

# Multistate systems and importance measures

Arne Bang Huseby  
University of Oslo, Norway

Martyna Kalinowska  
University of Oslo, Norway

Tobias Abrahamsen  
University of Oslo, Norway

## Abstract

We propose a new framework for modelling repairable multistate systems emphasising physical properties of components and systems. The Birnbaum measure is generalised and characterised along two axes: *forward-looking* versus *backward-looking*, and with respect to how criticality is measured. Forward-looking importance measures focus on the *next* component states: The most important component has the highest probability of changing the system state. Backward-looking importance measures focus on the *previous* component states: The most important component has the highest probability of having changed the system state. Moreover, two approaches to measuring criticality are considered: probability of criticality versus impact of criticality. Examples show that the chosen importance measure affects the ranking of the components. Both periodic life cycles as more general semi-Markov processes are considered.

SYSTEM RELIABILITY; REPAIRABLE MULTISTATE SYSTEMS; IMPORTANCE MEASURES

## 1 Introduction

One of the main problems in reliability theory is to find out how the reliability of a complex system can be determined from knowledge of the reliabilities of its components. A weakness of traditional binary reliability theory is that the system and the components are always described just as functioning or failed. This approach often represents an oversimplification in real-life situations where the system and their components are capable of assuming a whole range of levels of performance, varying from perfect functioning to complete failure.

Basic reliability theory for multistate systems was established in the mid 1980s, and has been developed continuously. In particular, many different tools for analysing the importance of components in multistate systems have been developed. There are two main reasons calculating importance of components in a system. Firstly, it permits the analyst to determine which components merit the most additional research and development to improve overall system reliability at minimum cost or effort. Secondly, it may be used in diagnostics as a way of generating a list of components ordered with respect to how likely they are to have caused the system failure.

It should be noted that no measure of importance can be expected to be universally best irrespective of usage purpose. Still comparing different measures is often of interest. The classical approaches to importance measures include Birnbaum, Barlow-Prochan and

Natvig measures of component importance, including the dual extensions of the latter measures. The measures are treated in details in Huseby and Natvig (2010), Huseby and Natvig (2012), Natvig and Gåsemeyr (2009), Natvig (2011a), Natvig (2011b), Natvig et al. (2011). Furthermore, a number of applications have been proposed, e.g in Natvig et al. (2009) the extended Natvig measure is applied to an offshore oil and gas production system. In particular, the Birnbaum measure of component importance provides a dynamic approach to determining importance of the components in the system. The Birnbaum measure of a given component is defined as the probability that such component is critical to the functioning of the system. Furthermore, time-independent Barlow-Proschan measure of importance can be expressed as a weighted average of the Birnbaum measure with respect to the component lifetime densities. The Barlow-Proschan measure implies that components with long lifetimes compared to the system lifetime will have a large reliability importance. An alternative measure can be defined by instead focussing on components which greatly reduce the remaining system lifetime by failing. This is reflected by the Natvig measures.

Recent work in this area includes generalisations of some well-established binary measures of component importance to multistate repairable systems. Importance measures of components in a multistate system have been studied from some new perspectives based on various real-life situations. E.g., a cost-based importance measure, as an extension of Birnbaum measure has been proposed in Wu and Coolen (2013). It is pointed out that existing importance measures have paid little attention to the costs incurred by maintaining a system and its components. This paper considers costs of improving component reliability, costs due to component failure and cost of system failure, and provides possible extensions and applications of importance measures.

Si et al. (2013) introduces an importance measure for multistate systems which is used to identify weak components which affect the reliability of a system. Traditionally, importance measures do not consider how transition rates between different component states may affect the system. The paper describe in which state a component should be kept in order to ensure a desired level of system performance. An application to an oil transportation system is presented to illustrate how the suggested importance measure can be used.

Finally, Wu et al. (2016) introduces a component maintenance priority measure. Performing preventive maintenance on a component in a system may affect system availability. To avoid such a reduction on availability, one may adopt the following method: if a component fails, preventive maintenance is carried out on a number of the other components while the failed component is being repaired. The importance measure can be used to select components for the preventive maintenance.

In the present paper we propose a new framework for modelling repairable multistate systems emphasising physical properties of components and systems. Within this framework we suggest four different generalisations of the classical Birnbaum measure. All these are true generalisations in the sense that they are reduced to the classical measure in the binary case. The measures are characterised along two axes: *forward-looking* versus *backward-looking*, and with respect to how criticality is measured. Forward-looking importance measures focus on the *next* component states. According to this approach the most important component is the one that has the highest probability of changing the system state. Backward-looking importance measures focus on the *previous* component states: According to this approach the most important component is the one that has the highest probability of having changed the system state. Furthermore, two approaches to measuring criticality are considered: probability of criticality versus impact of criticality.

In the first sections of the paper each component follows periodic life cycles, starting out

in the top state, and then transiting through the lower states until they fail. Then they are repaired or replaced, and a new life cycle starts. In the second part of the paper the life cycles of the components are modelled using general semi-Markov processes.

## 2 Binary systems

Before we introduce a framework for multistate systems, we review some basic concepts from the binary case. Assume that  $(C, \phi)$  is a binary system, where  $C = \{1, \dots, n\}$  is the component set, and  $\phi$  is the *structure function*. Moreover, let  $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))$  where  $X_i(t)$  represents the state of component  $i \in C$  at time  $t$ . Thus,  $X_i(t) = 1$  if component  $i$  is functioning at time  $t$ , and zero otherwise. We assume that all the components start out as functioning. When a component fails, it is either repaired back to its functioning state or replaced by a new component, and a new life cycle begins.

The structure function  $\phi = \phi(\mathbf{X}(t))$  represents the state of the system at time  $t$ . That is,  $\phi(\mathbf{X}(t)) = 1$  if the system is functioning at time  $t$ , and zero otherwise. According to standard conventions from reliability theory, the notation  $\phi(x_i, \mathbf{X})$  is interpreted as:

$$\phi(x_i, \mathbf{X}) = \phi(X_1, \dots, X_{i-1}, x_i, X_{i+1}, \dots, X_n).$$

In cases where the index  $i$  is obvious from the context, we simply write  $\phi(x, \mathbf{X})$ .

A component,  $i \in C$ , is said to be *critical* for the system at time  $t$  if:

$$\phi(0_i, \mathbf{X}(t)) \neq \phi(1_i, \mathbf{X}(t)). \quad (2.1)$$

The condition Eq. (2.1) can be rewritten in a way that makes it easy to extend it to multistate systems. We let  $X_i^+(t)$  and  $X_i^-(t)$  denote respectively the *next* and *previous* state of component  $i$ ,  $i = 1, \dots, n$ . Obviously, in a binary system each component only has two possible states. Thus, in such cases we must have:

$$X_i^+(t) = X_i^-(t) = \begin{cases} 0 & \text{for } X_i(t) = 1 \\ 1 & \text{for } X_i(t) = 0 \end{cases} \quad (2.2)$$

In systems where the components may have more than two states, however, we typically have that  $X_i^+(t) \neq X_i^-(t)$ . Using the above notation, we may write Eq. (2.1) as:

$$\phi(X_i(t), \mathbf{X}(t)) \neq \phi(X_i^+(t), \mathbf{X}(t)). \quad (2.3)$$

or alternatively:

$$\phi(X_i^-(t), \mathbf{X}(t)) \neq \phi(X_i(t), \mathbf{X}(t)). \quad (2.4)$$

Hence, component  $i$  is critical at time  $t$  if changing the component to its next state would result in a system state change as well. Alternatively, component  $i$  is critical at time  $t$  if changing the component to its previous state would result in a system state change as well.

According to Birnbaum (1969) the Birnbaum measure of importance of component  $i \in C$  at time  $t$ , denoted  $I_B^{(i)}(t)$ , is the probability that the component is critical at time  $t$ . Using Eq. (2.3) or Eq. (2.4) this implies that we have:

$$\begin{aligned} I_B^{(i)}(t) &= P[\phi(X_i(t), \mathbf{X}(t)) \neq \phi(X_i^+(t), \mathbf{X}(t))] \\ &= P[\phi(X_i^-(t), \mathbf{X}(t)) \neq \phi(X_i(t), \mathbf{X}(t))] \end{aligned} \quad (2.5)$$

### 3 Multistate systems

For an extensive introduction to multistate systems we refer to Natvig (2011a). In the present paper we define a multistate system similar to a binary system as an ordered pair  $(C, \phi)$ , where  $C = \{1, \dots, n\}$  is the component set, and  $\phi$  is the structure function. Moreover, we let  $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))$  where  $X_i(t)$  is the state variable of component  $i$  at time  $t$ . Contrary to a binary system, however, both the components and the system may be in multiple states (not just 0 and 1). More specifically, if  $i \in C$ , we let  $S_i = \{0, 1, \dots, r_i\}$  denote the set of states for component  $i$ .

We assume that each component starts out by being in its top-level state, i.e., state  $r_i$  for component  $i$ . Then at random points of time the component transits downwards through the state set until it reaches state 0. At this stage the component is repaired or replaced by a new component, and a new life cycle starts. In Section 4 we consider more general semi-Markov processes.

For each component  $i \in C$  we also introduce a function  $f_i : S_i \rightarrow \mathbb{R}$  representing the *physical state* of the component as a function of the state. Thus, if  $X_i(t) = x_i \in S_i$ , then the physical state of component  $i$  at time  $t$  is  $f_i(X_i(t)) = f_i(x_i)$ . If e.g., component  $i$  is a pipeline, then the physical state of the component at a given point of time may be the capacity of the pipeline at this point of time. Being a physical property of the pipeline, this may be any non-negative number, and the function  $f_i$  provides a convenient way of encoding this directly into the model.

Note that the functions  $f_1, \dots, f_n$  do not necessary need to be nondecreasing. By avoiding this restriction additional useful modeling flexibility is gained. This allows e.g., for the inclusion of burn-in phases, maintenance as well as minimal or partial repairs of a components as part of its life cycle before it reaches its failure state.

The structure function  $\phi$  represents the state of the system expressed as a function of the states of the components. It is common in multistate reliability theory to assume that  $\phi$  also assumes values in a set of non-negative integers. In this context, however, we let the structure function represent the physical state of the system and that:

$$\phi(\mathbf{X}(t)) = \phi(f_1(X_1(t)), \dots, f_n(X_n(t)))$$

Thus, the physical state of the system is a function of the physical states of the components. The advantage with this approach is that the system state is expressed in terms of physical quantities rather than being encoded more abstractly as non-negative integers.

#### 3.1 Criticality and importance in multistate systems

In order to extend the definition of criticality and importance we consider a multistate system  $(C, \phi)$ . Let  $i \in C$ , and let  $S_i = \{0, 1, \dots, r_i\}$  be the set of states for this component. We then introduce  $X_i^+(t)$  and  $X_i^-(t)$  as respectively the *next* and *previous* state of component  $i$ ,  $i = 1, \dots, n$ . With more than two possible states for each component, the mathematical expressions for  $X_i^+(t)$  and  $X_i^-(t)$  are different. More specifically, we have:

$$X_i^+(t) = \begin{cases} X_i(t) - 1 & \text{for } X_i(t) > 0 \\ r_i & \text{for } X_i(t) = 0 \end{cases}$$

$$X_i^-(t) = \begin{cases} X_i(t) + 1 & \text{for } X_i(t) < r_i \\ 0 & \text{for } X_i(t) = r_i \end{cases}$$

Based on  $X_i^+(t)$  and  $X_i^-(t)$  we introduce two notions of criticality. We say that component  $i$  is *n-critical* at time  $t$  if:

$$\phi(X_i(t), \mathbf{X}(t)) \neq \phi(X_i^+(t), \mathbf{X}(t)). \quad (3.1)$$

Hence, component  $i$  is n-critical at time  $t$  if changing the component to its *next* state would result in a system state change as well. Similarly we say that component  $i$  is *p-critical* at time  $t$  if:

$$\phi(X_i^-(t), \mathbf{X}(t)) \neq \phi(X_i(t), \mathbf{X}(t)). \quad (3.2)$$

Hence, component  $i$  is p-critical at time  $t$  if changing the component to its *previous* state would result in a system state change as well.

We then proceed by introducing two possible generalisations of the Birnbaum measure to multistate systems. We define the *n-Birnbaum measure* of importance of component  $i$  at time  $t$ , denoted  $I_{NB}^{(i)}(t)$ , as the probability that the component is n-critical at time  $t$ . Similarly, we define the *p-Birnbaum measure* of importance of component  $i$  at time  $t$ , denoted  $I_{PB}^{(i)}(t)$ , as the probability that the component is p-critical at time  $t$ . Hence, using Eq. (3.1) and Eq. (3.2) we get:

$$I_{NB}^{(i)}(t) = P[\phi(X_i(t), \mathbf{X}(t)) \neq \phi(X_i^+(t), \mathbf{X}(t))], \quad (3.3)$$

$$I_{PB}^{(i)}(t) = P[\phi(X_i^-(t), \mathbf{X}(t)) \neq \phi(X_i(t), \mathbf{X}(t))]. \quad (3.4)$$

Note that by Eq. (2.2) it follows that  $I_{NB}^{(i)}(t) = I_{PB}^{(i)}(t) = I_B^{(i)}(t)$  in the binary case. In the multistate case, however, we may have  $I_{NB}^{(i)}(t) \neq I_{PB}^{(i)}(t)$ . In order to take a closer look at the difference between the two importance measures, we rewrite the expressions Eq. (3.3) and Eq. (3.4) by conditioning on the state of component  $i$  and get:

$$\begin{aligned} I_{NB}^{(i)}(t) &= \sum_{u=1}^{r_i} P[\phi(u, \mathbf{X}(t)) \neq \phi(u-1, \mathbf{X}(t))] \cdot P[X_i(t) = u] \\ &\quad + P[\phi(0, \mathbf{X}(t)) \neq \phi(r_i, \mathbf{X}(t))] \cdot P[X_i(t) = 0] \end{aligned} \quad (3.5)$$

$$\begin{aligned} I_{PB}^{(i)}(t) &= \sum_{u=0}^{r_i-1} P[\phi(u+1, \mathbf{X}(t)) \neq \phi(u, \mathbf{X}(t))] \cdot P[X_i(t) = u] \\ &\quad + P[\phi(0, \mathbf{X}(t)) \neq \phi(r_i, \mathbf{X}(t))] \cdot P[X_i(t) = r_i] \end{aligned} \quad (3.6)$$

Changing the summation index in Eq. (3.6) this expression becomes:

$$\begin{aligned} I_{PB}^{(i)}(t) &= \sum_{u=1}^{r_i} P[\phi(u, \mathbf{X}(t)) \neq \phi(u-1, \mathbf{X}(t))] \cdot P[X_i(t) = u-1] \\ &\quad + P[\phi(0, \mathbf{X}(t)) \neq \phi(r_i, \mathbf{X}(t))] \cdot P[X_i(t) = r_i] \end{aligned} \quad (3.7)$$

Comparing Eq. (3.5) and Eq. (3.7) we observe that  $P[X_i(t) = u]$  in Eq. (3.5) is replaced by  $P[X_i(t) = u-1]$  in Eq. (3.7),  $u = 1, \dots, r_i$ . Moreover,  $P[X_i(t) = 0]$  in Eq. (3.5) is replaced by  $P[X_i(t) = r_i]$  in Eq. (3.7). From this it follows that if  $P[X_i(t) = 0] = P[X_i(t) = 1] = \dots = P[X_i(t) = r_i]$ , we will have  $I_{NB}^{(i)}(t) = I_{PB}^{(i)}(t)$ . In general, however, the two importance measures will be different.

**Example 3.1** We consider a multistate system  $(C, \phi)$  where  $C = \{1, 2\}$ . Both components have three possible states, 0, 1, 2. For simplicity, we assume that these states are identical to the physical states, i.e.,  $f_i(u) = u$ ,  $u = 0, 1, 2$  and  $i = 1, 2$ . Moreover, the structure function is given by:

$$\phi(X_1(t), X_2(t)) = \min(f_1(X_1(t)), f_2(X_2(t))).$$

For a given  $t$  we assume that  $P[X_1(t) = u] = p_u$  and  $P[X_2(t) = u] = q_u$ ,  $u = 0, 1, 2$ . It is then easy to see that:

$$\begin{aligned} P[\phi(0, X_2(t)) \neq \phi(2, X_2(t))] &= q_1 + q_2, \\ P[\phi(1, X_2(t)) \neq \phi(0, X_2(t))] &= q_1 + q_2, \\ P[\phi(2, X_2(t)) \neq \phi(1, X_2(t))] &= q_2, \\ P[\phi(X_1(t), 0) \neq \phi(X_1(t), 2)] &= p_1 + p_2, \\ P[\phi(X_1(t), 1) \neq \phi(X_1(t), 0)] &= p_1 + p_2, \\ P[\phi(X_1(t), 2) \neq \phi(X_1(t), 1)] &= p_2. \end{aligned}$$

Inserting this into Eq. (3.5) and Eq. (3.7) we get after some simplifications that:

$$\begin{aligned} I_{NB}^{(1)}(t) &= q_1 + q_2 - p_2q_1, \\ I_{NB}^{(2)}(t) &= p_1 + p_2 - p_1q_2, \\ I_{PB}^{(1)}(t) &= q_1 + q_2 - p_1q_1, \\ I_{PB}^{(2)}(t) &= p_1 + p_2 - p_1q_1. \end{aligned}$$

We observe that  $I_{PB}^{(1)}(t) > I_{PB}^{(2)}(t)$  if and only if  $q_1 + q_2 > p_1 + p_2$ . However, assuming that  $q_1 + q_2 > p_1 + p_2$  and at the same time choosing  $p_1 < p_2$  and  $q_1 > q_2$ , it is possible to obtain the opposite ranking with respect to the  $n$ -Birnbbaum measure.

Assume e.g., that  $p_1 = 0.20$ ,  $p_2 = 0.35$ ,  $q_1 = 0.40$  and  $q_2 = 0.20$ . We then get  $I_{NB}^{(1)}(t) = 0.46$  while  $I_{NB}^{(2)}(t) = 0.51$ . Furthermore,  $I_{PB}^{(1)}(t) = 0.52$  while  $I_{PB}^{(2)}(t) = 0.47$ . That is,  $I_{NB}^{(1)}(t) < I_{NB}^{(2)}(t)$  while  $I_{PB}^{(1)}(t) > I_{PB}^{(2)}(t)$  ■

**Example 3.2** In this example we also consider a multistate system  $(C, \phi)$  where  $C = \{1, 2\}$ , and where both components have three possible states, 0, 1, 2. As in the previous example we let  $f_1(u) = u$ ,  $u = 0, 1, 2$ . However, in this case we let  $f_2(0) = 0$ ,  $f_2(1) = 2$  and  $f_2(2) = 1$ . Thus, component 2 starts out in its intermediate physical state 1 before its best physical state 2 is reached. The structure function is once again:

$$\phi(X_1(t), X_2(t)) = \min(f_1(X_1(t)), f_2(X_2(t))),$$

and for a given  $t$  we let  $P[X_1(t) = u] = p_u$  and  $P[X_2(t) = u] = q_u$ ,  $u = 0, 1, 2$ . We now get:

$$\begin{aligned} P[\phi(0, X_2(t)) \neq \phi(2, X_2(t))] &= q_1 + q_2, \\ P[\phi(1, X_2(t)) \neq \phi(0, X_2(t))] &= q_1 + q_2, \\ P[\phi(2, X_2(t)) \neq \phi(1, X_2(t))] &= q_1, \\ P[\phi(X_1(t), 0) \neq \phi(X_1(t), 2)] &= p_1 + p_2, \\ P[\phi(X_1(t), 1) \neq \phi(X_1(t), 0)] &= p_1 + p_2, \\ P[\phi(X_1(t), 2) \neq \phi(X_1(t), 1)] &= p_2. \end{aligned}$$

Inserting this into Eq. (3.5) and Eq. (3.7) we get:

$$\begin{aligned} I_{NB}^{(1)}(t) &= q_1 + q_2 - p_2 q_2, \\ I_{NB}^{(2)}(t) &= p_1 + p_2 - p_1 q_2, \\ I_{PB}^{(1)}(t) &= q_1 + q_2 - p_1 q_2, \\ I_{PB}^{(2)}(t) &= p_1 + p_2 - p_1 q_1. \end{aligned}$$

We now assume that  $P[f_1(X_1(t)) = u] = P[f_2(X_2(t)) = u]$ ,  $u = 0, 1, 2$ . This implies that  $p_2 = q_1$  and  $p_1 = q_2$ . Focussing first on the  $n$ -Birnbbaum measure, it follows that:

$$I_{NB}^{(1)}(t) > I_{NB}^{(2)}(t) \quad \text{if and only if} \quad p_2 < p_1.$$

Assume more specifically that  $p_2 = q_1 = 0.3$  and  $p_1 = q_2 = 0.4$ . We then get:

$$I_{NB}^{(1)}(t) = 0.58 \quad \text{and} \quad I_{NB}^{(2)}(t) = 0.54.$$

Thus, even though  $f_1(X_1(t))$  and  $f_2(X_2(t))$  has the same distribution, and the structure function is symmetric with respect to  $f_1$  and  $f_2$ , component 1 has higher importance than component 2 when the  $n$ -Birnbbaum measure is used. If instead  $p_2 > p_1$ , the ranking is reversed. Similar results hold for the  $p$ -Birnbbaum measure ■

### 3.2 Importance based on expected physical criticality

The importance measures introduced so far are based on a binary notion of criticality: either a component is critical, or it is *not* critical. At the same time our definition of the structure function  $\phi$  is assumed to represent some physical quantity. In this section we utilise this as a basis for two new importance measures. More specifically, for a given multistate system  $(C, \phi)$  we introduce  $\Delta_N^{(i)}(t)$  and  $\Delta_P^{(i)}(t)$  defined for  $t > 0$  and  $i \in C$  as:

$$\begin{aligned} \Delta_N^{(i)}(t) &= |\phi(X_i(t), \mathbf{X}(t)) - \phi(X_i^+(t), \mathbf{X}(t))| \\ \Delta_P^{(i)}(t) &= |\phi(X_i(t)^-, \mathbf{X}(t)) - \phi(X_i(t), \mathbf{X}(t))| \end{aligned}$$

We observe that  $\Delta_N^{(i)}(t)$  measures the absolute value of the change in system state as a result of component  $i$  changing from its current state  $X_i(t)$  to its next state  $X_i^+(t)$ . Similarly,  $\Delta_P^{(i)}(t)$  measures the absolute value of the change in system state as a result of component  $i$  changing from its previous state  $X_i^-(t)$  to its current state  $X_i(t)$ .

The  $n^*$ -Birnbbaum measure of importance of component  $i$  at time  $t$ , denoted  $I_{NB}^{*(i)}(t)$ , is defined as the expected value of  $\Delta_N^{(i)}(t)$ :

$$I_{NB}^{*(i)}(t) = E |\phi(X_i(t), \mathbf{X}(t)) - \phi(X_i^+(t), \mathbf{X}(t))|. \quad (3.8)$$

Similarly, the  $p^*$ -Birnbbaum measure of importance of component  $i$  at time  $t$ , denoted  $I_{PB}^{*(i)}(t)$ , is defined as the expected value of  $\Delta_P^{(i)}(t)$ :

$$I_{PB}^{*(i)}(t) = E |\phi(X_i^-(t), \mathbf{X}(t)) - \phi(X_i(t), \mathbf{X}(t))|. \quad (3.9)$$

Note that in the binary case all the different measures are the same:

$$I_{NB}^{(i)}(t) = I_{PB}^{(i)}(t) = I_{NB}^{*(i)}(t) = I_{PB}^{*(i)}(t) = I_B^{(i)}(t).$$

In order to show how these measures can be computed, we expand the expressions Eq. (3.8) and Eq. (3.9) by conditioning on the state of component  $i$  and get:

$$\begin{aligned} I_{NB}^{*(i)}(t) &= \sum_{u=1}^{r_i} E|\phi(u, \mathbf{X}(t)) - \phi(u-1, \mathbf{X}(t))| \cdot P[X_i(t) = u] \\ &\quad + E|\phi(0, \mathbf{X}(t)) - \phi(r_i, \mathbf{X}(t))| \cdot P[X_i(t) = 0] \end{aligned} \quad (3.10)$$

$$\begin{aligned} I_{PB}^{*(i)}(t) &= \sum_{u=0}^{r_i-1} E|\phi(u+1, \mathbf{X}(t)) - \phi(u, \mathbf{X}(t))| \cdot P[X_i(t) = u] \\ &\quad + E|\phi(0, \mathbf{X}(t)) - \phi(r_i, \mathbf{X}(t))| \cdot P[X_i(t) = r_i] \end{aligned} \quad (3.11)$$

Again we change the summation index in Eq. (3.11) and get:

$$\begin{aligned} I_{PB}^{(i)}(t) &= \sum_{u=1}^{r_i} E|\phi(u, \mathbf{X}(t)) - \phi(u-1, \mathbf{X}(t))| \cdot P[X_i(t) = u-1] \\ &\quad + E|\phi(0, \mathbf{X}(t)) - \phi(r_i, \mathbf{X}(t))| \cdot P[X_i(t) = r_i] \end{aligned} \quad (3.12)$$

Using the same arguments as in the previous section, it follows that  $I_{NB}^{(i)}(t) = I_{PB}^{(i)}(t)$  if  $P[X_i(t) = 0] = P[X_i(t) = 1] = \dots = P[X_i(t) = r_i]$ . In general, however, the two importance measures will be different.

**Example 3.3** Consider a multistate system  $(C, \phi)$  where  $C = \{1, 2\}$ , and where  $S_1 = \{0, 1\}$  and  $S_2 = \{0, 1, 2\}$ . Moreover, we assume that:

$$\begin{aligned} f_1(u) &= 2u, \quad u \in S_1, \\ f_2(u) &= u, \quad u \in S_2. \end{aligned}$$

As before, the structure function is given by:

$$\phi(X_1(t), X_2(t)) = \min(f_1(X_1(t)), f_2(X_2(t))).$$

Finally, we again assume that the component state processes are independent, and that for a given  $t$  we have:

$$\begin{aligned} P[X_1(t) = u] &= p_u > 0, \quad u \in S_1, \\ P[X_2(t) = u] &= q_u > 0, \quad u \in S_2. \end{aligned}$$

Noting that component 1 only has two states, we get:

$$E|\phi(1, X_2(t)) - \phi(0, X_2(t))| = E|\phi(0, X_2(t)) - \phi(1, X_2(t))| = q_1 + 2q_2$$

Inserting this into Eq. (3.10) and using that  $p_0 + p_1 = 1$  we get:

$$I_{NB}^{*(1)}(t) = (q_1 + 2q_2)p_1 + (q_1 + 2q_2)p_0 = q_1 + 2q_2$$



For component 2 we have:

$$\begin{aligned} E|\phi(X_1(t), 1) - \phi(X_1(t), 0)| &= p_1 \\ E|\phi(X_1(t), 2) - \phi(X_1(t), 1)| &= p_1 \\ E|\phi(X_1(t), 0) - \phi(X_1(t), 2)| &= 2p_1 \end{aligned}$$

Inserting this into Eq. (3.10) and using that  $q_0 + q_1 + q_2 = 1$  we get:

$$I_{NB}^{*(2)}(t) = p_1q_1 + p_1q_2 + 2p_1q_0 = (1 + q_0)p_1$$

We also have:

$$\begin{aligned} E[f_1(X_1(t))] &= 0 \cdot p_0 + 2 \cdot p_1 = 2p_1 \\ E[f_2(X_2(t))] &= 0 \cdot q_0 + 1 \cdot q_1 + 2 \cdot q_2 = q_1 + 2q_2. \end{aligned}$$

In order to make the comparison between the components as fair as possible, we assume that their distributions are so that  $E[f_1(X_1(t))] = E[f_2(X_2(t))]$ , i.e.,  $2p_1 = q_1 + 2q_2$ . This implies that:

$$\begin{aligned} I_{NB}^{*(1)}(t) &= 2p_1 \\ I_{NB}^{*(2)}(t) &= (1 + q_0)p_1 \end{aligned}$$

Since we have assumed that  $q_1 > 0$  and  $q_2 > 0$ , it follows that  $q_0 < 1$ . Hence, we conclude that even though both components have the same expected performance we can still have  $I_{NB}^{*(1)}(t) > I_{NB}^{*(2)}(t)$  ■

## 4 Importance measures for semi-Markov processes

In the previous sections we assumed that each life cycle of a component was deterministic with respect to the states the component transited through. As a result the *next* and *previous* states at a given point of time, denoted respectively  $X_i^+(t)$  and  $X_i^-(t)$  were both determined with probability one by the current state  $X_i(t)$ . We now relax this assumption, and allow the components to follow a general *semi-Markov* process, where the state transitions follow a Markov chain, referred to as the *built-in Markov chain*. Thus, each time component  $i$  enters a state  $u \in S_i$ , it remains there for a random amount of time, and then makes a transition into state  $v \in S_i$  with probability  $P_{uv}^{(i)}$ . The full matrix of transition probabilities for the built-in Markov chain for component  $i$  is denoted  $\mathbf{P}^{(i)}$ ,  $i \in C$ . Given this matrix we have that:

$$P(X_i^+(t) = v | X_i(t) = u) = P_{uv}^{(i)}, \quad u, v \in S_i. \quad (4.1)$$

In order to find a similar expression for the conditional distribution of  $X_i^-(t)$ , we need the transition matrix for the backwards version of the built-in Markov chain, which we denote by  $\mathbf{Q}^{(i)}$ . It then follows that:

$$P(X_i^-(t) = v | X_i(t) = u) = Q_{uv}^{(i)}, \quad u, v \in S_i. \quad (4.2)$$

Within this more general context the definitions of  $I_{NB}^{(i)}(t)$  and  $I_{PB}^{(i)}(t)$  given in Eq. (3.3) and Eq. (3.4) are still valid. However, the equations Eq. (3.5) and Eq. (3.6) have to be modified as follows:

$$I_{NB}^{(i)}(t) = \sum_{u,v \in S_i} P[\phi(u, \mathbf{X}(t)) \neq \phi(v, \mathbf{X}(t))] \cdot P[X_i(t) = u] \cdot P_{uv}^{(i)} \quad (4.3)$$

$$I_{PB}^{(i)}(t) = \sum_{u,v \in S_i} P[\phi(u, \mathbf{X}(t)) \neq \phi(v, \mathbf{X}(t))] \cdot P[X_i(t) = u] \cdot Q_{uv}^{(i)} \quad (4.4)$$

We henceforth focus on the asymptotic properties of the processes and omit the time  $t$  from the notation. For component  $i \in C$  we denote the stationary probabilities of the built-in Markov chain by  $\pi_u^{(i)}$ ,  $u \in S_i$ . We then have the following well-known relation between the transition matrices  $\mathbf{P}^{(i)}$  and  $\mathbf{Q}^{(i)}$  (see e.g., Ross (2014)):

$$Q_{uv}^{(i)} = \frac{\pi_v^{(i)}}{\pi_u^{(i)}} P_{vu}^{(i)}, \quad u, v \in S_i. \quad (4.5)$$

Note that if the stationary distribution of the built-in Markov chain is *uniform*, i.e., if  $\pi_u^{(i)} = 1/(r_i + 1)$ , for all  $u \in S_i$ , we have:

$$\mathbf{Q}^{(i)} = (\mathbf{P}^{(i)})^T, \quad i \in C.$$

It is well-known that an irreducible aperiodic finite Markov chain has a uniform stationary distribution if and only if  $\mathbf{P}^{(i)}$  is a *doubly stochastic matrix*, i.e., all row sums and column sums are equal to 1.

In order to proceed we now introduce the times spent in each state between the transitions. More specifically, we let:

$$W_{ku}^{(i)} = \text{The } k\text{th waiting time in state } u \text{ for component } i.$$

We assume that all the waiting times are independent, and that for all components  $i \in C$  and states  $u \in S_i$  the waiting times  $W_{1u}^{(i)}, W_{2u}^{(i)}, \dots$  are identically distributed with finite mean  $\mu_u^{(i)}$ . Then it follows from standard renewal theory (see e.g., Ross (2014)) that the stationary distribution of  $X_i$  is given by:

$$P[X_i = u] = \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}}, \quad u \in S_i, i \in C. \quad (4.6)$$

Combining Eq. (4.6) with Eq. (4.3) and Eq. (4.4) we get the following expressions for the stationary importance measures:

$$I_{NB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \mathbf{X}) \neq \phi(v, \mathbf{X})] \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot P_{uv}^{(i)} \quad (4.7)$$

$$I_{PB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \mathbf{X}) \neq \phi(v, \mathbf{X})] \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot Q_{uv}^{(i)} \quad (4.8)$$

Using these formulas it is easy to establish a sufficient condition for when  $I_{NB}^{(i)} = I_{PB}^{(i)}$ .

**Theorem 4.1** Assume that  $\mu_0^{(i)} = \dots = \mu_{r_i}^{(i)}$ . Then  $I_{NB}^{(i)} = I_{PB}^{(i)}$ .

**Proof:** If  $\mu_0^{(i)} = \dots = \mu_{r_i}^{(i)}$ , the stationary distribution given in Eq. (4.6) is simplified to:

$$P[X_i = u] = \pi_u^{(i)}, \quad u \in S_i, i \in C.$$

Inserting this into Eq. (4.12) and Eq. (4.13), we get:

$$\begin{aligned} I_{NB}^{(i)} &= \sum_{u,v \in S_i} P[\phi(u, \mathbf{X}) \neq \phi(v, \mathbf{X})] \cdot \pi_u^{(i)} \cdot P_{uv}^{(i)} \\ I_{PB}^{(i)} &= \sum_{u,v \in S_i} P[\phi(u, \mathbf{X}) \neq \phi(v, \mathbf{X})] \cdot \pi_u^{(i)} \cdot Q_{uv}^{(i)} \end{aligned}$$

We then insert the expression for  $Q_{uv}^{(i)}$  given in Eq. (4.5) into the last equation and get:

$$I_{PB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \mathbf{X}) \neq \phi(v, \mathbf{X})] \cdot \pi_v^{(i)} \cdot P_{vu}^{(i)}$$

Hence, by interchanging  $u$  and  $v$  we get that  $I_{NB}^{(i)} = I_{PB}^{(i)}$  as stated ■

Another special case occurs when the transition matrix  $\mathbf{P}^{(i)}$  is doubly stochastic.

**Theorem 4.2** Assume that the transition matrix  $\mathbf{P}^{(i)}$  is doubly stochastic. Then we have:

$$\begin{aligned} I_{NB}^{(i)} &= \sum_{u,v \in S_i} P[\phi(u, \mathbf{X}) \neq \phi(v, \mathbf{X})] \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{uv}^{(i)} \\ I_{PB}^{(i)} &= \sum_{u,v \in S_i} P[\phi(u, \mathbf{X}) \neq \phi(v, \mathbf{X})] \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{vu}^{(i)} \end{aligned}$$

**Proof:** If the transition matrix  $\mathbf{P}^{(i)}$  is doubly stochastic, it is easy to see that the stationary distribution of the built-in Markov chain is uniform. Hence, the stationary distribution given in Eq. (4.6) is simplified to:

$$P[X_i = u] = \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}}, \quad u \in S_i, i \in C. \quad (4.9)$$

Moreover, the transition matrix  $\mathbf{Q}^{(i)}$  is equal to  $(\mathbf{P}^{(i)})^T$ . That is:

$$Q_{uv}^{(i)} = P_{vu}^{(i)}, \quad \text{for all } u, v \in S_i.$$

By combining these observations the stated result follows ■

We observe that by Eq. (4.12) and Eq. (4.13)  $I_{NB}^{(i)}$  and  $I_{PB}^{(i)}$  depends both on the stationary distributions and the transition matrices  $\mathbf{P}^{(i)}$  and  $\mathbf{Q}^{(i)}$ . Thus, even if two components have equal stationary distributions, they may still have different importance. The following example illustrates this.

**Example 4.3** We consider once again a multistate system  $(C, \phi)$  where  $C = \{1, 2\}$ , and where both components have three possible states, 0, 1, 2. For simplicity we again let  $f_i(u) = u$ ,  $u = 0, 1, 2$ ,  $i = 1, 2$ . Moreover, as before the structure function is given by:

$$\phi(X_1(t), X_2(t)) = \min(f_1(X_1(t)), f_2(X_2(t))).$$

The transition matrices of the built-in Markov chains are:

$$\mathbf{P}^{(1)} = \begin{bmatrix} 0.1, & 0.3, & 0.6 \\ 0.6, & 0.1, & 0.3 \\ 0.3, & 0.6, & 0.1 \end{bmatrix}, \quad \mathbf{P}^{(2)} = \begin{bmatrix} 0.7, & 0.1, & 0.2 \\ 0.2, & 0.7, & 0.1 \\ 0.1, & 0.2, & 0.7 \end{bmatrix},$$

while the mean waiting times are:

$$\mu_0^{(i)} = 2.5, \quad \mu_1^{(i)} = 3.5, \quad \mu_2^{(i)} = 4.0, \quad i = 1, 2.$$

It is easy to see that both  $\mathbf{P}^{(1)}$  and  $\mathbf{P}^{(2)}$  are doubly stochastic. Thus, the stationary distributions of the built-in Markov chains are uniform. Thus, we may calculate importance using Theorem 4.2. In particular, the stationary distributions can be calculated using the simplified formula given in Eq. (4.9), and we get:

$$P[X_i = 0] = 0.25, \quad P[X_i = 1] = 0.35, \quad P[X_i = 2] = 0.40, \quad i = 1, 2.$$

We now focus on  $I_{NB}^{(i)}$ . The corresponding results for  $I_{PB}^{(i)}$  are obtained in a similar fashion and are approximately the same in this case. In order to calculate  $I_{NB}^{(i)}$ , we need to compute a sum over all  $u, v \in S_i$ . Since, however, we obviously have:

$$P[\phi(u, X_2) \neq \phi(v, X_2)] = 0 \quad \text{if } u = v,$$

only the terms where  $u \neq v$  need to be included. Moreover, by symmetry we of course also have:

$$P[\phi(u, X_2) \neq \phi(v, X_2)] = P[\phi(v, X_2) \neq \phi(u, X_2)].$$

Using this we get the following non-zero probabilities for component 1:

$$\begin{aligned} P[\phi(0, X_2) \neq \phi(1, X_2)] &= P[X_2 = 1] + P[X_2 = 2] = 0.75, \\ P[\phi(0, X_2) \neq \phi(2, X_2)] &= P[X_2 = 1] + P[X_2 = 2] = 0.75, \\ P[\phi(1, X_2) \neq \phi(0, X_2)] &= P[\phi(0, X_2) \neq \phi(1, X_2)] = 0.75, \\ P[\phi(1, X_2) \neq \phi(2, X_2)] &= P[X_2 = 2] = 0.40, \\ P[\phi(2, X_2) \neq \phi(0, X_2)] &= P[\phi(0, X_2) \neq \phi(2, X_2)] = 0.75, \\ P[\phi(2, X_2) \neq \phi(1, X_2)] &= P[\phi(1, X_2) \neq \phi(2, X_2)] = 0.40. \end{aligned}$$

Moreover, since these probabilities only depend on the stationary distribution of component 2, and both components have the same stationary distribution, we get exactly the same probabilities for component 2. Thus, we have all the quantities needed in order to compute the importance measures using the formula for  $I_{NB}^{(i)}$  given in Theorem 4.2, and we get:

$$I_{NB}^{(1)} = 0.55425, \quad I_{NB}^{(2)} = 0.18475$$

We observe that component 1 is much more important than component 2, even though they both have exactly the same stationary distributions. The reason for this is that component 2 has a much higher probability of staying in the same state when a transition happens compared to component 1, i.e., 0.7 versus 0.1. Hence, for component 2 most of the weight from the transition probabilities are put on terms where  $u = v$ , and for these terms  $P[\phi(X_1, u) \neq \phi(X_1, v)] = 0$ . For component 1 on the other hand most of the weight from the transition probabilities are put on terms where  $u \neq v$ , and for these terms  $P[\phi(u, X_2) \neq \phi(v, X_2)] > 0$ .

In many applications transitions from one state directly back to the same state does not make sense. In order to avoid this we may e.g., replace the matrices of transition probabilities of the built-in Markov chains with the following:

$$\mathbf{P}^{(1)} = \begin{bmatrix} 0.0, & 0.1, & 0.9 \\ 0.9, & 0.0, & 0.1 \\ 0.1, & 0.9, & 0.0 \end{bmatrix}, \quad \mathbf{P}^{(2)} = \begin{bmatrix} 0.0, & 0.9, & 0.1 \\ 0.1, & 0.0, & 0.9 \\ 0.9, & 0.1, & 0.0 \end{bmatrix},$$

We observe that in these transition matrices all the diagonal terms are 0.0. Thus, the probability of staying in the same state after a transition has happened is zero.

It is again easy to verify that both  $\mathbf{P}^{(1)}$  and  $\mathbf{P}^{(2)}$  are doubly stochastic. This implies that the stationary probabilities are the same as in the previous case. The change in transition probabilities, however, implies that the importance measures change considerably. In this case we get:

$$I_{NB}^{(1)} = 0.61175, \quad I_{NB}^{(2)} = 0.62575,$$

implying that component 2 is now slightly more important than component 1.

By comparing transition matrices we also notice that  $\mathbf{P}^{(1)} = (\mathbf{P}^{(2)})^T$ . By Theorem 4.2 this implies that:

$$I_{PB}^{(1)} = I_{NB}^{(2)}, \quad \text{and} \quad I_{PB}^{(2)} = I_{NB}^{(1)}.$$

Hence, the importance ranking is reversed if we apply the  $p$ -Birnbbaum measure instead of the  $n$ -Birnbbaum measure ■

#### 4.1 Expected physical criticality for semi-Markov processes

We close this section by briefly showing how the importance measures based on expected physical criticality, introduced in Subsection 3.2 can be extended to semi-Markov processes. We observe that Eq. (3.8) and Eq. (3.9) are valid also in the general case. In order to calculate the  $n^*$ -Birnbbaum measure and the  $p^*$ -Birnbbaum measure, we again expand Eq. (3.8) and Eq. (3.9) by conditioning on the state of component  $i$  and get formulas similar to Eq. (4.3) and Eq. (4.4):

$$I_{NB}^{*(i)}(t) = \sum_{u,v \in S_i} E|\phi(u, \mathbf{X}(t)) - \phi(v, \mathbf{X}(t))| \cdot P[X_i(t) = u] \cdot P_{uv}^{(i)} \quad (4.10)$$

$$I_{PB}^{*(i)}(t) = \sum_{u,v \in S_i} E|\phi(u, \mathbf{X}(t)) - \phi(v, \mathbf{X}(t))| \cdot P[X_i(t) = u] \cdot Q_{uv}^{(i)} \quad (4.11)$$

Focussing on the asymptotic properties and using the same arguments as we did for  $I_{NB}^{(i)}$  and  $I_{PB}^{(i)}$ , we get the following analogues to Eq. (4.12) and Eq. (4.13):

$$I_{NB}^{*(i)} = \sum_{u,v \in S_i} E|\phi(u, \mathbf{X}) - \phi(v, \mathbf{X})| \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot P_{uv}^{(i)} \quad (4.12)$$

$$I_{PB}^{*(i)} = \sum_{u,v \in S_i} E|\phi(u, \mathbf{X}) - \phi(v, \mathbf{X})| \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot Q_{uv}^{(i)} \quad (4.13)$$

Having these formulas the following results can easily be proved using exactly the same arguments as we used for the corresponding results for  $I_{NB}^{(i)}$  and  $I_{PB}^{(i)}$ :

**Theorem 4.4** *Assume that  $\mu_0^{(i)} = \dots = \mu_{r_i}^{(i)}$ . Then  $I_{NB}^{*(i)} = I_{PB}^{*(i)}$ .*

**Theorem 4.5** *Assume that the transition matrix  $\mathbf{P}^{(i)}$  is doubly stochastic. Then we have:*

$$I_{NB}^{*(i)} = \sum_{u,v \in S_i} E|\phi(u, \mathbf{X}) - \phi(v, \mathbf{X})| \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{uv}^{(i)}$$

$$I_{PB}^{*(i)} = \sum_{u,v \in S_i} E|\phi(u, \mathbf{X}) - \phi(v, \mathbf{X})| \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{vu}^{(i)}$$

We close this section by considering an example.

**Example 4.6** *We consider the multistate system  $(C, \phi)$  introduced in Example 4.3. In order to compute  $I_{NB}^{*(1)}$  we start out by determining  $E|\phi(u, X_2) - \phi(v, X_2)|$  for all  $u, v \in S_1$ . We note again that:*

$$E|\phi(u, X_2) - \phi(v, X_2)| = 0 \quad \text{if } u = v,$$

*implying that only the terms where  $u \neq v$  need to be included. Moreover, by symmetry we have:*

$$E|\phi(u, X_2) - \phi(v, X_2)| = E|\phi(v, X_2) - \phi(u, X_2)|.$$

*Using this we get the following non-zero expectations for component 1:*

$$E|\phi(0, X_2) - \phi(1, X_2)| = 1 \cdot P[X_2 = 1] + 1 \cdot P[X_2 = 2] = 0.75,$$

$$E|\phi(0, X_2) - \phi(2, X_2)| = 1 \cdot P[X_2 = 1] + 2 \cdot P[X_2 = 2] = 1.15,$$

$$E|\phi(1, X_2) - \phi(0, X_2)| = E|\phi(0, X_2) - \phi(1, X_2)| = 0.75,$$

$$E|\phi(1, X_2) - \phi(2, X_2)| = 1 \cdot P[X_2 = 2] = 0.40,$$

$$E|\phi(2, X_2) - \phi(0, X_2)| = E|\phi(0, X_2) - \phi(2, X_2)| = 1.15,$$

$$E|\phi(2, X_2) - \phi(1, X_2)| = E|\phi(1, X_2) - \phi(2, X_2)| = 0.40.$$

*Moreover, since these expectations only depend on the stationary distribution of component 2, and both components have the same stationary distribution, we get exactly the same expectations for component 2. Thus, we have all the quantities needed in order to compute*

the importance measures using the formula for  $I_{NB}^{*(1)}$  and  $I_{NB}^{*(2)}$  given in Theorem 4.5. If the transition matrices are:

$$\mathbf{P}^{(1)} = \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.6 & 0.1 & 0.3 \\ 0.3 & 0.6 & 0.1 \end{bmatrix}, \quad \mathbf{P}^{(2)} = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.2 & 0.7 \end{bmatrix},$$

we get that:

$$I_{NB}^{*(1)} = 0.66225, \quad I_{NB}^{*(2)} = 0.22075$$

In particular component 1 is much more important than component 2, even though they both have exactly the same stationary distributions. If the transition matrices are:

$$\mathbf{P}^{(1)} = \begin{bmatrix} 0.0 & 0.1 & 0.9 \\ 0.9 & 0.0 & 0.1 \\ 0.1 & 0.9 & 0.0 \end{bmatrix}, \quad \mathbf{P}^{(2)} = \begin{bmatrix} 0.0 & 0.9 & 0.1 \\ 0.1 & 0.0 & 0.9 \\ 0.9 & 0.1 & 0.0 \end{bmatrix},$$

we get that:

$$I_{NB}^{*(1)} = 0.71775, \quad I_{NB}^{*(2)} = 0.77975$$

Since  $\mathbf{P}^{(1)} = (\mathbf{P}^{(2)})^T$ , Theorem 4.5 implies that:

$$I_{PB}^{*(1)} = I_{NB}^{*(2)}, \quad \text{and} \quad I_{PB}^{*(2)} = I_{NB}^{*(1)}.$$

Hence, the importance ranking is reversed if we apply the  $p^*$ -Birnbaum measure instead of the  $n^*$ -Birnbaum measure.

We observe that these results are very similar to the corresponding results for  $I_{NB}^{(1)}$  and  $I_{NB}^{(2)}$  found in Example 4.3, except that the importance measures are somewhat greater in this case ■

## 5 A numerical example

The examples presented in the previous sections are of course extremely simple and carefully chosen in order to illustrate the theoretical results. For these examples it was very easy to calculate the importance measures manually. In this section we present a somewhat larger example where the importance measures are estimated using Monte Carlo simulations.

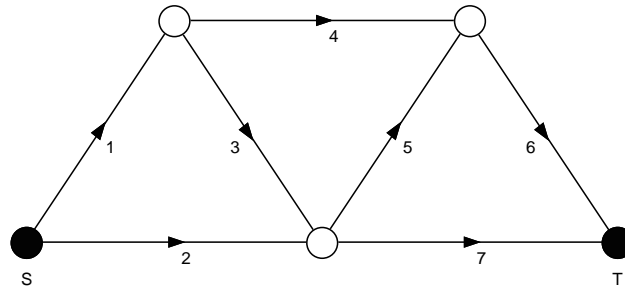


Figure 5.1: A network flow system

In this example we consider a directed network flow system consisting of 7 components representing the directed edges of the network. The system is illustrated in Figure 5. The physical state functions of the components,  $f_1, \dots, f_7$ , represent the flow capacity of the components. The physical state of the system is the amount of flow that can be sent through the network from the source node  $S$  to the terminal node  $T$ .

In order to express the system state as a function of the component states we identify the minimal cut sets in the network. These are  $K_1 = \{1, 2\}$ ,  $K_2 = \{1, 5, 7\}$ ,  $K_3 = \{2, 3, 4\}$ ,  $K_4 = \{4, 5, 7\}$ ,  $K_5 = \{2, 3, 6\}$ ,  $K_6 = \{6, 7\}$ . According to the well-known *max-flow-min cut* theorem (see Ford and Fulkerson (1956)) we then have<sup>1</sup>:

$$\phi(\mathbf{X}(t)) = \min_{1 \leq j \leq 6} \sum_{i \in K_j} f_i(X_i(t)).$$

The component state functions in this example are given by:

$$\begin{aligned} f_1(u) &= f_6(u) = 2.5 \cdot u, & u &= 0, 1, 2. \\ f_2(u) &= f_7(u) = 1.5 \cdot u, & u &= 0, 1, 2. \\ f_3(u) &= f_5(u) = 5.0 \cdot u, & u &= 0, 1. \\ f_4(u) &= 1.0 \cdot I(u = 1) + 2.5 \cdot I(u = 2), & u &= 0, 1, 2. \end{aligned}$$

We observe that if all components are in their respective top state, the maximal flow through the network is 8. The mean waiting times for the different states of the components are given in Table 5.1.

Comp.	$\mu_0^{(i)}$	$\mu_1^{(i)}$	$\mu_2^{(i)}$
1	2.5	3.5	4.5
2	2.5	3.5	4.5
3	4.5	5.5	-
4	2.5	3.5	4.5
5	4.5	5.5	-
6	2.5	3.5	4.5
7	2.5	3.5	4.5

Table 5.1: Mean waiting times

Finally, the transition probabilities for the components are:

$$\begin{aligned} \mathbf{P}^{(1)} &= \mathbf{P}^{(2)} = \mathbf{P}^{(6)} = \mathbf{P}^{(7)} = \begin{bmatrix} 0.0, & 0.0, & 1.0 \\ 1.0, & 0.0, & 0.0 \\ 0.4, & 0.6, & 0.0 \end{bmatrix}, \\ \mathbf{P}^{(3)} &= \mathbf{P}^{(5)} = \begin{bmatrix} 0.0, & 1.0 \\ 1.0, & 0.0 \end{bmatrix}, & \mathbf{P}^{(4)} &= \begin{bmatrix} 0.0, & 0.0, & 1.0 \\ 1.0, & 0.0, & 0.0 \\ 0.6, & 0.4, & 0.0 \end{bmatrix}. \end{aligned}$$

In the simulation all waiting times were sampled from exponential distributions with the respective mean values. For the asymptotic results, however, the choice of distribution is of course not important.

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<sup>1</sup>It should be noted that there exists efficient algorithms for calculating the system state without having to identify all the minimal cut sets. However, in this simple case with only 6 minimal cut sets we just use the standard formula.



We ran 10000 simulations of the system over a time frame of 500 time units. The resulting estimated importance measures are given in Table 5.2.

Comp.	$I_{NB}^{(i)}$	$I_{PB}^{(i)}$	$I_{NB}^{*(i)}$	$I_{PB}^{*(i)}$
1	0.53	0.56	1.34	1.48
2	0.64	0.66	1.24	1.39
3	0.34	0.34	0.83	0.83
4	0.40	0.40	0.77	0.83
5	0.34	0.34	0.83	0.83
6	0.53	0.56	1.34	1.48
7	0.64	0.66	1.24	1.39

Table 5.2: Estimated importance

Due to the structural symmetry in the network as well as the chosen distribution parameters and transition probabilities, the components are grouped into four sets:  $C_1 = \{1, 6\}$ ,  $C_2 = \{2, 7\}$ ,  $C_3 = \{3, 5\}$ ,  $C_4 = \{4\}$  with respect to importance. Within these groups the importance measures are equal regardless of which measure we use. However, we observe that the overall ranking depends on the chosen importance measure. If we choose  $I_{NB}^{(i)}$  or  $I_{PB}^{(i)}$ , the components in  $C_2$  are the most important components followed by  $C_1$ ,  $C_4$  and  $C_3$ . On the other hand if we choose  $I_{NB}^{*(i)}$ , the components in  $C_1$  are the most important components followed by  $C_2$ ,  $C_3$  and  $C_4$ . Finally, if we choose  $I_{PB}^{*(i)}$ , the components in  $C_1$  are still the most important components followed by  $C_2$ . In this case, the components in  $C_3$  and  $C_4$  are tied.

This example shows that there is a difference between just being critical and the physical effect of being critical. The components in  $C_2$  have a very high probability of being critical to the system. However, the components in this set have lower flow capacities compared to the components in  $C_1$ . Thus, the ranking of these two sets is reversed when the measures  $I_{NB}^{*(i)}$  or  $I_{PB}^{*(i)}$  are used. The same effect can be seen in the ranking between  $C_3$  and  $C_4$ .

The suggested measures allow the analyst to examine criticality from different viewpoints which is of interest both in a diagnostic setting as well as when the analysis is done in order to support decisions regarding improvement of the system.

## 6 Conclusions

In the present paper we have described a framework for modelling repairable multistate systems emphasising the physical properties of the components and the system. Within this framework we have generalised the Birnbaum measure to such systems in four different ways. Two of the suggested measures define component criticality at a given point of time relative to the *next* state of the component. These measures have a forward-looking focus which is often useful when the analysis is done in order to support decisions regarding improvement of the system. The two other measures define component criticality at a given point of time relative to the *previous* state of the component. These measures have a backward-looking focus which is often useful in a diagnostic setting.

The suggested measures can also be classified with respect to how criticality is measured. Two of the suggested measures focus on whether a component is *critical* or not, while the other two measures include information about the physical effects of being critical. Exam-

ples presented in the paper show that these viewpoints may result in different importance ranking of the components.

The paper considers both simple periodic component life cycles as more general semi-Markov processes and presents results on when the various measures are equal or can be simplified computationally.

The proposed measures can be interpreted as *time-dependent* measures reflecting component importance at a given point of time. Alternatively, the measures can be interpreted as *asymptotic* measures representing component importance given that the system has reached a stationary status. In cases where the component life cycles are long compared to the time frame under consideration or when the component processes are not time-homogenous, it may be of interest to find non-asymptotic, *time-independent* importance measures. Future work within this area will focus on this problem.

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