UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in STK4400 — Risk and reliability analysis

Day of examination: Wednesday June 15, 2016

Examination hours: 14.30 – 18.30

This problem set consists of 5 pages.

Appendices: None

Permitted aids: Calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

(a) Give the definitions of a multistate strongly coherent system, a multistate coherent system and a multistate weakly coherent system. Explain the mutual relations between these three.

(b) Give the definition of a binary type multistate monotone system. What is the idea behind this measure?

Problem 2

(a) Assume that $X_1, \ldots, X_n$ are associated random variables. Show that for $j \in \{1, \ldots, M\}$ we have for the multistate series system.

$$P[\min_{1 \le i \le n} X_i \ge j] \ge \prod_{i=1}^{n} p_i^j \tag{1}$$

(b) Let ϕ be a multistate structure function that is nondecreasing in each argument and we assume that

$$\min_{1 \le i \le n} x_i \le \phi(\boldsymbol{x}) \tag{2}$$

Assume that $X_1, \ldots, X_n$ are associated random variables. Show that we have

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$$\prod_{i=1}^{n} p_i^j \leq p_\phi^j \tag{3}$$

Comment on the result.

Problem 3

Let $\boldsymbol{X}$ be a vector of environmental variables with sample space $\mathcal{X} \subseteq \mathbb{R}^n$. Furthermore let $P_e \in (0, 0.5)$ be a given exceedence probability. We shall in this exercise look into how one can identify a convex set $\mathcal{B} \subset \mathcal{X}$ such that for all hyperplanes Π that are tangents to $\mathcal{B}$, we have $P[\boldsymbol{X} \in \Pi^+] = P_e$, where Π^+ denotes the halfspace that is bounded by the hyperplane Π and not containing $\mathcal{B}$. We also introduce Π^- denoting the halfspace that is complementary to Π^+, such that $\mathcal{B} \subseteq \Pi^-$. The boundary of $\mathcal{B}$ is denoted by $\partial\mathcal{B}$ and is what is called an *environmental contour*.

(a) Explain how such an environmental contour can be used to check whether a certain mechanical structure is safe.

We assume in the rest of the exercise that $\boldsymbol{X} = (T, H)$. For $\theta \in [0, 2\pi)$ we define the random variable $Y(\theta) = T\cos(\theta) + H\sin(\theta)$. We also introduce

the function $C(\theta)$ defined for $\theta \in [0, 2\pi)$:

$$C(\theta) = \inf\{C : P[T\cos(\theta) + H\sin(\theta) > C] = P_e\}.$$

(b) What property does $C(\theta)$ have related to the probability distribution of $Y(\theta)$?

We also introduce:

$$\begin{aligned}
\Pi(\theta) &= \{(t, h) : t\cos(\theta) + h\sin(\theta) = C(\theta)\}, \\
\Pi^+(\theta) &= \{(t, h) : t\cos(\theta) + h\sin(\theta) > C(\theta)\}, \\
\Pi^-(\theta) &= \{(t, h) : t\cos(\theta) + h\sin(\theta) \leq C(\theta)\}.
\end{aligned}$$

Then it can be shown that the set $\mathcal{B}$ can can be written as:

$$\mathcal{B} = \bigcap_{\theta \in [0, 2\pi)} \Pi^-(\theta).$$

We consider the intersection point $(t(\theta_1, \theta_2), h(\theta_1, \theta_2))$ between the hyperplanes $\Pi(\theta_1)$ $\Pi(\theta_2)$.

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(c) Show the first of the two following relations:

$$t(\theta_1, \theta_2) = \frac{\sin(\theta_2)C(\theta_1) - \sin(\theta_1)C(\theta_2)}{\sin(\theta_2 - \theta_1)}$$

$$h(\theta_1, \theta_2) = \frac{-\cos(\theta_2)C(\theta_1) + \cos(\theta_1)C(\theta_2)}{\sin(\theta_2 - \theta_1)}$$

Then let $(t(\theta), h(\theta)) = \lim_{\delta \to 0^+}(t(\theta, \theta + \delta), h(\theta, \theta + \delta))$.

(d) Show that:

$$\begin{pmatrix} t(\theta) \\ h(\theta) \end{pmatrix} = \begin{bmatrix} C(\theta) & -C'(\theta) \\ C'(\theta) & C(\theta) \end{bmatrix} \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix},$$

where $C'(\theta)$ denotes the derivative of $C(\theta)$, and explain in a short way how this expression can be used to identify $\partial\mathcal{B}$.

(e) What sort of contour is $\partial\mathcal{B}$ if $C(\theta) = R$ for all $\theta \in [0, 2\pi)$? Give a reason for your answer.

Assume we have carried through a Monte Carlo simulation based on the joint probability distribution of (T, H), and that we have generated n vectors:

$$(T_1, H_1), \dots, (T_n, H_n) \tag{4}$$

For a given $\theta \in [0, 2\pi)$ we then calculate:

$$Y_i(\theta) = T_i \cos(\theta) + H_i \sin(\theta), \quad i = 1, \dots, n \tag{5}$$

(f) Explain how we can use $Y_1(\theta), \dots, Y_n(\theta)$ to estimate $C(\theta)$.

Problem 4

Consider events occuring on a time axis starting from time $t = 0$. Let $0 < S_1 < S_2 < \dots$ be the event times and let $T_i = S_i - S_{i-1}$ be the times between events, where $S_0 = 0$. Let further $N(t)$ for $t > 0$ denote the number of events in the time interval $(0, t]$.

(a) Define what it means that the process $S_1, S_2, \dots$ is a non-homogeneous Poisson process (NHPP) with intensity function $w(t)$. Define also the cumulative intensity function $W(t)$ and write down the probability distribution of $N(t)$ for a given $t > 0$.

A software system is subject to failures at random times caused by errors present in the code. Let $N(t)$ be the cumulative number of failures experienced by time t. We

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assume that each failure is caused by exactly one error in the code, and that this error is successfully removed from the software after each failure. Hence $N(t)$ also represents the cumulative number of errors detected and removed by time t.

A classical model for software reliability is the *time dependent error detection model of Goel and Okumoto* (the GO-model), where one assumes that $N(t)$ is an NHPP with intensity function of the form

$$w(t) = \alpha\beta e^{-\beta t}$$

for parameters $\alpha > 0, \beta > 0$.

(b) Calculate the cumulative intensity function $W(t)$. What is the interpretation of $W(t)$?

What is the limit of $W(t)$ as $t \to \infty$? Explain why this limit suggests an interpretation of the parameter α as the "initial number of errors" in the code.

(c) Suppose that the software system has been run until time $s > 0$. The *conditional reliability function* $R(t|s)$ of the software at time s is defined to be the probability that the software operates without failures for at least time t beyond time s. Show that

$$R(t|s) = \exp\left(-e^{-\beta s}W(t)\right). \tag{6}$$

The GO-model is in particular used in software testing. Suppose that values of α and β are estimated from previous data and hence can be assumed known. Then (6) can be used to calculate an *optimal testing time* s_0 by the requirement, on the testing time s, that the conditional reliability $R(t_0|s)$ should be at least equal to a given value r ($0 < r < 1$), for a given time t_0.

(d) Derive the corresponding expression for s_0 in terms of α, β, r and t_0. Discuss in particular the choice $t_0 = \infty$.

Recall the interpretation of the parameter α as the "initial number of errors" in the code. In the *Jelinski-Moranda model for software reliability* (the JM-model) the *true* initial number of errors, a, is an unknown parameter, which is hence a non-negative integer which we shall assume is at least one.

The basic assumption of the JM-model is that the time T_i between failure number $i - 1$ and failure number i is exponentially distributed with hazard rate

$$\lambda_i = (a - i + 1)b$$

for $i = 1, 2, \dots, a$, where $b > 0$ is the second parameter of the model. It is further assumed that $T_1, T_2, \dots, T_a$ are *stochastically independent.*

(e) Explain why the time S_a of the ath failure can be interpreted as the time when all errors in the code are found.

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Derive an exact expression for $E(S_a)$, i.e., the expected time when the software is error-free. Then verify the approximation

$$E(S_a) \approx \frac{\ln a}{b}$$

when a is large.