8. Operations on modelled distributions

In order to give a meaning to the KPZ equation

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi - C \qquad (8.1)$$

from the viewpoint of regularity structures, we want to introduce the concepts of differential operators and products on such structures.

In fact, if α is a multi-index and $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$, then the derivative of the distribution Λ, i.e. $D^\alpha \Lambda$ given by

$|\alpha| = \alpha_1 + \dots + \alpha_d$

$$(D^\alpha \Lambda)(\phi) = (-1)^{|\alpha|} \Lambda(D^\alpha \phi), \quad \phi \in \mathcal{D}(\mathbb{R}^d) \qquad (8.2)$$

defines a linear functional on $\mathcal{D}(\mathbb{R}^d)$, which is a distribution, again.

Hence,

$$\mathcal{L}\Lambda \in \mathcal{D}'(\mathbb{R}^d)$$

for the homogeneous diff. operator $\mathcal{L}$ of degree m

$$\mathcal{L} = \sum_{|\alpha| = m} a_\alpha D^\alpha$$

→ What is the corresponding concept of $\mathcal{L}$ in the context of regularity structures?

→ **Def. 8.1** (Differential operators on sectors)
Let (A, T, G) be a regularity structure and $V \subseteq T$ be a subspace. Then V is called a sector if

$$V = \bigoplus_{\alpha \in A} V_\alpha \quad \text{with } V_\alpha \subseteq T_\alpha$$

and V is invariant under the actions of G (structure group), that is

$$\Gamma(V) \subseteq V \quad \text{for all } \Gamma \in G$$

<u>Further</u>, we say a linear operator $\partial : V \longrightarrow T$ <u>realizes $\mathcal{L}$</u> (of degree m) with respect to the model (Π, Γ) if

(i) $\partial \tau^* \in T_{\alpha - m}$ for all $\tau^* \in V_\alpha$

(ii) $\Gamma \partial \tau^* = \partial \Gamma \tau^*$ for all $\tau^* \in V$

(iii) $\Pi_x \partial \tau^* = \mathcal{L} \Pi_x \tau^*$ for all $\tau^* \in V$, $x \in \mathbb{R}^d$

→ <u>Prop. 8.2</u>: Denote by $\mathcal{D}^\gamma(V)$ the space of modelled distributions $\mathcal{D}^\gamma_M$ taking values in $V \subset T$.
Let $\gamma > m$. Then $\partial f^* \in \mathcal{D}^{\gamma - m}(V)$ and

$$\mathcal{R} \partial f^* = \mathcal{L} \mathcal{R} f^*$$

for all $f^* \in \mathcal{D}^\gamma(V)$

reconstruction operator

<u>Def. 8.3</u> (Pointwise multiplication of abstract Taylor expansions)
Let $V, \bar{V} \subset T$ be sectors. Then a <u>product on $(V, \bar{V})$</u> is a bilinear map $\diamond : V \times \bar{V} \longrightarrow T$ s.t. for all $\tau^* \in V_\alpha$, $\bar{\tau}^* \in \bar{V}_\beta$

(i) $\tau^* \diamond \bar{\tau}^* \in T_{\alpha + \beta}$ ← degrees of monomials add up under multiplication

(ii) $\Gamma(\tau^* \diamond \bar{\tau}^*) = \Gamma(\tau^*) \diamond \Gamma(\bar{\tau}^*)$ for all $\Gamma \in G$

→ <u>Th. 8.4</u> (Multiplication of modelled distributions)
Denote by $\mathcal{D}^\gamma_\alpha$ the space of all $f^* \in \mathcal{D}^\gamma$ s.t.

$$f^*(x) \in T_{\geq \alpha} := \{\tau^* \in T : \tau^*(\beta) = 0 \text{ for } \beta < \alpha\}$$

for all $x \in \mathbb{R}^d$.

Let $\diamond$ be a product on $(V, \bar{V})$ and $f_1^* \in \mathcal{D}^{\gamma_1}_{\alpha_1}(V)$, $f_2^* \in \mathcal{D}^{\gamma_2}_{\alpha_2}(\bar{V})$. Then f^* defined by

$$f^*(x) = f_1^*(x) \diamond f_2^*(x)$$

is contained in $\mathcal{D}^\gamma_\alpha$,
where $\alpha = \alpha_1 + \alpha_2$ and $\gamma := (\gamma_1 + \alpha_2) \overset{\text{minimum}}{\wedge} (\gamma_2 + \alpha_1)$.

Def. 8.5: We say a sector $V \subset T$ is *function-like* if $V_\alpha = 0$ for $\alpha < 0$ and $V_0 = \langle 1^* \rangle$

From now on, we consider models (Π, Γ), which are *normal*, that is models satisfying

$$(\Pi_x 1^*)(y) = 1 \quad \text{for all } y$$

$\longrightarrow$ One shows: If $f^* \in \mathcal{D}^\gamma(V)$ then $\mathcal{R} f^*$ is a cont. funct. and $(\mathcal{R} f^*)(x) = (f^*(x))(0)$

$\longrightarrow$ Analogue to L. 4.1!

Prop. 8.6: Let $\diamond : V \times V \longrightarrow \overset{\text{function-like}}{V}$ be a product and $f^* \in \mathcal{D}^\gamma(V)$, $\gamma > 0$. Assume a function $G : \mathbb{R} \longrightarrow \mathbb{R}$ in C^K with $K \geq (\gamma/\alpha_0) \overset{\text{max}}{\vee} 1$ for some $\overset{\text{which by assumption exists}}{\alpha_0} > 0$ s.t.

$$\tilde{f}^*(x) := f^*(x) - \bar{f}(x) \cdot 1^* \in T_{\geq \alpha_0}$$

for all x with

$$\bar{f}(x) := (f^*(x))(0)$$

Define

$$(G \circ f^*)(x) := \overset{\text{projection to } T_{<\gamma}}{Q_\gamma^-} \sum_{l=0}^{K} \frac{G^{(l)}(\bar{f}(x))}{l!} \, (\tilde{f}^*(x))^{\diamond l} \in T_{\geq l\alpha_0}$$

for $(\tilde{f}^*)^{\diamond l} = \underbrace{\tilde{f}^* \diamond \cdots \diamond \tilde{f}^*}_{l \text{ times}}$, $G^{(0)} = G$, $(\tilde{f}^*)^0 = 1^*$

Then

$$\mathcal{G} : \mathcal{D}^\gamma(V) \longrightarrow \mathcal{D}^\gamma(V), \; f^* \longmapsto G \circ f^*.$$

If $K > (\gamma/\alpha_0 \vee 1) + 1$ then it is locally Lipschitz cont.

89 If h is a "solution" to the KPZ equation (8.1), then h satisfies the heat equation

$$\partial_t h = \partial_x^2 h + f \quad (8.3)$$

space-time white noise

where $f = (\partial_x h)^2 + \xi + C$

Then

$$h = \mathcal{G} * \left((\partial_x h)^2 + \xi \right) + \mathcal{G} h_0 , \quad (8.4)$$

where $\mathcal{G}$ is Green's function, $\mathcal{G} h_0$ the solution to the heat equation with initial condition and where

$$(g * r)(z) := \int_{\mathbb{R}^m} g(z-y)\, r(y)\, dy$$

is the convolution of two functions g and r

In fact

$$(4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

$$\mathcal{G} = K + \hat{K} , \quad (8.5)$$

where K is smooth except for a singularity at the origin and where $\hat{K}$ is smooth and bounded

→ <u>objective</u>: We want <u>to give a meaning to K in the context of regularity structures</u> by using an operator $\mathcal{K}$ on $\mathcal{D}^\gamma$

To this end, we make the following assumptions

<u>Assumption 8.7</u>: For all $n \in \mathbb{N}$, $T_n = \bar{T}_n$, where $\bar{T}_n = \langle X^{j} : |j| = n \rangle$ are the monomial spaces of the canonical polynomial regularity structure in Ex. 7.3 and elements $\Gamma \in G$ act on X^{j} as before.

→ canonical polynomial regularity structure is a part of the regularity structure

corresponds to $f \mapsto K * f$

We also need the existence of a map $\mathcal{I} : T \longrightarrow T$, which gives an abstract representation of $\mathcal{K}$:

<u>Assumption 8.8</u>: There exists a linear map $\mathcal{I} : T \longrightarrow T$ (abstract integration map of order $\beta > 0$) s.t.

(i) $\mathcal{J} T_\alpha \subset T_{\alpha+\beta}$ for all α

(ii) $\mathcal{J} \overline{T} = 0$, where $\overline{T}$ is the sector spanned by elements of T_n, $n \in \mathbb{N}$

(iii) $\Gamma \mathcal{J} \tau^* - \mathcal{J} \Gamma \tau^* \in \overline{T}$ for all $\Gamma \in G$, $\tau \in T$

The next assumption makes precise the meaning of a singularity of K at the origine, which approximately homogeneous of degree $\beta - d$:

Assumption 8.9: K can be decomposed as

$$K(x) = \sum_{n \geq 0} K_n(x),$$

where each of the kernels $K_n : \mathbb{R}^d \longrightarrow \mathbb{R}$ is smooth with compact support in $\{x : |x| \leq 2^{-n}\}$. Further, require that for every multi-index k there is a constant C s.t.

$$\sup_x |D^k K_n(x)| \leq C 2^{n(d-\beta+|k|)}$$

uniformly in n as well as

$$\int_{\mathbb{R}^d} K_n(x) P(x) \, dx = 0$$

for every polynomial P of degree $\leq N$, for some $N > 0$

Rem. 8.10: K in (8.5) satisfies Assumption 8.9 for $\beta = 2$

Def. 8.11 (Compatibility of the model with the kernel K)
Let K be a kernel as in Assumption 8.9 and an abstract integration operator $\mathcal{J}$ on a regularity structure as in Assumption 8.7 and 8.8. Then a model (Π, Γ) is admissible, if

(i) $(\Pi_x X^{k,*})(y) = (y-x)^k$, $y \in \mathbb{R}^d$

(ii) for all $\tau^* \in T_\alpha$, $x \in \mathbb{R}^d$

converges absolutely for any α with $\alpha + \beta \notin \mathbb{N}$

$$(\Pi_x \mathcal{J} \tau^*)(\varphi) = \sum_{n \geq 0} \int_{\mathbb{R}^d} \varphi(y) (\Pi_x \tau^*)(K^\alpha_{n,x}) \, dy, \qquad (8.6)$$

where

$$K_{n,x}^{\alpha}(z) := K_n(y-z) - \sum_{|k|<\alpha+\beta} \frac{(y-x)^k}{k!} D^k K_n(x-z)$$

ensures that $\Pi_x \mathcal{J} \tau^*$ vanishes at sufficiently high order

Th. 8.12 (Schauder estimate)

Suppose that Assumptions 8.7, 8.8 and 8.9 hold.
Let (Π, Γ) be an admissible model.
Define for $f^* \in \mathcal{D}^\gamma$

$$(\mathcal{K} f^*)(x) = \mathcal{J} f^*(x) + \mathcal{J}(x) f^*(x) + (\mathcal{N} f^*)(x), \qquad (8.7)$$

where
$$\mathcal{J}(x)\tau^* := \sum_{|k|<\alpha+\beta} \sum_{n\geq 0} \frac{X^{k,*}}{k!} (\Pi_x \tau^*)(D^k K_n(x-\cdot)),$$

summable for $|k|<\alpha+\beta$

$\tau^* \in T_\alpha$

and
$$(\mathcal{N} f^*)(x) := \sum_{|k|<\gamma+\beta} \sum_{n \geq 0} \frac{X^{k,*}}{k!} (\mathcal{R} f^* - \Pi_x f^*(x))(D^k K_n(x-\cdot)).$$

summable

Then for all $f^* \in \mathcal{D}^\gamma$ with $\gamma \in (0, N-\beta)$ and $\gamma+\beta \notin \mathbb{N}$
$\mathcal{K} f^*$ well-def in (8.5) and belongs to $\mathcal{D}^{\gamma+\beta}$.

Moreover
$$\mathcal{R}\mathcal{K} f^* = K * \mathcal{R} f^* \qquad (8.8)$$

Rem.: $(\mathcal{R}\mathcal{K} f^*)(\varphi) = \underbrace{(K * \mathcal{R} f^*)(\varphi)}$
$$= \sum_{n\geq 0} \int_{\mathbb{R}^d} \mathcal{R} f^*)(K_n(x-\cdot))\, \varphi(x)\, dx$$

Application to the KPZ equation

→ objetive: We want to solve the KPZ equation based on a suitable notion of solution by using the previously developed concepts from the theory of regularity structures

→ rough description of this program:

1. step: Construction of the regularity structure T:

Recall from (8.5) in connection with (8.4) that

$$\mathcal{G} = K + \hat{K}$$

We already gave a meaning to the kernel in the context of regularity structures by $\mathcal{K}$ in Th. 8.12.

What about $\hat{K}$?

→ $\hat{\mathcal{K}} : \mathcal{D}^{\gamma} \longrightarrow \mathcal{D}^{\infty}$ given by

$$(\hat{\mathcal{K}} f^{*})(z) = \sum_{k} \frac{X^{k}}{k!} R f^{*}(D^{k} \hat{K}(z - \cdot)),$$

where $z = (x,t) \in \mathbb{R}^2$, k is a 2-dim. multi-index

(8.4) → The KPZ equation (8.4) has the following formulation in our (not yet defined) regularity structure:

$$h \longleftrightarrow H^{*} = \mathcal{K}((\partial H^{*})^2 + \Xi^{*}) + \hat{\mathcal{K}}((\partial H^{*})^2 + \Xi^{*}) + \mathcal{G} h_0 \quad (8.9)$$

Using the decomposition of $\mathcal{K}$ in terms of the integration map $\mathcal{I}$ in Th. 8.12, we can recast (8.9) as

$$H^{*} = \mathcal{I}((\partial H^{*})^2 + \Xi^{*}) + R, \quad (8.10)$$

where $R \in T \cong \mathbb{R}[X_0^{*}, X_1^{*}]$

time-variable
space-variable

By iterating the equation (8.10), we obtain a description of H^{*} by means of symbols in (8.10):

So e.g. we need Λ^*, X^{1*} $\leftarrow X_0^{1*}, X_1^{1*}$, $(\mathcal{I}\Lambda)^*$, $(\mathcal{I}'\Lambda)^*$, $\ldots$, where $\mathcal{I}' = \partial\mathcal{I}$

$\longrightarrow$ <u>regularity structure</u> T vector space spanned by linear combinations of elements from

$$W := \mathcal{U} \cup \{\Lambda^*\} \cup \{\partial\tau_1^*\partial\tau_2^* : \tau_i^* \in \mathcal{U}\}, \quad (8.11)$$

where $\mathcal{U}$ is a set of expressions of symbols, which is the smallest set containing $\{1^*, X^*, (\mathcal{I}\Lambda)^*\}$ s.t.

$$\text{if } \tau_1^*, \tau_2^* \in \mathcal{U} \text{ then } \mathcal{I}(\partial\tau_1\partial\tau_2)^* \in \mathcal{U}$$

Here we set $(\mathcal{I}X^{1k})^* = 0$ for all k.

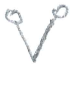
Further, we define the degree $|\Lambda^*|$ of Λ^* by $-\frac{3}{2}$ $\leftarrow$ "roughness" of ξ, naturally and the degree of other symbol expressions are naturally determined by the rules

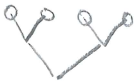
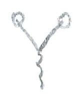
Ass. 8.8 $\rightarrow$ $|\underbrace{\mathcal{I}\tau^*}_{=:(\mathcal{I}\tau)^*}| = |\tau^*| + 2$, $|\underbrace{\partial\tau^*}_{=:(\partial\tau)^*}| = |\tau| - 1$ $\leftarrow$ Def. 8.1(c), $|\underbrace{\tau^*\bar{\tau}^*}_{=:(\tau\bar{\tau})^*}| = |\tau^*| + |\bar{\tau}^*|$ $\leftarrow$ corresponds to multiplication

$$\longrightarrow \quad T = \bigoplus_{\alpha \in A} T_\alpha, \quad (8.12)$$

where $A := \{|\tau^*| : \tau^* \in W\}$, $T_\alpha = \langle \tau^* : |\tau^*| = \alpha \rangle$ $\leftarrow$ finite-dimensional

<u>Rem</u>. 8.13: Graphical shorthand notation of the symbols in terms of "trees":

$\Lambda^* \Longleftrightarrow \circ$, $\mathcal{I} \Longleftrightarrow \wr$, $\mathcal{I}' \Longleftrightarrow |$, $(\mathcal{I}'\Lambda)^{2,*} \Longleftrightarrow$ ,

$(\mathcal{I}'(\mathcal{I}'(\Lambda)^2))^{2,*} \Longleftrightarrow$ , $\mathcal{I}(\mathcal{I}'(\Lambda)^2)^* \Longleftrightarrow$  $\ldots$

no special notation for X^{1*}: e.g. $X_0^*\mathcal{I}'(\Lambda)^{2,*} = X_0^*$ 

2. step: Construction of the structure group G

The properties of $\Gamma \in G$ in connection with (8.6) and (8.7) indicate that the definition of such Γ should be based on information coming from monomials of the form

$$X^i, \mathcal{J}_\ell(\hat{\tau})$$

$$\longrightarrow \quad T^+ := \underbrace{\mathbb{R}[\mathcal{A}]}_{\text{commutative ring}} \tag{8.12}$$

where

$$\overbrace{\mathcal{A}}^{\text{set of indeterminants}} := \{X^i, \mathcal{J}_\ell(\hat{\tau}) : i \in \{1, \dots, d\}, \tau^* \in W \text{ with } |\tau^*| < |\ell|\}$$

Note that $\mathcal{J}_\ell(\hat{\tau})$ is the ℓ-th component of $\mathcal{J}$ with $|\ell| < |\tau^*| + \beta^{\wedge 2}$ for $\tau^* \in T_\alpha$ (see Th. 8.12)

$$\longrightarrow \quad T^+ = \bigoplus_{\alpha \in A^+} T^+_\alpha, \tag{8.13}$$

where $A^+ := \{|\tau^*| : \tau^* \in T^+_\alpha\} \ (\subset [0, \infty))$

$$T^+_\alpha := \langle \tau^* : |\tau^*| = \alpha \rangle$$

and where (naturally)

$$|\mathcal{J}_\ell(\hat{\tau})| := |\hat{\tau}^*| + 2 - |\ell|, \quad |X^k| := |k|, \quad |\hat{\tau}\bar{\tau}| = |\hat{\tau}| + |\bar{\tau}|$$

In order to define G w.r.t. an admissible model (Π, Γ), let us consider the linear maps $f_x : T^+ \to \mathbb{R}$ given by

$$f_x(X^k) = (-x)^k, \quad f_x(\tau\bar{\tau}) = f_x(\tau) f_x(\bar{\tau}) \quad \text{and}$$

$$f_x(\mathcal{J}_{\ell_i}(\hat{\tau}_i)) = (\Pi_x \hat{\tau}_i^*)(D^{\ell_i} K(x - \cdot)) \tag{8,14}$$

Further, define

$$\Delta : T \to T \otimes T^-$$

by

$$\Delta 1^* = 1^* \otimes 1, \quad \Delta \Lambda^* = \Lambda^* \otimes 1, \quad \Delta X_i^* = X_i^* \otimes 1 + 1^* \otimes X_i$$

and then recursively for all elements in T by

$$\Delta(\tau^*\bar\tau^*) = \Delta\tau^*\cdot\Delta\bar\tau^*$$

$$\Delta\mathcal{J}(\tau^*) = (\mathcal{J}\otimes \underbrace{I}_{\text{identity}})\Delta\tau^* + \sum_{l,m}\frac{X^{l,*}}{l!}\otimes\frac{X^m}{m!}\mathcal{J}_{l+m}(\tau)$$

$$\Delta\mathcal{J}'(\tau^*) = (\mathcal{J}'\otimes I)\Delta\tau^* + \sum_{l,m}\frac{X^{l,*}}{l!}\otimes\frac{X^m}{m!}\mathcal{J}_{l+m+\overset{\text{multi-index}}{(0,1)}}(\tau)$$

Here: (i) $(\mathcal{J}\otimes I)(a\otimes b) = \mathcal{J}a\otimes Ib = b$, similarly $\mathcal{J}'\otimes I$

(ii) $\mathcal{J}_k : T\to T^+ ;\ \tau^*\mapsto \mathcal{J}_k(\tau^*) \overset{\text{def}}{=} \mathcal{J}_k(\tau)$ is a linear map, if we set $\mathcal{J}_k(\tau^*) = 0$ for $\tau^*\in W$ with $|\tau^*| < |k|-2$ (see (8.7.))

Denote by G_+ the set of linear maps $g: T^+\to\mathbb{R}$ s.t.

$$g(\sigma\bar\sigma) = g(\sigma)g(\bar\sigma)$$

for all $\sigma,\bar\sigma\in T^+$

Then define for all $g\in G_+$: $(I\otimes g)(a\otimes b) = g(b)a$

$$\Gamma_g : T\to T ;\ \tau^*\mapsto (I\otimes g)\Delta\tau^* \tag{8.15}$$

$\longrightarrow$ $(G_+,\circ)$ is <u>a group</u> with the group operation $\circ$ defined as follows:

There ex. a linear map $\Delta^+ : T^+\to T^+\otimes T^+$ s.t.

$$(\Delta\otimes I)\Delta = (I\otimes\Delta^+)\Delta,\quad \Delta^+(\sigma\bar\sigma) = \Delta^+\sigma\cdot\Delta^+\bar\sigma$$

$\longrightarrow$
$$(f\circ g)(\sigma) := (f\otimes g)\Delta^+\sigma \tag{8.16}$$
for $f,g\in G$

$\longrightarrow$ composition of maps
$$\Gamma_{f\circ g} = \Gamma_f\circ\Gamma_g \tag{8.17}$$

$\longrightarrow$ $G := \{\Gamma_g : g\in G_+\}$

$\longrightarrow$
$$\Gamma_{xy} := \Gamma_{f_x^{-1}\circ f_y} \tag{8.18}$$

<u>Rem. 8.14</u>: (i) $\Pi_z\Gamma_{f_z}^{-1}$ is independent of z

(ii) T^+ forms a so called <u>Hopf algebra</u>. See Hairer (2014).

3. step: Canonical lifts and construction of the renormalisation group $\mathfrak{R}$

Let ξ be a continuous space-time function

$\longrightarrow$ canonical lif $\mathcal{L}\xi$ of ξ is defined as

$$\mathcal{L}\xi = (\Pi, \Gamma), \tag{8.19}$$

where $(\Pi_z \mathbb{1}^*)(\bar{z}) = \xi(\bar{z})$, $(\Pi_z X^{1*,k})(\bar{z}) = (\bar{z}-z)^k$

and then recursively by

$$(\Pi_z \tau^* \bar{\tau}^*)(\bar{z}) = (\Pi_z \tau^*)(\bar{z})\,(\Pi_z \bar{\tau}^*)(\bar{z})$$

$\longrightarrow$ $\mathcal{L}\xi$ admissible model with Γ_{xy} as in (8.18)

$\xrightarrow{\text{Rem. 7.13 (ii)}}$

$$\mathcal{R}(f^* g^*) = \mathcal{R}(f^*) \cdot \mathcal{R}(g^*) \tag{8.20}$$

as long as $f^* g^*, f^*, g^* \in \mathcal{D}^\gamma$ for some $\gamma > 0$

Further, because of (8.8) we have that

$$\mathcal{R}\mathcal{K} f^* = K * \mathcal{R} f^*$$

$\xrightarrow[\text{both sides of (8.9)}]{\mathcal{R}\text{ applied to}}$ $h := \mathcal{R}H^*$ solution to the KPZ eq.

Rem.: $(h_0, \mathcal{L}\xi) \longmapsto h$ continuous (in some topology)

$\longrightarrow$ natural question: If $\xi^\varepsilon \longrightarrow \xi$ for smooth ξ^ε, does $\mathcal{L}\xi^\varepsilon$ converge to an admissibe model yielding a solution to the KPZ eq.?

$\longrightarrow$ answere: no

In order to overcome this problem one can construct a renormalization group $\mathfrak{R}$ of (continuous) transformations on the space of admissible models, from which one can select transformations M_ε s.t.

$$M_\varepsilon \mathcal{L}\xi^\varepsilon \longrightarrow (\hat{\Pi}, \hat{\Gamma})$$

w.r.t. a canonical distance in the space of models $\mathcal{M}$ (i.e. given by the "norms" $\|\Pi\|_{\gamma,K} := \sup_{\beta < \gamma \in A} \sup_{\tau^* \in T_\beta} \sup_{x \in K, \lambda \in (0,1]} \frac{|\Pi_x \tau^*(\varphi_x^\lambda)|}{\lambda^\beta \|\tau^*\|_\beta}$)

$\longrightarrow$ <u>Prop. 8.15</u>: For all linear maps $M: T \to T$ commuting with $\mathcal{J}$ and multiplication with $X^{k,*}$, there ex. unique linear maps $\Delta^M : T \to T \otimes T^+$ and $\hat{\Delta}^M : T^+ \to T^+ \otimes T^+$ s.t. the maps Π_x^M and Γ_{xy}^M defined by

$$\Pi_x^M \hat{\tau}^* = (\Pi_x \otimes f_x)\Delta^M \hat{\tau}^* \qquad \Gamma_{xy}^M(\hat{\tau}^*) = \Gamma_{\gamma_{xy}^M}(\hat{\tau}^*)$$

where

$$\gamma_{xy}^M := ((f_x^{-1} \circ f_y) \otimes f_x)\hat{\Delta}^M \quad (8.16)'$$

satisfy Def. 8.11 (i), (ii) and $\Pi_x^M \Gamma_{xy}^M = \Pi_y^M$

Def. 7.8 $\longrightarrow$ <u>Def. 8.16</u> (Renormalization group R)
The <u>renormalization group R</u> is the set of all linear maps $M: T \to T$ commuting with $\mathcal{J}$, $\mathcal{J}'$ and with multiplication by $X^{k,*}$ s.t. for all $\hat{\tau}^* \in T_\alpha$:

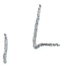
$$\Delta^M \hat{\tau}^* - \hat{\tau}^* \otimes 1 \in T_{>\alpha} \otimes T^+ \quad (8.21)$$

The actions of R on the space of admissible models are given by Prop. 8.15.

Consider now the family of maps from T to T given by

$$M = \exp\Big(-\sum_{i=0}^{3} \underbrace{c_i}_{\text{constants}} L_i\Big), \quad (8.22)$$

where $L_i : T \to T$, $i = 0, \ldots, 3$ a linear maps defined as

L_0:  $\mapsto 1^*$, L_1:  $\mapsto 1^*$, L_2:  $\mapsto 1^*$, L_3:  $\mapsto 1^*$ (8.23)

$\longrightarrow$ in $\hat{\tau}^*$ the corresponding symbols are replaced by 1^*

$\longrightarrow$ <u>Prop. 8.17</u>: The family of maps (8.22) forms a 4-dimensional subgroup of R. Further, if (Π, Γ) is an admissible model s.t. $\Pi_x \hat{\tau}^*$ is a continuous function for all $\hat{\tau}^* \in T$, then

$$(\Pi_x M_{\hat{\tau}^*})(x) = (\Pi_x M \hat{\tau}^*)(x)$$

same x

→ <u>Prop. 15.12</u> (Renomalized KPZ-equation)
Let $M = \exp(-\sum_{i=0}^{3} c_i L_i)$ as in (8.22) and (Π^M, Γ^M) as in Prop. 8.15 w.r.t. the canonical model lift $L\xi$ for some smooth function ξ.
Let H^* be the solution to (8.9) w.r.t. (Π^M, Γ^M).
Then $h(t,x) := (\mathcal{R}^M H^*)(t,x)$ solves

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 - 4 c_0 \partial_x h + \xi - (c_1 + c_2 + 4 c_3) \quad (8.24)$$

<u>4. step</u>: One can find constants c_i^ε, $i = 0, \dots, 3$ s.t. $(\Pi^{M^\varepsilon}, \Gamma^{M^\varepsilon})$ converges to a model $(\hat{\Pi}, \hat{\Gamma})$

→ solution concept and result:

<u>Th. 15.13</u> (KPZ equation)
Consider

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + (\partial_x h_\varepsilon)^2 + \xi_\varepsilon - C_\varepsilon,$$

where $\xi_\varepsilon = \delta_\varepsilon * \xi$ with $\delta_\varepsilon(t,x) = \varepsilon^{-3} \rho(\varepsilon^{-2} t, \varepsilon^{-1} x)$ for some smooth and compactly supported function ρ and ξ a space-time white noise.
Then there ex. diverging constants C_ε s.t.

$$h_\varepsilon \xrightarrow[\varepsilon \downarrow 0]{} h$$

in probability.
The limiting process h, which is called a solution to the KPZ equation, is independent of the choice of the mollifier ρ