Optimal design

Arne Bang Huseby and Kristina Rognlien Dahl

University of Oslo, Norway

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Value-at-risk

Let X be some risk, and introduce $S_X(x) = P(X > x)$. The α-level value-at-risk associated with the risk X, denoted by $V_\alpha[X]$, is given by $S_X^{-1}(\alpha)$. More formally, we define:

$$V_\alpha[X] = S_X^{-1}(\alpha) = \inf\{x : P(X > x) \leq \alpha\}. \tag{1}$$

In the special case where X is absolutely continuously distributed, we have:

$$V_\alpha[X] = S_X^{-1}(\alpha) = x \text{ if and only if } P(X > x) = \alpha.$$

More generally, if S_X is strictly decreasing, we have that:

$$V_\alpha[X] = x \text{ if and only if } P(X > x) \leq \alpha \leq P(X \geq x). \tag{2}$$

Finally, if X is a discrete random variable, we have that:

$$V_\alpha[X] = x \text{ if and only if } P(X > x) \leq \alpha < P(X \geq x). \tag{3}$$

Value-at-risk (cont.)

Proposition (Monotonicity)

For any strictly increasing continuous function ϕ we have:

$$V_\alpha[\phi(X)] = S^{-1}_{\phi(X)}(\alpha) = \phi(S_X^{-1}(\alpha)) \tag{4}$$

PROOF: We note that since ϕ is strictly increasing, it follows by (1) that:

$$\begin{aligned} V_\alpha[\phi(X)] &= \inf\{y : P(\phi(X) > y) \leq \alpha\} \\ &= \inf\{y : P(X > \phi^{-1}(y)) \leq \alpha\}. \end{aligned}$$

We then substitute $y = \phi(x)$ and $\phi^{-1}(y) = x$, and get:

$$\begin{aligned} V_\alpha[\phi(X)] &= \inf\{\phi(x) : P(X > x) \leq \alpha\} \\ &= \phi(\inf\{x : P(X > x) \leq \alpha\}) \\ &= \phi(S_X^{-1}(\alpha)). \end{aligned}$$

Value-at-risk (cont.)

Corollary (Linearity)

For $a > 0$ and $b \in \mathbb{R}$ we have:

$$V_\alpha[aX + b] = aV_\alpha[X] + b.$$

PROOF: The result follows directly from the monotonicity property by noting that:

$$\phi(X) = aX + b$$

is a strictly increasing function for all $a > 0$ and $b \in \mathbb{R}$.

Value-at-risk and optimal design

Let $\boldsymbol{V} = (V_1, \ldots, V_n) \in \mathcal{V}$ be a vector of environmental variables and let $\alpha \in (0, 1)$ be a given probability representing an acceptable level of risk. We assume that we have determined a function $C(\boldsymbol{u})$ defined for all unit vectors $\boldsymbol{u} \in \mathbb{R}^n$ such that:

$$P[\boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})] = \alpha, \text{ for all } \boldsymbol{u} \in \mathbb{R}^n. \tag{5}$$

We also introduce the following notation:

$$\Pi(\boldsymbol{u}) = \{\boldsymbol{V} \in \mathcal{V} : \boldsymbol{u}'\boldsymbol{V} = C(\boldsymbol{u})\},$$

$$\Pi^+(\boldsymbol{u}) = \{\boldsymbol{V} \in \mathcal{V} : \boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})\},$$

$$\Pi^-(\boldsymbol{u}) = \{\boldsymbol{V} \in \mathcal{V} : \boldsymbol{u}'\boldsymbol{V} \leq C(\boldsymbol{u})\}$$

Hence, we have:

$$P[\boldsymbol{V} \in \Pi^+(\boldsymbol{u})] = P[\boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})] = \alpha, \text{ for all } \boldsymbol{u} \in \mathbb{R}^n. \tag{6}$$

Value-at-risk and optimal design (cont.)

Next we let $\boldsymbol{x} = (x_1, \ldots, x_m)$ be a vector of design variables for a given system representing various parameters such as capacity, thickness, strength etc.

Every design is referred to simply by its corresponding vector of design variables, i.e., $\boldsymbol{x}$. The set of possible designs is denoted by $\mathcal{X}$.

The performance function of a system is denoted by g, and is assumed to be a function of both $\boldsymbol{V}$ and $\boldsymbol{x}$:

$$g = g(\boldsymbol{V}, \boldsymbol{x}).$$

The performance function is used to identify environmental conditions where the system fails. More specifically, the system fails if and only if $g(\boldsymbol{V}, \boldsymbol{x}) > 0$.

Value-at-risk and optimal design (cont.)

The cost of a system failure is denoted by K. We also introduce a deterministic function $\kappa = \kappa(\boldsymbol{x})$ representing the cost of the design $\boldsymbol{x}$, and assume that:

$$\kappa(\boldsymbol{x}) < K \text{ for all } \boldsymbol{x} \in \mathcal{X}.$$

The total cost, denoted H, is then given by:

$$H(\boldsymbol{V}, \boldsymbol{x}) = K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] + \kappa(\boldsymbol{x}).$$

The α-level value-at-risk of a given design, denoted $V_\alpha(H)$, is given by:

$$V_\alpha(H) = S_H^{-1}(\alpha),$$

where $S_H(h) = 1 - F_H(h) = P(H > h)$. Thus, $V_\alpha(H)$ is the $(1 - \alpha)$-percentile of the distribution of H.

Our main objective is to choose a design $\boldsymbol{x}$ so that $V_\alpha(H)$ is minimised.

Value-at-risk and optimal design (cont.)

Since $\kappa(\boldsymbol{x})$ is deterministic, it follows by the linearity of V_α that:

$$V_\alpha[H] = V_\alpha[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]] + \kappa(\boldsymbol{x}).$$

We observe that $K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]$ is a discrete random variable with only two possible values, 0 and K. Its distribution is given by:

$$P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] = K] = P[g(\boldsymbol{V}, \boldsymbol{x}) > 0],$$

$$P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] = 0] = P[g(\boldsymbol{V}, \boldsymbol{x}) \leq 0].$$

By (3) we know that:

$$V_\alpha[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]] = y,$$

if and only if:

$$P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > y] \leq \alpha < P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \geq y]$$

Value-at-risk and optimal design (cont.)

In particular, we have $P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > K] = 0 < \alpha$ implying that:

$$V_\alpha[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]] = K,$$

if and only if:

$$P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \geq K] = P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > \alpha$$

Furthermore, we have $P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \geq 0] = 1 > \alpha$ implying that:

$$V_\alpha[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]] = 0,$$

if and only if:

$$P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > 0] = P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq \alpha$$

Value-at-risk and optimal design (cont.)

Summarising this we get:

$$V_\alpha(K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]) = \begin{cases} K & \text{if } P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > \alpha \\ 0 & \text{if } P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq \alpha \end{cases}$$

From this it follows that:

$$V_\alpha(H) = \begin{cases} K + \kappa(\boldsymbol{x}) & \text{if } P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > \alpha \\ \kappa(\boldsymbol{x}) & \text{if } P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq \alpha \end{cases}$$

Since we have assumed that $\kappa(\boldsymbol{x}) < K$ for all $\boldsymbol{x} \in \mathcal{X}$, it follows that an optimal design $\boldsymbol{x}$ must be chosen so that:

$$P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq \alpha \tag{7}$$

Value-at-risk and optimal design (cont.)

Theorem (Halfspace condition)

A sufficient condition for (7) to hold is that $g(\boldsymbol{V}, \boldsymbol{x}) \leq 0$ for all $\boldsymbol{V}$ such that $\boldsymbol{u}'\boldsymbol{V} \leq C(\boldsymbol{u})$, where $\boldsymbol{u} \in \mathbb{R}^n$ is a suitably chosen unit vector.

PROOF: The condition implies that if $g(\boldsymbol{V}, \boldsymbol{x}) > 0$, then $\boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})$.

Hence, by (5) we get that:

$$P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq P[\boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})] = \alpha.$$

Hence, we conclude that (7) is satisfied.

Value-at-risk and optimal design (cont.)

We then let $\boldsymbol{u} \in \mathbb{R}^n$ be a unit vector and consider the following subclass of designs:

$$\mathcal{X}(\boldsymbol{u}) = \{\boldsymbol{x} \in \mathcal{X} : g(\boldsymbol{V}, \boldsymbol{x}) \leq 0 \text{ for all } \boldsymbol{V} \in \Pi^{-}(\boldsymbol{u})\}.$$

By the halfspace condition theorem we know that the condition (7) is satisfied for all designs $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{u})$.

Hence, an optimal design within the subclass $\mathcal{X}(\boldsymbol{u})$ can be found by minimising $\kappa(\boldsymbol{x})$ with respect to $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{u})$.

Different choices of the unit vector $\boldsymbol{u}$ will generate different optimal designs. However, the choice of $\boldsymbol{u}$ may often be a result of initial concept decisions related to the system of interest. Thus, it may not be necessary to consider multiple subclasses of design.

Example: Structural reliability

We consider a system whose performance depends on the non-negative environmental variables, $\boldsymbol{V} = (V_1, \ldots, V_n) \in \mathcal{V}$. The system fails if:

$$A\boldsymbol{V} > \boldsymbol{x}$$

where $A = A^{m \times n}$ is a matrix, and the design $\boldsymbol{x} = (x_1, \ldots, x_m)$ is a vector of *strengths*.

The cost of the design $\boldsymbol{x}$ is given by:

$$\kappa(\boldsymbol{x}) = c_1 x_1 + \cdots + c_m x_m.$$

We want to minimise $\kappa(\boldsymbol{x})$ subject to $P[A\boldsymbol{V} > \boldsymbol{x}] \leq \alpha$. Since this failure probability may be difficult to compute, we instead minimise $\kappa(\boldsymbol{x})$ subject to:

$$\{\boldsymbol{V} \in \mathcal{V} : A\boldsymbol{V} > \boldsymbol{x}\} \subseteq \{\boldsymbol{V} \in \mathcal{V} : \boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})\}. \tag{8}$$

Example: Structural reliability

It follows that if the design $\boldsymbol{x}$ satisfies (8), then:

$$P[A\boldsymbol{V} > \boldsymbol{x}] \leq P[\boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})] = \alpha.$$

For a given design $\boldsymbol{x}$, we can then check if it satisfies (8) by solving the following LP-problem:

$$\text{Minimise } \boldsymbol{u}'\boldsymbol{V} \text{ subject to } A\boldsymbol{V} \geq \boldsymbol{x}. \tag{9}$$

Let $\boldsymbol{V}_0$ denote the solution to (9). Then $\boldsymbol{x}$ satisfies (8) if and only if:

$$\boldsymbol{u}'\boldsymbol{V}_0 > C(\boldsymbol{u}).$$

By using a suitable iteration method one can then find a design $\boldsymbol{x}$ which minimises $\kappa(\boldsymbol{x})$ subject to (8).