Multistate systems and importance measures

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Binary systems

A *binary system* is an ordered pair (C, ϕ) where:

$$C = \{1, \ldots, n\} \text{ is the component set.}$$

$$X_i(t) = I(\text{Component } i \text{ is functioning at time } t), \quad i \in C$$

$$\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$$

$$\phi(t) = I(\text{The system is functioning at time } t)$$

$$\phi(t) = \phi(\boldsymbol{X}(t))$$

Criticality

A component, $i \in C$, is said to be *critical* for the system at time t if:

$$\phi(0_i, \boldsymbol{X}(t)) \neq \phi(1_i, \boldsymbol{X}(t)).$$

Now introduce the following notation:

$$X_i^+(t) = \text{The } \textit{next state} \text{ of component } i \text{ at time } t$$

$$X_i^-(t) = \text{The } \textit{previous state} \text{ of component } i \text{ at time } t$$

Since each component only has two possible states, we have:

$$X_i^+(t) = X_i^-(t) = \begin{cases} 0 & \text{for } X_i(t) = 1 \\ 1 & \text{for } X_i(t) = 0 \end{cases}$$

NOTE: For $t = 0$, the notion of a previous state is *not defined*. However, we ignore this problem.

Criticality, cont.

It follows that component i is critical for the system at time t if:

$$\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t)). \tag{1}$$

or alternatively:

$$\phi(X_i^-(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t)). \tag{2}$$

That is, component i is critical at time t if changing the component to its next state would result in a system state change as well.

Alternatively, component i is critical at time t if changing the component to its previous state would result in a system state change as well.

The Birnbaum measure of component importance

The *Birnbaum measure* of importance of component $i \in C$ at time t, denoted $I_B^{(i)}(t)$, is the probability that the component is critical at time t.

Using our notation we have:

$$
\begin{aligned}
I_B^{(i)}(t) &= P[\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t))] \\
&= P[\phi(X_i^-(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t))]
\end{aligned}
$$

Multistate systems

A *multistate system* is an ordered pair (C, ϕ) where:

$$C = \{1, \ldots, n\} \text{ is the component set.}$$

$$\phi = \phi(t) = \text{The state of the system at time } t$$

In a multistate system the components have multiple states:

$$S_i = \{0, 1, \ldots, r_i\} = \text{The state set of component } i,\ i \in C.$$

We also introduce the component state processes:

$$X_i(t) = \text{The state of component } i \text{ at time } t, \quad i \in C$$

$$\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$$

The function ϕ is called the *structure function* of the system, and we assume that:

$$\phi(t) = \phi(\boldsymbol{X}(t)).$$

Component life cycles

At this stage we simplify the model by assuming that the components are *repairable* and have the following life cycles:

Each component starts out by being in the top state r_i:

$$X_i(0) = r_i, \quad i \in C.$$

At random points of time $0 < T_{1,r_i}^{(i)} < T_{1,r_i-1}^{(i)} < \cdots < T_{1,0}^{(i)}$ the component degrades through the entire state set until it reaches state 0:

$$X_i(T_{1,r_i}^{(i)}) = r_{i-1} \quad X_i(T_{1,r_i-1}^{(i)}) = r_{i-2} \quad \cdots \quad X_i(T_{1,1}^{(i)}) = 0.$$

At time $T_{1,0}^{(i)}$ the component is repaired or replaced, and a new life cycle starts. For this life cycle the state changes occur at times: $T_{2,r_i}^{(i)} < T_{2,r_i-1}^{(i)} < \cdots < T_{2,0}^{(i)}$, etc.

Physical component states

For $i \in C$ we introduce the functions:

$$f_i : S_i \to \mathbb{R} = \text{The } \textit{physical state} \text{ of component } i.$$

Thus, the physical state of component i at time t is $f_i(X_i(t))$, $i \in C$.

The functions $f_1, \ldots, f_n$ provide a convenient and intuitive way of encoding physical properties into a model.

EXAMPLE: Let component i be a pipeline. Then the physical state of the component at a given point of time may be the capacity of the pipeline at this point of time.

Physical component states, cont.

NOTE: A physical property of a component may be any real number (not just integers). In most cases, however, such properties will be *non-negative* numbers.

NOTE: The functions $f_1, \ldots, f_n$ do not need to be nondecreasing. By omitting this restriction, additional useful modeling flexibility is gained.

This allows e.g., for the inclusion of burn-in phases, maintenance as well as minimal or partial repairs of a components as part of its life cycle before it reaches its failure state.

Physical system states

It is common in multistate reliability theory to define ϕ such that it takes values in a set of *non-negative integers*.

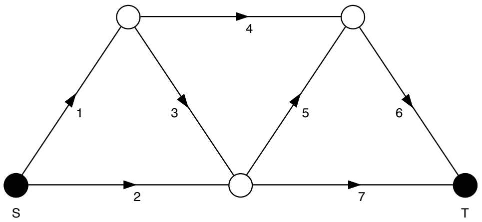
Here, however, we avoid this extra layer of abstraction, and let the structure function represent the *physical state* of the system.

Moreover, we assume that ϕ has the following form:

$$\phi(\boldsymbol{X}(t)) = \phi(f_1(X_1(t)), \ldots, f_n(X_n(t)))$$

Since the structure function represents a physical quantity, it is easier both to model and to interpret than if it had to be encoded more abstractly in terms of non-negative integers.

EXAMPLE: Flow network



$f_i(X_i(t))$ = Capacity of component i, $i = 1, \ldots, 7$.

Minimal cut sets: $K_1 = \{1, 2\}$, $K_2 = \{1, 5, 7\}$, $K_3 = \{2, 3, 4\}$, $K_4 = \{4, 5, 7\}$, $K_5 = \{2, 3, 6\}$, $K_6 = \{6, 7\}$

$$\phi(\boldsymbol{X}(t)) = \min_{1 \leq j \leq 6} \sum_{i \in K_j} f_i(X_i(t)).$$

Importance measures

For multistate systems there are many ways of defining importance measures. There is no such thing as the *best importance measure*.

Traditional uses of importance measures include:

- In design: Identifying components that should be improved
- In diagnostics: Identifying the components that are most likely to have failed

Other uses of importance measures:

- Understanding the structural and probabilistic properties of a system
- Understanding how each component affects various aspects of system performance

Criticality in multistate systems

We introduce the following notation:

$$X_i^+(t) = \text{The } \textit{next state} \text{ of component } i \text{ at time } t$$

$$X_i^-(t) = \text{The } \textit{previous state} \text{ of component } i \text{ at time } t$$

Given the life cycles of the components, these quantities are well defined, and we have:

$$X_i^+(t) = \begin{cases} X_i(t) - 1 & \text{for } X_i(t) > 0 \\ r_i & \text{for } X_i(t) = 0 \end{cases}$$

$$X_i^-(t) = \begin{cases} X_i(t) + 1 & \text{for } X_i(t) < r_i \\ 0 & \text{for } X_i(t) = r_i \end{cases}$$

NOTE: In the binary case $X_i^+(t) = X_i^-(t)$. For components with more than two possible states, however, $X_i^+(t)$ and $X_i^-(t)$ are *not* equal.

Criticality in multistate systems, cont.

We say that component i is *n-critical* at time t if:

$$\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t)).$$

Thus, component i is n-critical at time t if changing the component to its *next* state would result in a system state change as well.

We say that component i is *p-critical* at time t if:

$$\phi(X_i^-(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t)).$$

Thus, component i is p-critical at time t if changing the component to its *previous* state would result in a system state change as well.

Multistate importance

The *n-Birnbaum measure* of importance of component i at time t, denoted $I_{NB}^{(i)}(t)$, is the probability that component i is n-critical at time t:

$$I_{NB}^{(i)}(t) = P[\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t))].$$

The *p-Birnbaum measure* of importance of component i at time t, denoted $I_{PB}^{(i)}(t)$, is the probability that component i is p-critical at time t:

$$I_{PB}^{(i)}(t) = P[\phi(X_i^-(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t))].$$

NOTE: In the binary case we have $I_{NB}^{(i)}(t) = I_{PB}^{(i)}(t) = I_B^{(i)}(t)$. In the multistate case, however, we may have $I_{NB}^{(i)}(t) \neq I_{PB}^{(i)}(t)$.

Multistate importance, cont.

Assuming independent component state processes and conditioning on the state of component i at time t we get:

$$I_{NB}^{(i)}(t) = \sum_{j=1}^{r_i} P[\phi(j_i, \boldsymbol{X}(t)) \neq \phi((j-1)_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = j] \\ + P[\phi(0_i, \boldsymbol{X}(t)) \neq \phi((r_i)_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = 0]$$

$$I_{PB}^{(i)}(t) = \sum_{j=0}^{r_i - 1} P[\phi((j+1)_i, \boldsymbol{X}(t)) \neq \phi(j_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = j] \\ + P[\phi(0_i, \boldsymbol{X}(t)) \neq \phi((r_i)_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = r_i]$$

Changing the summation index in the last expression we get:

$$I_{PB}^{(i)}(t) = \sum_{j=1}^{r_i} P[\phi(j_i, \boldsymbol{X}(t)) \neq \phi((j-1)_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = j-1] \\ + P[\phi(0_i, \boldsymbol{X}(t)) \neq \phi((r_i)_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = r_i]$$

Multistate importance, cont.

We observe that $P[X_i(t) = j]$ in the formula for $I_{NB}^{(i)}(t)$ is replaced by $P[X_i(t) = j - 1]$ in the formula for $I_{PB}^{(i)}(t)$, $j = 1, \ldots, r_i$.

Moreover, $P[X_i(t) = 0]$ in the formula for $I_{NB}^{(i)}(t)$ is replaced by $P[X_i(t) = r_i]$ in in the formula for $I_{PB}^{(i)}(t)$.

From this it follows that in the special case where the component state processes are independent and:

$$P[X_i(t) = 0] = P[X_i(t) = 1] = \cdots = P[X_i(t) = r_i],$$

we will have $I_{NB}^{(i)}(t) = I_{PB}^{(i)}(t)$.

In general, however, the two importance measures will not be equal, and may even result in different rankings.

Example 1

Consider a multistate system (C, ϕ) where $C = \{1, 2\}$.

Both components have three possible states: $S_1 = S_2 = \{0, 1, 2\}$.

In this case the component states are identical to the physical states:

$$f_i(j) = j, \quad j \in S_i \quad \text{and } i \in C.$$

The structure function is given by:

$$\phi(X_1(t), X_2(t)) = \min(f_1(X_1(t)), f_2(X_2(t))).$$

Finally, we assume that the component state variables are independent, and that for a given t we have:

$$P[X_1(t) = j] = p_j > 0, \quad j \in S_1.$$
$$P[X_2(t) = j] = q_j > 0, \quad j \in S_2.$$

Example 1, cont.

We assume that the component state processes are independent, and that for a given t we have:

$$
\begin{aligned}
P[X_1(t) = j] &= p_j > 0, \quad j \in S_1. \\
P[X_2(t) = j] &= q_j > 0, \quad j \in S_2.
\end{aligned}
$$

It is then easy to see that:

$$
\begin{aligned}
P[\phi(0, X_2(t)) \neq \phi(2, X_2(t))] &= q_1 + q_2, \\
P[\phi(1, X_2(t)) \neq \phi(0, X_2(t))] &= q_1 + q_2, \\
P[\phi(2, X_2(t)) \neq \phi(1, X_2(t))] &= q_2,
\end{aligned}
$$

$$
\begin{aligned}
P[\phi(X_1(t), 0) \neq \phi(X_1(t), 2)] &= p_1 + p_2, \\
P[\phi(X_1(t), 1) \neq \phi(X_1(t), 0)] &= p_1 + p_2, \\
P[\phi(X_1(t), 2) \neq \phi(X_1(t), 1)] &= p_2.
\end{aligned}
$$

Example 1, cont.

Hence, we get

$$I_{NB}^{(1)}(t) = p_0(q_1 + q_2) + p_1(q_1 + q_2) + p_2 q_2 \;=\; q_1 + q_2 - p_2 q_1,$$

$$I_{NB}^{(2)}(t) = q_0(p_1 + p_2) + q_1(p_1 + p_2) + q_2 p_2 \;=\; p_1 + p_2 - p_1 q_2,$$

$$I_{PB}^{(1)}(t) = p_2(q_1 + q_2) + p_0(q_1 + q_2) + p_1 q_2 \;=\; q_1 + q_2 - p_1 q_1,$$

$$I_{PB}^{(2)}(t) = q_2(p_1 + p_2) + q_0(p_1 + p_2) + q_1 p_2 \;=\; p_1 + p_2 - p_1 q_1.$$

We observe that $I_{PB}^{(1)}(t) > I_{PB}^{(2)}(t)$ if and only if $q_1 + q_2 > p_1 + p_2$.

However, assuming that $q_1 + q_2 > p_1 + p_2$ and that $p_1 < p_2$ and $q_2 < q_1$, it is possible to obtain the opposite ranking with respect to the n-Birnbaum measure.

Example 1, cont.

Assume e.g., that $p_1 = 0.20$, $p_2 = 0.35$, $q_1 = 0.40$ and $q_2 = 0.20$. Then we have:

$$q_1 + q_2 = 0.60 > p_1 + p_2 = 0.55$$
$$p_2 q_1 = 0.14 > p_1 q_2 = 0.04$$
$$p_1 q_1 = 0.08$$

Hence, we get:

$$I_{NB}^{(1)}(t) = q_1 + q_2 - p_2 q_1 = 0.46,$$
$$I_{NB}^{(2)}(t) = p_1 + p_2 - p_1 q_2 = 0.51,$$

$$I_{PB}^{(1)}(t) = q_1 + q_2 - p_1 q_1 = 0.52,$$
$$I_{PB}^{(2)}(t) = p_1 + p_2 - p_1 q_1 = 0.47.$$

That is, $I_{NB}^{(1)}(t) < I_{NB}^{(2)}(t)$ while $I_{PB}^{(1)}(t) > I_{PB}^{(2)}(t)$.

Example 1, cont.

For $i \in C$, $j \in S_i$, and $k = 1, 2, \ldots$ we introduce:

$$W_{kj}^{(i)} = k\text{th waiting time in state } j \text{ for component } i,$$

We assume that all waiting times are independent and exponentially distributed with:

$$E[W_{kj}^{(1)}] = \begin{cases} 4.5 & \text{for } j = 0 \\ 2.0 & \text{for } j = 1 \\ 3.5 & \text{for } j = 2 \end{cases}$$

$$E[W_{kj}^{(2)}] = \begin{cases} 4.0 & \text{for } j = 0 \\ 4.0 & \text{for } j = 1 \\ 2.0 & \text{for } j = 2 \end{cases}$$

Example 1, cont.

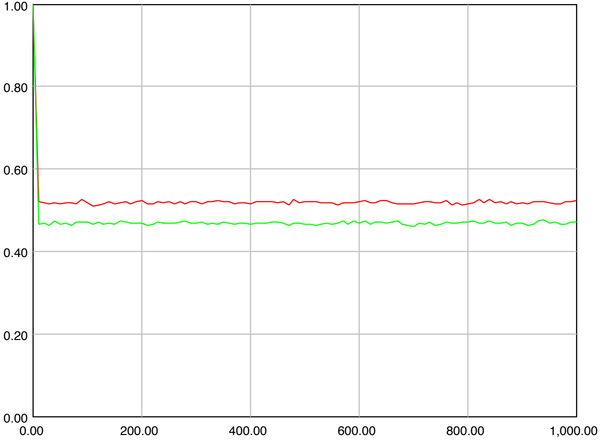


Figure: $I_{PB}^{(1)}(t)$ (red curve) $I_{PB}^{(2)}(t)$ (green curve)

Example 1, cont.

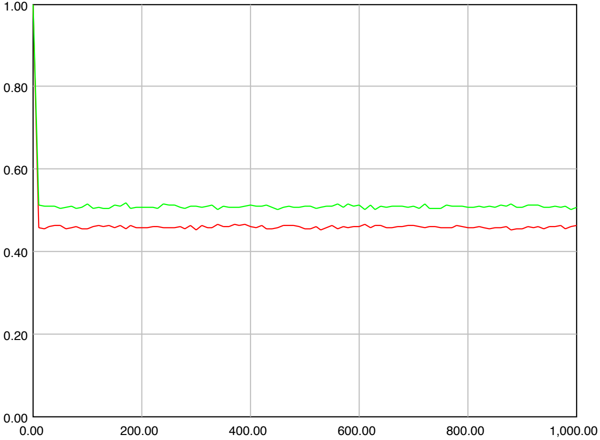


Figure: $I_{NB}^{(1)}(t)$ (red curve) $I_{NB}^{(2)}(t)$ (green curve)

Importance based on expected physical criticality

The *n*-Birnbaum measure* of importance of component i at time t, denoted $I_{NB}^{*(i)}(t)$, is the expected effect of changing i to its next state at time t:

$$I_{NB}^{*(i)}(t) = E\left|\phi(X_i(t), \boldsymbol{X}(t)) - \phi(X_i^+(t), \boldsymbol{X}(t))\right|.$$

The *p*-Birnbaum measure* of importance of component i at time t, denoted $I_{NB}^{*(i)}(t)$, is the expected effect of changing i to its previous state at time t:

$$I_{PB}^{*(i)}(t) = E\left|\phi(X_i^-(t), \boldsymbol{X}(t)) - \phi(X_i(t), \boldsymbol{X}(t))\right|.$$

NOTE: In the binary case all the different measures are the same:

$$I_{NB}^{*(i)}(t) = I_{NB}^{(i)}(t) \quad = \quad I_{PB}^{*(i)}(t) = I_{PB}^{(i)}(t) \quad = \quad I_{B}^{(i)}(t).$$

Example 2

Consider a multistate system (C, ϕ) where $C = \{1, 2\}$, and where $S_1 = \{0, 1\}$ and $S_2 = \{0, 1, 2\}$.

Moreover, we assume that:

$$f_1(j) = 2j, \quad j \in S_1,$$
$$f_2(j) = j, \quad j \in S_2.$$

As before, the structure function is given by:

$$\phi(X_1(t), X_2(t)) = \min(f_1(X_1(t)), f_2(X_2(t))).$$

Finally, we again assume that the component state processes are independent, and that for a given t we have:

$$P[X_1(t) = j] = p_j > 0, \quad j \in S_1,$$
$$P[X_2(t) = j] = q_j > 0, \quad j \in S_2.$$

Example 2, cont.

We then get:

$$E|\phi(0, X_2(t)) - \phi(1, X_2(t))| = q_1 \cdot 1 + q_2 \cdot 2 = q_1 + 2q_2$$

$$E|\phi(1, X_2(t)) - \phi(0, X_2(t))| = q_1 \cdot 1 + q_2 \cdot 2 = q_1 + 2q_2$$

$$E|\phi(X_1(t), 0) - \phi(X_1(t), 2)| = 2p_1$$

$$E|\phi(X_1(t), 1) - \phi(X_1(t), 0)| = p_1$$

$$E|\phi(X_1(t), 2) - \phi(X_1(t), 1)| = p_1$$

Hence, it follows that:

$$I_{NB}^{*(1)}(t) = p_0 \cdot (q_1 + 2q_2) + p_1 \cdot (q_1 + 2q_2) \ = \ q_1 + 2q_2$$
$$I_{NB}^{*(2)}(t) = q_0 \cdot 2p_1 + q_1 \cdot p_1 + q_2 \cdot p_1 \ = \ (1 + q_0)p_1$$

Example 2, cont.

We also have:

$$E[f_1(X_1(t))] = p_0 \cdot 0 + p_1 \cdot 2 \; = \; 2p_1$$
$$E[f_2(X_2(t))] = q_0 \cdot 0 + q_1 \cdot 1 + q_2 \cdot 2 \; = \; q_1 + 2q_2.$$

We then assume that $E[f_1(X_1(t))] = E[f_2(X_2(t))]$. This implies that:

$$I_{NB}^{*(1)}(t) = q_1 + 2q_2 = 2p_1$$
$$I_{NB}^{*(2)}(t) = (1 + q_0)p_1$$

Hence, since we must have $q_0 < 1$, we conclude that $I_{NB}^{*(1)}(t) > I_{NB}^{*(2)}(t)$.

Conclusions and further work

SUMMARY:

- Multistate systems defined with emphasis on physical interpretation
- Importance measures defined relative to next and previous component states
- Framework allowing a physical interpretation of importance measures
- Simulation software available for calculating availability and importance

FURTHER WORK:

- Extensions to more complex life-cycles
- Time independent versions of importance measures

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