\longrightarrow$

$$X_t^{s,x} = x + \int_s^t f(X_u^{s,x})\, dZ_u, \quad Z_t = (B_t, W_t)^T \in \mathbb{R}^2, \; f = (\sigma, \beta)$$

Let $\Phi_{s,T}$ be the flow of $X_\cdot^{s,x}$ and set

$$Y_t = \langle u_t, \bar{g} \rangle, \quad Y_t' = -\langle u_t, \Gamma^* g \rangle, \quad \bar{g} := \Gamma^* g,$$

where $\Gamma^* g \overset{\text{def}}{=} -\operatorname{div}(\beta g) = -\frac{\partial}{\partial x}(\beta g)$

Then

$$Y_{T-t} - Y_{T-s} - Y_{T-t}' W_{T-s, T-t} =$$

$$E\Big[\int_{\mathbb{R}} \{g(\Phi_{T-t,T}(x)) - g(\Phi_{T-s,T}(x))\}\bar\varsigma(x) + g(\Phi_{T-t,T}(x))\,\Gamma^*\bar\varsigma(x)\, W_{T-s,T-t}\, dx\Big]$$

$$\overset{\text{substitution, Th. 6.4}}{=} E\Big[\int_{\mathbb{R}} g(y)\{\bar\varsigma(\Phi^{-1}_{T-t,T}(y))\,D\Phi^{-1}_{T-t,T}(y) - \bar\varsigma(\Phi^{-1}_{T-s,T}(y))\,D\Phi^{-1}_{T-s,T}(y) + \Gamma^*\bar\varsigma(\Phi^{-1}_{T-t,T}(y))\,D\Phi^{-1}_{T-t,T}(y)\,W_{T-s,T-t}\}\,dy\Big]$$

$$= E\Big[\int_{\mathbb{R}} g(y)\underbrace{\{\bar\varsigma(\Phi^{-1}_{T-t,T}(y))\,D\Phi^{-1}_{T-t,T}(y) - \underbrace{\bar\varsigma(\Phi^{-1}_{T-s,T}(y))}_{=:A_s}\underbrace{D\Phi^{-1}_{T-s,T}(y)}_{=:B_s} + \Gamma^*\bar\varsigma(\Phi^{-1}_{T-t,T}(y))\,D\Phi^{-1}_{T-t,T}(y)\,W_{T-s,T-t} - \frac{\partial}{\partial x}(f\bar\varsigma)(\Phi^{-1}_{T-t,T}(y))\,D\Phi^{-1}_{T-t,T}(y)\,\underbrace{B_{T-s,T-t}}_{\text{indep}}\}}_{=:F}\,dy\Big] \qquad (*)$$

$\overleftarrow{X}_s := Z_{T-s} \xrightarrow[\text{L. 4.1}]{\text{Th. 6.4}} (A,A'),\ (B,B') \in \mathcal{D}^{2\alpha}_{\overleftarrow{X}}$, where

$$A'_s = D\bar\varsigma(\Phi^{-1}_{T-s,T}(y))\,\underbrace{f(\Phi^{-1}_{T-s,T}(y))}_{=(\Phi^{-1}_{T-s,T}(y))'}$$

and
$$B'_s = Df(\Phi^{-1}_{T-s,T}(y))\,D\Phi^{-1}_{T-s,T}(y)$$

$$\longrightarrow\quad F =: A_tB_t - A_sB_s - (A'_tB_t + A_tB'_t)\overleftarrow{X}_{s,t}$$

$$= (A_t - A_s + A'_t\overleftarrow{X}_{t,s})B_t + A_s(B_t - B_s + B'_t\overleftarrow{X}_{t,s}) + (A_s - A_t)B'_t\overleftarrow{X}_{s,t} \qquad (+)$$

Prop. 4.2 (a priori estimate) yields

$$\|(\Phi^{-1}_{T-\cdot,T}(y), (\Phi^{-1}_{T-\cdot,T}(y))')\|_{\overleftarrow{X},2\alpha} \le C_\alpha[(\|f\|_{C^2_b}|||\overleftarrow{\mathbf{X}}|||_\alpha) \vee (\|f\|_{C^2_b}|||\overleftarrow{\mathbf{X}}|||_\alpha)^{1/\alpha}]$$

$$\overset{\text{L. 4.1}}{\Longrightarrow} \|A,A'\|_{\overleftarrow{X},2\alpha} \le C_{\alpha,T}[\|f\|_{C^2_b}|||\overleftarrow{\mathbf{X}}|||_\alpha \vee (\|f\|_{C^2_b}|||\overleftarrow{\mathbf{X}}|||_\alpha)^{1/\alpha}]\cdot$$
$$\cdot\,\|\bar\varsigma\|_{C^2_b}(1+|||\overleftarrow{\mathbf{X}}|||_\alpha)^2(\|f\|_\infty + C_\alpha[(\|f\|_{C^2_b}|||\overleftarrow{\mathbf{X}}|||_\alpha) \vee (\|f\|_{C^2_b}|||\overleftarrow{\mathbf{X}}|||_\alpha)^{1/\alpha}]$$

$$\le C_{\alpha,T,f,\bar\varsigma}(1+|||\overleftarrow{\mathbf{X}}|||_\alpha)^8$$

Similarly, we can find an estimate for $\|(B,B')\|_{\overleftarrow{X},2\alpha}$

$$\xrightarrow{(+)}\quad |F| \le C_{\alpha,T,f,\bar\varsigma}\overbrace{\exp(C N_{1,[0,T]})(1+|||\overleftarrow{\mathbf{X}}|||_\alpha)^{20}}^{L:=}|t-s|^{2\alpha}, \qquad (++)$$

where $N_{1,[0,T]}$ is a r.v. (with Gaussian tail)

$$\xrightarrow[\text{Fernique th.}]{\text{generalized}}\quad E[L] < \infty$$

(75) Suppose $g \in L^1$.

$\overset{(*)}{\Longrightarrow}$ $|Y_{T-t} - Y_{T-s} - Y'_{T-t} W_{T-s,T-t}| \lesssim |t-s|^{2\kappa}$

Similarly, one also shows that $Y \in C^\kappa$, $Y' \in C^\kappa$

$\Longrightarrow$ $(Y, Y') \in \mathcal{D}_W^{2\kappa}$

(++) implies that $\sup_m \| R^{Y^{\varepsilon_m}} \|_{2\kappa} < \infty$ for $\varepsilon_m \downarrow 0$ w.r.t W^{ε_m}.

Similarly we get that $\sup_m \| (Y^{\varepsilon_m})' \|_\kappa < \infty$

Th. 6.2, version of the proof of L. 2.14 w.r.t. pointwise conv. $\Longrightarrow$ $\| (Y^{\varepsilon_m})' - Y' \|_\kappa + \| R^{Y^{\varepsilon_m}} - R^Y \|_{2\kappa} \xrightarrow[m \to \infty]{} 0$

Th. 3.10, $\Longrightarrow$ Th. 6.2 $u_t(x) = E[g(X_T^{s,x})]$ solution to (6.8)

Rem. 6.5: One can now choose e.g. $\mathbb{W} = (B, \mathbb{B}^{Strat})$ in (6.7) to obtain an analytically weak (adapted) solution u to the corresponding stochastic PDE.

7. Regularity structures

$\longrightarrow$ generalization of the theory of rough paths based on abstract Taylor expansions

motivation :

Let $\gamma \in (k, k+1)$ for some $k \in \mathbb{N}$ and let $f : \mathbb{R} \to \mathbb{R}$ be in C^γ (i.e. $f \in C^k$, $D^k f \in C^{\gamma-k}$, $\|f\|_{C^\gamma} := \sum_{j=0}^{k} \|D^j f\|_\infty + \|D^k f\|_{\gamma-k}$)

$\longrightarrow$ Taylor expansion :

$$f(t) \longrightarrow f_t = f_s + \sum_{\ell=1}^{k} f_s^{(\ell)} (t-s)^\ell + \mathcal{O}(|t-s|^\gamma), \qquad (7.1)$$

where $f_s^{(\ell)} = (D^\ell f)(s)/\ell!$

On the other hand, if $(Y, Y') \in \mathcal{D}_W^{2\alpha}$, then

$$Y_t = Y_s + Y_s' W_{s,t} + \underbrace{\mathcal{O}(|t-s|^{2\alpha})}_{R^Y_{s,t}} \qquad (7.2)$$

$\longrightarrow$ looks like a first-order Taylor expansion

$\longrightarrow$ elements in C^γ and $\mathcal{D}_W^{2\alpha}$ admit Taylor (-like) expansions (7.1), (7.2) with "monomials" $(t-s)^m$ and $W_{s,t}$, resp.

observation :

$$(x-x_0)^m = \sum_{k+\ell=m} \frac{m!}{k!\,\ell!} (x_1-x_0)^k \cdot (x-x_1)^\ell \qquad (7.3)$$

(if $x \in \mathbb{R}^d$, $k = (k_1, \dots, k_d)$ multi-index $\longrightarrow k! := k_1! \cdots k_d!$)

and

$$\underbrace{W_{s_0,t}}_{W_t - W_{s_0}} = W_{s_0,s_1} \cdot 1 + 1 \cdot W_{s_1,t} \qquad (7.4)$$

$\longrightarrow$ monomials around points x_0, s_0 can be written as Taylor expansions around points x_1, s_1

$\longrightarrow$ Taylor expansions in (7.1), (7.2) around x_0, s_0 can be re-expanded around any other points x_1, s_1, resp.

$\longrightarrow$ The operation Γ_{z_1, z_0} of re-expansion from z_0 to

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Further observation: monomial of degree m

$$\Gamma_{x_1,x_0}((x-x_0)^m) - (x-x_0)^m \overset{(7.3)}{=} -\sum_{\substack{k+l=m \\ l<m}} \frac{m!}{k!\,l!}(x_1-x_0)^k \cdot (x-x_1)^l$$ ← polynomial of degree less than m (7.5)

monomial of degree α

and $$\Gamma_{s_1,s_0}(W_{s_0,t}) - W_{s_0,t} \overset{(7.4)}{=} W^{R}_{s_0,s_1}$$ ← polynomial of degree less than α

→ What could be a unifying structure or abstract definition for elements in $C^{\gamma,2\alpha}_{1,W}$?

→ it is reasonable to consider vector spaces T_α, which are spanned by elements corresponding to monomials of degree α (e.g. x^α)

→ the (inner or) outer direct sum

$$T := \bigoplus_{\alpha\in A} T_\alpha = \{ f : A \to \bigcup_{\alpha\in A} T_\alpha : f(\alpha)\in T_\alpha,\ f(\alpha)\neq 0 \text{ for at most finitely many } \alpha\in A\},\ A\subset\mathbb{R} \text{ index set} \quad (7.6)$$

represents all Taylor expansions

Further, in view of (7.5) we should have a family of "re-expansion operators" Γ s.t.

$$\Gamma\tau^*_\alpha - \tau^*_\alpha \in T_{<\alpha} := \{ f\in T : f(\beta) = 0 \text{ for } \beta\geq\alpha\}$$

for all $\alpha\in A$ (7.7)

→ Def. 7.1 (Regularity structure $\mathcal{J} = (A,T,G)$)

$\mathcal{J} = (A,T,G)$ is called a regularity structure, if

(i) $\mathbb{R}\supset A\ni 0$, A bounded from below, A locally finite (i.e. $\gamma\in A \Rightarrow \exists$ open neighbourhood U of γ s.t. $U\cap A$ has finitely many elements)

(ii) the model space $T = \bigoplus_{\alpha\in A} T_\alpha$, where T_α is a Banach space with norm $\|\cdot\|_\alpha$ for all α. Elements of T_α are said to have homogeneity (or degree) α.

Further, $T_0 = \langle 1^* \rangle \cong \mathbb{R}$ 1-dim. space spanned by the element 1^*

For $\tau^* \in T$, we define $\|\tau^*\|_\alpha = \|\tau^*_\alpha\|_\alpha$ $\tau^*(\alpha) \in T_\alpha$

(iii) G is a group (structure group) of continuous linear operators on $\Gamma: T \longrightarrow T$ (w.r.t. to compositions of maps) s.t. for all $\alpha \in A$, $\tau^*_\alpha \in T_\alpha$:

$$\Gamma \tau^*_\alpha - \tau^*_\alpha \in T_{<\alpha} \tag{7.8}$$

Further, $\Gamma 1^* = 1^*$ for all $\Gamma \in G$

Rem. 7.2: (i) Elements $a \in T_\beta$ are identified with $f \in T$ given by $f(\alpha) = \begin{cases} a, & \text{if } \alpha = \beta \\ 0 & \text{else} \end{cases}$

(ii) A topology on T can be e.g. induced by the system of norms $\|\cdot\|_B$ defined as $\|\tau^*\|_B = \max_{\alpha \in B} \|\tau^*\|_\alpha$, $B \subset A$ finite

$\xRightarrow{A \text{ finite}}$ canonical topology

(iii) convention: "abstract" elements in T are denoted w.r.t. $*$, i.e. $\tau^* \in T$ (in contrast to "concrete" τ, see later)

(iv) convention: If $T = \langle \tau_1^*, \tau_2^*, \dots \rangle$, then spanned by $\tau_1^*, \tau_2^*, \dots$ $\tau_i^* \in T_{\alpha_i}$ with $\alpha_1 \leq \alpha_2 \leq \dots$ (listed in order of increasing degrees)

$\longrightarrow$ Ex. 7.3 (Canonical polynomial regularity structure on $\mathbb{R}^d$)

$\longrightarrow$ abstract characterization of elements in C^γ

$A = \mathbb{N} = \{0, 1, 2, \dots\}$

$T = \mathbb{R}[X_1^*, \dots, X_d^*] = \{ a_0^{\in\mathbb{R}} + a_{j_1}^{\in\mathbb{R}} X^{j_1,*} + \dots + a_{j_m}^{\in\mathbb{R}} X^{j_m,*} : j_i \in J, m \in \mathbb{N}\}$ commutative ring (Euclidean ring) indeterminants

J set of multi-indices $j = (l_1, \dots, l_d)$, $l_i \in \mathbb{N}$

$X^{j,*} := (X_1^*)^{l_1} \cdots (X_d^*)^{l_d}$

$T_\alpha := \langle X^{j,*} : |j| = \alpha \rangle$, $|j| := l_1 + \dots + l_d$ spanned by monomials $X^{j,*}$, $|j| = \alpha$ in T

$\Rightarrow$ $T = \bigoplus_{\alpha \in A} T_\alpha$

(79) $G := \{\Gamma_h : h=(h_1,\dots,h_d)\in\mathbb{R}^d,\ \Gamma_h P(X_1^*,\dots,X_d^*) = P((X_1^*+h_1),\dots,(X_d^*+h_d))$ for any polynomial $P\}$

e.g. $P(X_1^*,\dots,X_d^*) = X_1^* X_2^* \Longrightarrow \Gamma_h P(X_1^*,\dots,X_d^*) = (X_1^*+h_1)(X_2^*+h_2)$

$\longrightarrow \quad \Gamma_h \circ \Gamma_g = \Gamma_{h+g} \Longrightarrow G \cong (\mathbb{R}^d,+)$

<u>Ex. 7.4</u> :

spanned in $\mathbb{R}[X^*]$, $d=1$

$A=\{0,1,2\}$, $T = \langle 1^*, X^*, (X^*)^2\rangle = T_0 \oplus T_1 \oplus T_2 \cong \mathbb{R}^3$,

$T_0 = \langle 1^*\rangle$, $T_1 = \langle X^*\rangle$, $T_2 = \langle (X^*)^2\rangle$

$\Gamma_h P(X^*) = P(X^*+h)$, $h\in\mathbb{R}$

Choose $f:\mathbb{R}\to\mathbb{R}$ in $C^{2+\beta}$ $\overset{(7.1)}{\Longrightarrow}$ $f^*:\mathbb{R}\to T$ given by $f^*(x) = f_x 1^* + Df_x X^* + \frac{1}{2}D^2 f_x (X^*)^2$ corresponds to f

<u>Ex. 7.5</u> (Regularity structure for $\mathcal{D}_W^{2\alpha}$)

$Y_t \approx Y_s + Y_s' W_{s,t}$ monomial of degree α

corresponds to W

$\longrightarrow \quad A = \{0,\alpha\}$, $T_0 = \langle 1^*\rangle \cong \mathbb{R}$, $T_1 = \langle W^*\rangle$,

$T := T_0 \oplus T_1 \cong \mathbb{R}^2$, $\Gamma_h W^* = W^* + h$, $\Gamma_h 1^* = 1^*$,

$h\in\mathbb{R}$

or $\mathbb{R}$

$\longrightarrow \quad Y^*: [0,T] \longrightarrow T$, $s \mapsto Y_s 1^* + Y_s' W^*$ corresponds to Y

<u>Ex. 7.6</u> (Regularity structure for α-Hölder rough paths on $\mathbb{R}$)

Let $\mathbf{W} = (W,\mathbb{W}) \in \mathscr{C}^\alpha$, $(Y,Y')\in \mathcal{D}_W^{2\alpha}$

$\xrightarrow{\text{Th. 3.8}} \quad Z_{s,t} = \int_s^t Y_u \, dW_u \approx Y_s W_{s,t} + Y_s' \mathbb{W}_{s,t}$

$\xrightarrow{\text{formally}} \quad \dot{Z}_t \approx Y_s \dot{W}_t + Y_s' \dot{\mathbb{W}}_s$

monomial of degree $\alpha-1$

where $\dot{W}_t = \frac{\partial}{\partial t} W_t$, $\dot{\mathbb{W}}_s = \frac{\partial}{\partial t} \underbrace{\mathbb{W}_{s,t}}_{=\frac{1}{2}(W_{s,t})^2} = \frac{\partial}{\partial t}\int_s^t W_{s,r}\dot{W}_r \, dr$

$= W_{s,t}\dot{W}_t \longleftarrow$ "monomial" of degree $2\alpha - 1$ $\qquad (*)$

$\longrightarrow \quad A = \{\alpha-1, 2\alpha-1, 0, \alpha\}$, negative values in A stand for distributions (see later)

$T = T_{\alpha-1} \oplus T_{2\alpha-1} \oplus T_0 \oplus T_\alpha$

$T_{\alpha-1} = \langle \overset{\dot W}{\dot W^*} \rangle$, $T_{2\alpha-1} = \langle \overset{\dot{\mathbb{W}}}{\dot{\mathbb{W}}^*} \rangle$, $T_0 = \langle 1^* \rangle$, $T_\alpha = \langle \overset{W}{W^*} \rangle$

$\longrightarrow$ $\dot Z^* : \overset{\text{or } \mathbb{R}}{[0,T]} \longrightarrow T$; $s \longmapsto Y_s \dot W^* + Y_s' \dot{\mathbb{W}}^*$ represents $\dot Z$

$$\Gamma_h 1^* = 1^*, \quad \Gamma_h W^* = W^* + h 1^* \xrightarrow{\text{reasonable}} \Gamma_h \dot W^* = \overset{\text{derivative}}{(W^* + h1^*)^{\bm\cdot}} = \dot W, \quad \Gamma_h \dot{\mathbb{W}}^* \overset{(*)}{=} (W^* + h)\underbrace{(W^* + h1^*)^{\bm\cdot}}_{= \dot W^*} =$$

$$W^* \dot W^* + h \dot W^* \overset{(*)}{=} \dot{\mathbb{W}}^* + h \dot W^*, \quad h \in \mathbb{R} \qquad (**)$$

$\longrightarrow$ Γ_h acts on $\dot W^*$, $\dot{\mathbb{W}}^*$ via W^* $\longrightarrow$ reason, why we need $\dot W^*$ in T

$\longrightarrow$ generalization:

<u>Ex. 7.7</u> (Regularity structure for α-Hölder rough paths on $\mathbb{R}^d$)

$A = \{\alpha - 1, 2\alpha - 1, 0, \alpha\}$,

$T = T_{\alpha-1} \oplus T_{2\alpha-1} \oplus T_0 \oplus T_\alpha \cong \mathbb{R}^{d+d^2+1+d}$,

where $T_0 = \langle 1^* \rangle$, $T_\alpha = \langle W^{1,*}, \dots, W^{d,*} \rangle$,

$T_{\alpha-1} = \langle \dot W^{1,*}, \dots, \dot W^{d,*} \rangle$, $T_{2\alpha-1} = \langle \dot{\mathbb{W}}^{ij,*}, 1 \le i,j \le d \rangle$

$G \cong (\mathbb{R}^d, +)$ given by the maps

$\Gamma_h 1^* = 1^*$, $\Gamma_h W^{i,*} = W^{i,*} + h_i 1^*$, $\Gamma_h \dot W^{i,*} = \dot W^{i,*}$,

$\Gamma_h \dot{\mathbb{W}}^{ij,*} = \dot{\mathbb{W}}^{ij,*} + h_i \dot W^{j,*}$

$\longrightarrow$ $W = B$ Bm, $B \in C^\alpha$, $\alpha = \frac{1}{2} - \kappa$ with prob. 1 for any $\kappa \in (0, \frac{1}{2})$ $\longrightarrow$ $A = \{\underbrace{-\frac{1}{2} - \kappa}_{= \alpha - 1}, \underbrace{-2\kappa}_{2\alpha - 1}, 0, \frac{1}{2} - \kappa\}$

<u>Next objective</u>:

1. Let $\tau^* \in T$ (i.e. "abstract Taylor polynomial" or "jet"), $x_0 \in \mathbb{R}^d$ $\longrightarrow$ We want to associate to (τ^*, x_0) a concrete "Taylor polynomial at x_0" by means of distributions (in $\mathcal{D}'(\mathbb{R}^d)$)
2. We want to introduce the notion of "vanishing at order α" w.r.t. such distributions

$\longrightarrow$ concept of a model M for $\mathcal{T}$ (see Def. 7.8)

(81) For this purpose we need some definitions and notions:

$K \subseteq \mathbb{R}^d$ compact, $\mathcal{D}_K$ space of all $f \in C_0^\infty(\mathbb{R}^d)$ with support in K

$\mathcal{D}(\mathbb{R}^d) := C_0^\infty(\mathbb{R}^d)$

$\mathcal{D}'(\mathbb{R}^d)$ space of linear functionals $\Lambda : \mathcal{D}(\mathbb{R}^d) \longrightarrow \mathbb{R}$ s.t. for all compact $K \subseteq \mathbb{R}^d$ there exists a $N \in \mathbb{N}$ and $C < \infty$:

$$|\Lambda \phi| \leq C \|\phi\|_N \qquad (7.9)$$

for all $\phi \in \mathcal{D}_K$,

where $\|\phi\|_N := \max \{ |D^\alpha \phi(x)| : x \in \mathbb{R}^d, |\alpha| \leq N \}$,

$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$, $|\alpha| = \alpha_1 + \dots + \alpha_d$

Elements of $\mathcal{D}'(\mathbb{R}^d)$ are called distributions

each f with $\int_K |f(x)|\,dx < \infty$ for every compact $K \subset \mathbb{R}^d$ can be identified with the distribution Λ_f given by

e.g. cont. functions

$$\Lambda_f(\phi) = \int_{\mathbb{R}^d} \phi(x) f(x)\,dx$$

Let $r > 0$ integer. $\mathcal{B}_r$ is defined as the space of all $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$ in C_b^r with support in the unit ball at Θ s.t. $\|\varphi\|_{C_b^r} \leq 1$

We also use the shorthand notation

$$\varphi_x^\lambda(y) = \lambda^{-d} \varphi(\lambda^{-1}(y-x))$$

approx. of the Dirac delta δ_y if $\int_{\mathbb{R}^d} \overset{\geq 0}{\varphi}(x)\,dx = 1$

<u>Def. 7.8</u> (Model $M = (\Pi, \Gamma)$ for $\mathcal{J}$)

$M = (\Pi, \Gamma)$ is a model for the regularity structure $\mathcal{J}$ on $\mathbb{R}^d$ for maps

$\Pi : \mathbb{R}^d \longrightarrow L(T, \mathcal{D}'(\mathbb{R}^d))$, $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow G$

structure group

<u>if</u>

$$\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz} \quad \text{and} \quad \Pi_x \Gamma_{xy} = \Pi_y \quad , x,y,z \in \mathbb{R}^d \qquad (7.10)$$

and if for the smallest integer $r > |\min A| \geq 0$ and all compact $\mathcal{K} \subset \mathbb{R}^d$, $\gamma > 0$ there exists a $C = C(\mathcal{K}, \gamma) < \infty$ s.t.

for all $\varphi \in \mathcal{B}_r$, $x, y \in \mathcal{K}$, $\lambda \in (0,1]$, $\tau^* \in T_\alpha$ with $\beta < \alpha \leq \gamma$:

$$|(\Pi_x \tau^*)(\varphi_x^\lambda)| \leq C \lambda^\alpha \|\tau^*\|_\alpha \qquad (7.11)$$

and

$$\|\Gamma_{xy}\tau^*\|_\beta \leq C |x-y|^{\alpha-\beta} \|\tau^*\|_\alpha \qquad (7.12)$$

We say Π_x realizes elements in T as distributions in $\mathcal{D}'(\mathbb{R}^d)$.

Rem. 7.9

(i) The space $\mathcal{M}$ of all models w.r.t. a given $\mathcal{J}$ is not a linear space

(ii) Γ_{xy} re-expands monomials around "$y-x$"
Given an abstract Taylor polynomial $\tau^* \in T$, $\Pi_x \tau^* \in \mathcal{D}'(\mathbb{R}^d)$ represents the corresponding "concrete" Taylor polynomial re-expanded around x

(iii) interpretation of (7.11): $\tau^* \in T_\alpha$ represented by $\Pi_x \tau^*$ "vanishes" at order α around x and α describes the speed at which distributions $\Pi_x \tau^*$ for $\tau^* \in T_\alpha$ vanish around x

(iv) E.g. $\tau^* = (X^{1*})^{\alpha=2} \longrightarrow \Gamma_{xy}\tau^* = (X^{1*} - (y-x))^{\alpha=2} =$
$= (X^{1*})^2 - 2(y-x)X^{1*} + (y-x)^2 \xrightarrow{\beta=1} \|\Gamma_{xy}\tau^*\|_\beta = 2|y-x| \leq$
$C|y-x|^{\overset{=1}{\alpha-\beta}} \underbrace{\|\tau^*\|_\alpha}_{=1}$
$\longrightarrow$ (7.12) natural condition

We need the following concept of Hölder spaces w.r.t. regularity structures $\mathcal{J}$!

Def. 7.10 : Let $M = (\Pi, \Gamma)$ be a model for $\mathcal{J}$ on $\mathbb{R}^d$.
Then $\mathcal{D}^\gamma_M = \mathcal{D}^\gamma_M(\mathcal{J})$ is the set of functions
$f^*: \mathbb{R}^d \longrightarrow T_{<\gamma}$ s.t. for all compact $\mathcal{K}$ and $\alpha < \gamma$
there exists a constant C s.t.

$$\| f^*(x) - \Gamma_{xy} f^*(y) \|_\alpha \leq C |x-y|^{\gamma-\alpha} \qquad (7.13)$$

uniformly in $x, y \in \mathcal{K}$
Functions $f^* \in \mathcal{D}^\gamma_M$ are called modelled distributions

Rem. 7.11 :
(i) For fixed $\mathcal{K}$ one can define semi-norms $\|\cdot\|_{M,\gamma,\mathcal{K}}$ given by $\|f^*\|_{M,\gamma,\mathcal{K}}$ = smallest C in (7.13)
$\longrightarrow$ $\mathcal{D}^\gamma_M$ Fréchet space w.r.t. these semi-norms
(ii) $\mathcal{M} \ltimes \mathcal{D}^\gamma \overset{\text{def.}}{=} \bigcup_{M \in \mathcal{M}} \{M\} \times \mathcal{D}^{\gamma}_M$ (compare Def. 3.5)
$\longrightarrow$ total space with base space $\mathcal{M}$ and "fibres" $\mathcal{D}^\gamma_M$

We are coming to the most fundamental result of the theory of regularity structures

Th. 7.12 (Reconstruction theorem)
Let $M = (\Pi, \Gamma)$ be a model for $\mathcal{J}$ on $\mathbb{R}^d$ and let $f^* \in \mathcal{D}^\gamma_M$ with $\gamma > 0$. Then there exists a continuous linear map

$$\mathcal{R} : \mathcal{D}^\gamma_M \longrightarrow \mathcal{D}'(\mathbb{R}^d)$$

s.t.

$$| (\mathcal{R} f^* - \Pi_x f^*(x))(\varphi^\lambda_x) | \lesssim \lambda^\gamma \qquad (7.14)$$

uniformly for all $\varphi \in \mathcal{B}_r$, $\lambda \in (0,1]$, locally uniformly in x
The bound (7.14) determines $\mathcal{R} f^*$ uniquely.
If $\gamma \leq 0$, the result still holds true, but uniqueness may fail.

Rem. 7.13 :
(i) interpretation of Th. 7.12 : $\forall f^* \in \mathcal{D}^\gamma_M$ there ex. a

(84) unique distribution $\mathcal{R}f^* \in \mathcal{D}'(\mathbb{R}^d)$ s.t. for all $x \in \mathbb{R}^d$: $\mathcal{R}f^*$ "looks" like $\Pi_x f^*(x)$ near x (because of (7.14))

(ii) If $(\Pi_x \tau^*)(\varphi) = \int_{\mathbb{R}^d} g^{\tau^*,x}(y)\,\varphi(y)\,dy$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$ for all $\tau^* \in T$, $x \in \mathbb{R}^d$, where $g^{\tau^*,x} : \mathbb{R}^d \longrightarrow \mathbb{R}$ are continuous functions then $\mathcal{R}f^*$ is cont. and given by

$$(\mathcal{R}f^*)(x) = g^{f^*(x),x}(x), \quad x \in \mathbb{R}^d$$

<u>Ex. 7.14</u> (Canonical polynomial model $M = P$)

$\longrightarrow$ model for the canonical polynomial regularity structure (see Ex. 7.3):

$T = \mathbb{R}[X_1^*, \dots, X_d^*]$ $\quad = (X_1^*)^{l_1} \cdots (X_d^*)^{l_d}$, $j = (l_1, \dots, l_d)$

$\longrightarrow$ Π_x given by $\Pi_x X^{j,*} = (y \longmapsto \underbrace{(y-x)^j}_{= \prod_{i=1}^d (y_i - x_i)^{l_i}})$, $x = (x_1, \dots, x_d)$

Γ_{xy} given by $\Gamma_{xy} P(X_1^*, \dots, X_d^*) = (\Gamma_h P(X_1^*, \dots, X_d^*))|_{h = x-y}$

$= P(X_1^* + x_1 - y_1, \dots, X_d^* + x_d - y_d)$

$\longrightarrow$ <u>Prop. 7.15</u> (Characterization of C^β by means of $\mathcal{D}_P^\beta$)

Let $\beta = n + \gamma$ with $n \in \mathbb{N}$, $\gamma \in (0,1)$. Then $f \in C^\beta$ iff there ex. a $f^* \in \mathcal{D}_P^\beta$ s.t. $(f^*(x))(0) = f(x)$ for all x

<u>Ex. 7.16</u> (Model $M = M_W$ for the rough path regularity structure)

$A = \{\alpha-1, 2\alpha-1, 0, \alpha\}$, $T = T_{\alpha-1} \oplus T_{2\alpha-1} \oplus T_0 \oplus T_\alpha \cong \mathbb{R}^{d+d^2+1+d}$,

$\Gamma_h 1^* = 1^*$, $\Gamma_h W^{i,*} = W^{i,*} + h_i 1^*$, $\Gamma_h \dot{W}^{i,*} = \dot{W}^{i,*}$, $\Gamma_h \dot{\mathbb{W}}^{ij,*} = \dot{\mathbb{W}}^{ij,*} + h_i \dot{W}^{j,*}$

$\longrightarrow$ Π_s given by

distribution because of the Young ineq. $\quad$ Young integral

$(\Pi_s 1^*)(t) = 1$, $(\Pi_s W^{j,*})(t) = W^j_{s,t}$, $(\Pi_s \dot{W}^{j,*})(\psi) = \int_{\mathbb{R}} \psi(t)\, dW_t^j$

$(\Pi_s \dot{\mathbb{W}}^{ij,*})(\psi) = \int_{\mathbb{R}} \psi(t)\, d\mathbb{W}^{ij}_{s,t}$ $\longleftarrow$ Young integration w.r.t. t

If W^j smooth $\longrightarrow$ $\dot{W}^j \longleftrightarrow \underbrace{(\psi \mapsto \int_{\mathbb{R}} \psi(t)\dot{W}_t^j\,dt)}_{= \int_{\mathbb{R}} \psi(t)\, dW_t^j} \in \mathcal{D}'(\mathbb{R})$

(85) $\Gamma_{t,s} = \Gamma_h|_{h=W_{s,t}}$

If $(Y, Y') \in \mathcal{D}_W^{2\alpha}$ then $Y^* \in \mathcal{D}_{M_W}^{2\alpha}$, where

$$Y^*(s) = Y(s)\cdot 1^* + \sum_{i=1}^{d} Y_i'(s)\, W^{i,*}$$

$\Pi_s \Gamma_{s,u} \tau^* = \Pi_u \tau^*$, since this is true for all $(\tau^* \in \{1^*, \dot{W}^{j,*}, \dot{W}^{ij,*}\})$

and $(\Pi_s \Gamma_{s,u} \dot{W}^{ij,*})(\psi) = (\Pi_s (\dot{W}^{ij,*} + W_{u,s}^i \cdot \dot{W}^{j,*}))(\psi)$

$$= \int_{\mathbb{R}} \psi(t)\, d(\mathbb{W}_{s,t}^{ij} + W_{u,s}^i W_{s,t}^j) \overset{\text{Chen's relation}}{=} \int_{\mathbb{R}} \psi(t)\, d\mathbb{W}_{u,t}^{ij}$$

$= (\Pi_u \dot{W}^{ij,*})(\psi)$

and all other cond. of Def. 7.8 and 7.10 are satisfied

Define $(Y\dot{W}^j)^*$ by

$$(Y\dot{W}^j)(s) = Y(s)\, \dot{W}^{j,*} + \sum_{i=1}^{d} Y_i'(s)\, \dot{W}^{ij,*}$$

then $(Y\dot{W}^j)^* \in \mathcal{D}_{M_W}^{3\alpha-1}$

$\xrightarrow[\text{for } \gamma = 3\alpha-1>0]{\text{Th. 7.12}}$ $\exists$ a unique distribution $\mathcal{R}(Y\dot{W}^j)^*$ s.t.

$$(\mathcal{R}Y\dot{W}^j)^*(\psi_s^\lambda) = \int_{\mathbb{R}} \psi_s^\lambda(t)\, Y(s)\, dW_t^j + \sum_{i=1}^{d} \int_{\mathbb{R}} \psi_s^\lambda(t)\, Y_i'(s)\, d\mathbb{W}_{s,t}^{ij} + O(\lambda^{3\alpha-1})$$

uniformly in $\psi \in \mathcal{B}_r$, $\lambda \in (0,1]$ and locally unif. in s

$\xrightarrow{\lambda = |t-s|}$ $(\mathcal{R}Y\dot{W}^j)(\psi) = \int_{\mathbb{R}} \psi(t)\, dZ(t)$ for a (unique) Z with

$$Z_{s,t} = Y(s) W_{s,t}^j + \sum_{i=1}^{d} Y_i'(s)\, \mathbb{W}_{s,t}^{ij} + O(|t-s|^{3\alpha})$$

$\longrightarrow$ Th. 3.4 and Th. 3.8 as a special case