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Valuing Risky Projects: Option Pricing Theory and Decision Analysis

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In the academic literature and professional practice, there are a number of alternative and apparently competing methods for valuing risky projects. In this paper, we compare and contrast three different approaches: risk-adjusted discount-rate analysis, option pricing analysis, and decision analysis, focusing on the last two. We show that, in contrast to some of the claims made in the "real options" literature, when both option pricing and decision analysis methods are correctly applied, they must give consistent results. We also explore ways in which option pricing and decision analysis methods can be profitably integrated. In particular, we show how option pricing techniques can be used to simplify decision analyses when some risks can be hedged by trading and, conversely, how decision analysis techniques can be used to extend option pricing techniques to problems with incomplete securities markets.
(*Valuation; Option Pricing Theory; Decision Analysis*)

1. Introduction

In the usual MBA curriculum, students are presented with a number of alternative and apparently competing methods for valuing risky projects. In their decision analysis course, students learn about decision trees and utility theory and are taught to think of values in terms of expected utilities and certainty equivalents. In their finance course, they learn about the discounted cash flow model and are taught to think of values in terms of net present values computed using a discount rate reflecting the risk of the project. In an advanced finance course, they might learn about option pricing methods and be taught to think of projects as being analogous to put and call options on a stock. The result of all this training is graduates who may understand each method but fail to appreciate the relationships between them and their relative strengths and weaknesses.

A similar gap between the decision analysis and finance disciplines exists in the academic literature and professional practice. This gap has become increasingly apparent with the development of option pricing techniques for valuing projects in which managerial flexibility or "real options" play an important role. As an example of a real option, suppose a firm is considering obtaining rights to a new chemical process. With these rights they could invest now and build a plant using the new process. Alternatively, they might obtain a one-year option on these rights and wait a year before deciding whether to build the plant. If conditions prove favorable in one year, they can build the plant; if conditions prove unfavorable, they can decline and avoid losses they would have incurred had they built the plant now. These kinds of options may have substantial value and, it is argued, are often ignored or undervalued in discounted cash flow analyses. (See, for example, Robichek and Van Horne 1968.)

In response to these criticisms, finance theorists have proposed the use of option pricing[1] techniques—like those used to value puts and calls on stocks—for valuing risky projects in which real options play an important role (see Myers 1984; see Pindyck 1991 for a recent review). Decision scientists, on the other hand, have suggested that these options can be readily incorporated

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[1] What we refer to as "option pricing analysis" is sometimes called "contingent claims analysis" (see, for example, Mason and Merton 1985 and Trigeorgis and Mason 1987) or "valuation by arbitrage" (see, for example, Huang and Litzenberger 1988).

into decision tree or dynamic programming models (see Bonini 1977). Now we find the advocates of option pricing methods claiming superiority over decision analysis methods: ". . . we have shown the option-pricing approach is superior to both the NPV technique and DTA [decision tree analysis] when naively applied" (Copeland et al. 1990, p. 353; see also Mason and Merton 1985 and Trigeorgis and Mason 1987).

This paper has two goals. Our first goal is to show that the shortcomings of decision tree analysis noted by Copeland et al. (and others) are artifacts of their "naive" analysis which overlooks market opportunities to borrow and trade that are considered in the options analysis, and which confounds time and risk preferences by using a single risk-adjusted discount rate. If these market opportunities are included in the decision tree, and if time and risk preferences are captured using a utility function, the two approaches give results that are consistent in the following sense. When option pricing methods give a unique project value and optimal strategy (for example, when markets are complete), a correct decision tree analysis will give the same value and optimal strategy. When option pricing methods give bounds on the project value and identify a set of potentially optimal strategies, a correct decision tree analysis will give a value that lies between these bounds and an optimal strategy that is a member of this potentially optimal set.

Our second goal is to describe ways in which decision analysis and option pricing techniques can be profitably integrated. If the securities market is complete in that every project risk can be perfectly hedged by trading securities, option pricing methods provide a convenient means for separating (à la Fisher 1930) the decision analysis problem into two simpler subproblems. The first subproblem—the "investment problem"—focuses exclusively on the project being valued (ignoring opportunities to borrow and trade) and is solved by option pricing methods using only market information. The second subproblem—the "financing problem"—focuses exclusively on opportunities to borrow and trade (ignoring the project), and is solved by decision analysis methods using subjective beliefs and preferences.

While this completeness assumption may be quite reasonable when valuing options on stocks or other derivative securities, most "real" risky projects can only be partially hedged by trading securities. Though the separation result does not (in general) apply with incomplete markets, if we are willing to restrict preferences and assume that markets are "partially complete," we can develop a valuation procedure that integrates decision analysis and option pricing methods and satisfies consistency and separation properties analogous to those obtained with complete markets. Here, unlike the complete markets case, subjective beliefs and preferences play a critical role in both the investment and financing decisions.

The paper is organized as follows. Section 2 introduces the basic framework and definitions common to all subsequent sections. In §3, we assume that securities markets are complete. In this setting, option pricing methods give precise values and we can state strong versions of the consistency and separation results. In §4, we consider the general case of incomplete markets. Here the option pricing methods give bounds on the project value and we can state only a weak version of the consistency theorem. In §5, we restrict preferences and consider "partially complete" markets. Here our integrated procedure gives precise values and we can state strong versions of the consistency and separation results. All proofs are in an appendix.

Sections 2–4 build on Nau and McCardle (1991) and integrate known results in decision analysis and finance. The results in §5 appear to be new.

2. Basic Framework and Definitions

Our basic model of the securities market is a discrete-time, discrete-space version of the standard models in the option pricing literature (see, e.g., Huang and Litzenberger 1988, pp. 223–257). Our main departure from the option pricing literature is that we explicitly model the beliefs of a single market participant, hereafter referred to as the "firm." We attribute beliefs and preferences to this firm as if it were privately owned and operated by a single owner/manager or, equivalently, as if its owners/managers were "of one mind." This is consistent with the decision analysis approach where the analyst works with the firm's top officers to develop a "corporate utility function that [is] viewed as a policy statement by top management" (Spetzler 1968, p. 299) and works with the firm's designated experts to assess

probabilities for relevant uncertainties. We reconsider this assumption in §5.6 below.

Uncertainties are resolved and trading takes place at times $t = 0, 1, \ldots, T$. Let $\mathcal{S} = \{\gamma_1, \gamma_2, \ldots, \gamma_S\}$ denote the (finite) set of possible states of the world. The true state of the world is revealed to the firm at time T. At intermediate times t, the firm possesses some information about this final state that we represent as the *time-t state of information* ω_t. Formally, these time-t states of information ω_t are defined as subsets of $\mathcal{S}$ that form a partition of $\mathcal{S}$ (the possible ω_t's are mutually exclusive and their union is $\mathcal{S}$) and become successively finer with increasing t (each ω_{t-1} is the union of states ω_t in the next time period). The firm's beliefs about the state of the world are captured by subjective probabilities which (without loss of generality) are assumed to be strictly positive. Given state of information ω_t, the firm's expected value is written $E[- | \omega_t]$; $E[- | \omega_0]$ is abbreviated as $E[-]$.

We will be evaluating *risky cash flow streams* $x(t, \omega_t)$ where $x(t, \omega_t)$ denotes the amount the firm receives at time t in state ω_t. Where convenient, we suppress the dependence on the state ω_t and write $x(t)$ in place of $x(t, \omega_t)$. The firm's preferences for cash flows are captured by a utility function, $U(x(0), x(1), \ldots, x(T))$, that is assumed to be continuous, strictly increasing, and strictly concave in $x(0), x(1), \ldots, x(T)$.

A *project* is a risky cash flow stream $c(t, \omega_t)$ that specifies the project's payoff at every time and in every possible state. Unlike securities, projects are lumpy, all-or-nothing-type investments that are not traded. When there are a number of alternative strategies available for managing a project, each strategy α defines a different project whose payoffs are denoted c_α. We assume that these *project management strategies* α are mutually exclusive and finite in number.

There are $N + 1$ traded securities. To simplify the notation and analysis, we will assume that these securities pay no dividends in the time frame of the model. The prices of the securities are given by a vector

$$\mathbf{s}(t, \omega_t) = (s_0(t, \omega_t), s_1(t, \omega_t), \ldots, s_N(t, \omega_t))$$

where $s_i(t, \omega_t)$ denotes the price of the ith security at time t in state ω_t. To highlight the role of borrowing and lending, we assume that there is a risk-free security

(the 0th security) whose time-t price $s_0(t, \omega_t)$ is $(1 + r_f)^t$ in all states ω_t; r_f is referred to as the *risk-free rate*.[2]

As is standard in the option pricing literature, we assume that the securities market is frictionless in that the firm can buy or sell as many shares of a security as desired (including fractional amounts) at the market price without incurring any transaction costs (commissions, taxes, etc.). We let

$$\beta(t, \omega_t) = (\beta_0(t, \omega_t), \beta_1(t, \omega_t), \ldots, \beta_N(t, \omega_t))$$

denote a *trading strategy* that specifies a portfolio of securities held from time t to time $t + 1$ given state ω_t. To prevent "borrowing from beyond the horizon," we require $\beta(T) = 0$.[3]

The securities market is *complete* if all project risks can be perfectly hedged by trading securities. Formally, the securities market is complete if, for every project c, there exists a *replicating trading strategy* β_r that generates cash flows which exactly match the project's future cash flows at all times and in all states, i.e., a trading strategy β_r such that

$$[\beta_r(t - 1, \omega_{t-1}) - \beta_r(t, \omega_t)]\mathbf{s}(t, \omega_t) = c(t, \omega_t)$$

for all $t > 0$ and ω_t. (The product here is an inner product or dot product.) Since the project generates no cash flows after time T, the final replicating portfolio $\beta_r(T, \omega_T)$ is equal to 0 for all ω_T. For earlier times, given $\beta_r(t, \omega_t)$ for all ω_t, we find $\beta_r(t - 1, \omega_{t-1})$ by solving

$$\beta_r(t - 1, \omega_{t-1})\mathbf{s}(t, \omega_t) = c(t, \omega_t) + \beta_r(t, \omega_t)\mathbf{s}(t, \omega_t).$$

$$(1)$$

For each ω_{t-1}, (1) defines a set of M equations in $N + 1$ unknowns where M is the number of states ω_t that are subsets of ω_{t-1}. To be able to solve these equations for $\beta_r(t - 1, \omega_{t-1})$ for any c, as required by completeness, the M by $N + 1$ matrix $\mathbf{s}(t, \omega_t)$ must have rank greater than or equal to M for each t and ω_t. Thus completeness

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[2] The assumption that there is a risk-free security is not as strong as it might first appear: provided there is a security (say the 0th security) whose price is always positive we could renormalize cash flows so that the zeroth security behaves like a risk-free security in the normalized price system.

[3] We could instead include terminal wealth as an argument in the utility function.

requires the price process $s(t, \omega_t)$ to "span" the space of all possible projects $c(t, \omega_t)$.

Throughout this paper, we will assume that the market is *arbitrage-free* in that the firm cannot profit by trading securities without taking some risk or expending some capital. Formally, we say the market is arbitrage-free if there is no trading strategy β with nonpositive current value (i.e., $\beta(0)s(0) \leq 0$), that always generates nonnegative cash flows (i.e., $[\beta(t - 1, \omega_{t-1}) - \beta(t, \omega_t)]s(t, \omega_t) \geq 0$ for all t and possible states ω_t and ω_{t-1}), and has some chance of generating a positive cash flow, (i.e., $[\beta(t - 1, \omega_{t-1}) - \beta(t, \omega_t)]s(t, \omega_t) > 0$ for some t and possible states ω_t and ω_{t-1}). Such a trading strategy, called an "arbitrage opportunity," could be implemented with no initial cash outlay and has some chance of generating profits and no chance of generating losses.

An equivalent, dual characterization of this no-arbitrage condition focuses on a probability distribution that supports the observed security prices: a securities market is arbitrage-free if and only if there exists a strictly positive probability distribution π such that for all t,

$$s(0) = \sum_{\omega_t} \frac{\pi(\omega_t)}{(1 + r_f)^t} s(t, \omega_t) = E_\pi\left[\frac{s(t)}{(1 + r_f)^t}\right] \quad (2)$$

where E_π denotes expectation with respect to π.[4] This distribution is unique if and only if the market is complete as well as arbitrage-free. (See Harrison and Kreps 1979 for proof and discussion.) The probabilities $\pi(\omega_t)$ are, in general, not equal to the probabilities of our firm or any other market participant, but can be interpreted as the probabilities used by a hypothetical risk-neutral "representative investor" to determine securities prices (Cox and Ross 1976). For this reason, π is referred to as a *risk-neutral distribution*. These risk-neutral probabilities can also be interpreted as "state prices" because $\pi(\omega_t)/(1 + r_f)^t$ is the current price of a claim that pays one dollar if and only if state ω_t prevails at time t.

3. Complete Markets: A Simple Capital Budgeting Example

We first examine the relationships between the different methods of valuation in the case where the securities

[4] Formally, the distribution π is defined on $\mathcal{S}$ and $\pi(\omega_t) = \sum_{\gamma \in \omega_t} \pi(\gamma)$.

market is complete. This is the usual assumption in the real options literature and allows us to state the strongest results. We relax this assumption and consider incomplete markets in the next two sections.

3.1. A Simple Capital Budgeting Example

We illustrate our results by considering a simple two-period capital budgeting problem due to Trigeorgis and Mason (1987) and used in Copeland et al. (1990, pp. 343–353) and Nau and McCardle (1991). The problem is illustrated in the decision tree of Figure 1. The firm is presented with the opportunity to invest $104 now to build a plant that a year later will have a payoff that depends on the uncertain "level of demand." In the "good" state, the plant pays $180 and in the "bad" state, it pays only $60. The firm believes that these two states are equally likely. Alternatively, for a fee to be negotiated, the firm may obtain a one-year license that allows them to defer construction of the plant until after the state is known. If they choose this option, one year from now they may invest $112.32 and receive a certain value of either $180 or $60, or decline to invest and let the option expire. (This $112.32 assumes that the $104 it costs to build the plant grows at the risk-free rate of 8 percent.) The firm may also decline to invest in the plant or license without paying or receiving any money.

Following Trigeorgis and Mason and Copeland et al., suppose that there are two securities, a risk-free security that allows the firm to borrow and lend at 8 percent and a "twin security" whose future values depend on the uncertain level of demand. As shown in Figure 2, the current price of the twin security is $20, in the good state it will be worth $36, and in the bad state it will be worth $12. These two securities are sufficient to

Figure 1   Decision Tree for the Simple Capital Budgeting Example

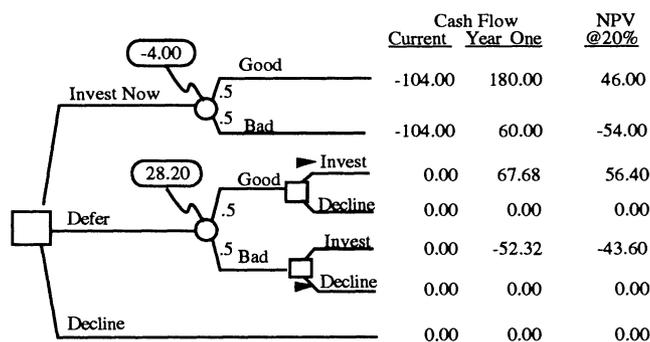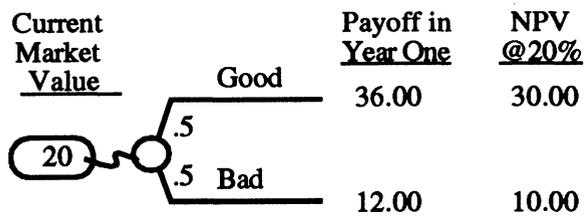|	Cash Flow		NPV
	Current	Year One	@20%
Good	-104.00	180.00	46.00
Bad	-104.00	60.00	-54.00
Good Invest	0.00	67.68	56.40
Decline	0.00	0.00	0.00
Bad Invest	0.00	-52.32	-43.60
Decline	0.00	0.00	0.00
Decline	0.00	0.00	0.00

Figure 2   Payoffs of the Twin Security

complete the market: there are two possible states of the world and there are two linearly independent securities, so the payoffs of every risky cash flow can be represented as a linear combination of the payoffs of these two securities.

We analyze this problem using three different methods. First, following Trigeorgis and Mason and Copeland et al., we present a "naive" decision tree analysis that uses discounted cash flow techniques in a simple decision tree format. We next present an option pricing analysis and a full decision tree analysis and then discuss the consistency and separation results.

3.2.   Naive Decision Tree Analysis

The fundamental idea of the discounted cash flow approach is that the value of a project is defined as "the future expected cash flows discounted at a rate that reflects the riskiness of the cash flow" (Copeland et al. 1990, p. 75). Typically, these discount rates are defined as "the equilibrium expected rate of return on securities equivalent in risk to the project being valued" (Myers 1984, p. 126). For example, suppose that the nth security is judged to be "equivalent in risk" to the project being valued.[5] Then we can determine the appropriate time-t discount rate $r(t)$ by computing the expected rate of return on the security (sometimes called the "market-required rate of return"), by solving

$$s_n(0) = E\left[\frac{s_n(t)}{(1 + r(t))^t}\right]$$

for $r(t)$. The discounted cash flow value of the project c_α is then defined as

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[5] Exactly what is meant by "equivalent in risk" will generally depend on some assumed underlying model of the securities market. For example in the capital asset pricing model, two securities are "equivalent in risk" if they have the same beta. More generally, the equivalent security might be a portfolio of securities rather than a single security.

$$E\left[\sum_{t=0}^{T} \frac{c_\alpha(t)}{(1 + r(t))^t}\right].$$

Note that, in general, the market-required rate of return will vary with the project management strategy α (because the equivalent security may change) as well as with time.

In our simple capital budgeting example, the end-of-period values of the twin security are perfectly proportional to the payoffs of the Invest Now alternative (they are equal to $\frac{1}{5}$ of the project payoffs); thus the Invest Now alternative and twin security are precisely "equivalent in risk." We can determine the market-required rate of return r for the twin security by solving

$$\$20 = \frac{0.5(\$36) + 0.5(\$12)}{1 + r} = \frac{\$24}{1 + r}$$

to obtain $r = 20$ percent. Using this 20 percent discount rate in the decision tree of Figure 1, we can "roll back" the tree to find an expected NPV of $-\$4.00$.

What discount rate should we use for the Defer alternative? Copeland et al. (1990, pp. 350–351) write: "The problem with the decision-tree approach is that we do not know the appropriate discount rate. The 20 percent rate derived from our NPV comparable is inappropriate, because the comparable security is not even approximately correlated with the payouts from the [Defer] option. But let us use it anyway just for the heck of it." Using this 20 percent discount rate in the decision tree of Figure 1, we find an expected NPV of $\$28.20$. Thus, if the cost of the one-year license is less than $\$28.20$, the naive analysis suggests that the optimal strategy is to Defer and wait a year before deciding whether to build the plant.

3.3.   Option Pricing Analysis

In the option pricing approach, rather than searching for one security that is "equivalent in risk" to the project being valued, one seeks a portfolio of securities that exactly replicates the project's payoffs. The value of the project is then given by the market value of this replicating portfolio. Formally, if β_r is the replicating strategy for a project c (such a strategy will always exist if the market is complete), the option-pricing value of the project is defined as the current value of this trading strategy plus any time-0 project cash flows: $c(0)$

$+ \beta_r(0)s(0)$. The fundamental assumption underlying this approach is that the value of a nontraded project is "the price the project would have if it *were* traded" (Mason and Merton 1985, pp. 38–39)—if the project had any price other than $c(0) + \beta_r(0)s(0)$, there would be an arbitrage opportunity.

We can easily construct replicating portfolios for the alternatives in our example. The payoffs of the Invest Now alternative can be replicated by purchasing exactly 5 shares of the twin security: in the good state, the value of 5 shares of the twin security is $5 \times \$36 = \$180$ and, in the bad state, the value of 5 shares of the twin security is $5 \times \$12 = \$60$. Since the market price of the twin security is $20, the value of the future payoffs of the Invest Now alternative is $5 \times \$20 = \$100$. Subtracting off the cost of investment, the option-pricing value of the Invest Now alternative is $\$100 - \$104 = -\$4$, as in the naive decision tree analysis. For the Defer alternative, letting β_0 and β_1 denote the number of shares of the risk-free and twin securities purchased, we construct a replicating portfolio by applying Equation (1) and equating the payoffs of the project and the portfolio in the good and bad states:

Good State: $\beta_0(\$1.08) + \beta_1(\$36) = \$67.68$

Bad State: $\beta_0(\$1.08) + \beta_1(\$12) = \$0.00$.

Solving these two equations gives $\beta_0 = -31.33$ and $\beta_1 = 2.82$, so the replicating portfolio consists of borrowing $31.33 and buying 2.82 shares of the twin security. The current value of this portfolio and, hence, the option-pricing value of the Defer alternative is given by $-\$31.33 + 2.82(\$20) = \$25.07$, which is less than the value ($28.20) given by the naive decision tree analysis.

Which value is right? The argument in favor of the option-pricing method is compelling: if one could buy a portfolio of securities that has the same payoffs as the Defer alternative for $25.07, it would be foolish to pay $28.20 for the license. So what is wrong with the naive analysis? Copeland et al. (1990, p. 352) note that, because the naive decision tree analysis used a discount rate based on a security that did not mimic the payoffs from the Defer option, the naive decision tree analysis was "comparing apples and oranges." The correct discount rate should be determined by computing the

market-required rate of return r for the replicating portfolio (a truly equivalent security); this gives $r = 35$ percent. Thus, with the risk-adjusted discount rate approach, we should have used a 35 percent discount rate for the Defer alternative and a 20 percent rate for the Invest Now alternative.[6]

As an alternative to using replicating trading strategies, we can determine option pricing values using risk-neutral probabilities. Just as the price of any security is equal to its discounted expected future value, the value of any project c is equal to the expected NPV of its future cash flows where, in both cases, expectations are computed using the risk-neutral probabilities and cash flows are discounted using the risk-free rate. More explicitly, the value of the project c is given by

$$E_\pi\left[\sum_{t=0}^{T} \frac{c(t)}{(1 + r_f)^t}\right], \qquad (3)$$

where E_π denotes expectation with respect to the unique risk-neutral distribution satisfying Equation (2).

To apply the risk-neutral valuation procedure in the example, we first use Equation (2) to determine the risk-neutral probabilities and then use Equation (3) to determine the value of the project. Letting π denote the probability of the good state, Equation (2) becomes

$$\frac{\pi(\$36) + (1 - \pi)(\$12)}{(1 + 0.08)} = \$20,$$

which implies $\pi = 0.4$. We then use these risk-neutral probabilities in the decision tree shown in Figure 3 to apply Equation (3): we first compute NPVs for all cash flows using the 8 percent risk free rate and then "roll back" the tree to compute expected values. This gives expected values of $-\$4.00$ and $\$25.07$ for the Invest Now and Defer alternatives, in exact agreement with the values derived by explicitly constructing the replicating portfolio.

3.4. Full Decision Tree Analysis

In the traditional decision analysis paradigm, rather than using risk-adjusted discount rates or market-based risk-

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[6] This result is true in general: given the replicating portfolio and its value, we can solve for the correct discount rate. Of course, if we are given the value of the replicating portfolio, we know the project's value and there is no point in determining the correct discount rate!

Figure 3   Risk-Neutral Decision Tree for the Simple Capital Budgeting Example

		Cash Flow Current	Cash Flow Year One	NPV @8%
Invest Now	Good .4	-104.00	180.00	62.67
	Bad .6	-104.00	60.00	-48.44
Defer	Good .4 ▸ Invest	0.00	67.68	62.67
	Decline	0.00	0.00	0.00
	Bad .6 Invest	0.00	-52.32	-48.44
	Decline	0.00	0.00	0.00
Decline		0.00	0.00	0.00

(-4.00) · (25.07)
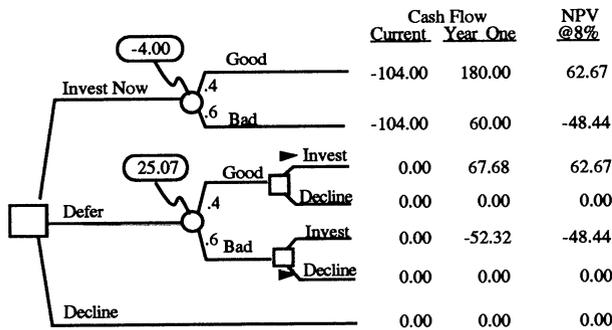
neutral probabilities, we use the firm's subjective probabilities and capture time and risk preferences using its utility function. Rather than defining the value of a project as "the price the project would have if it were traded," value is typically defined subjectively in terms of the firm's *breakeven buying price* or *breakeven selling price* (the latter is also called the certainty equivalent). In general, the breakeven buying price and breakeven selling prices are not equal (e.g., see Raiffa 1968, pp. 89–91) and the appropriate definition of value will depend on whether the firm is buying or selling the project.

The key to reconciling the option pricing and decision analysis approaches is to explicitly recognize market opportunities to trade by including them in the decision analysis model. To formalize this approach, suppose the firm is considering a project c. If the firm undertakes the project and invests in securities following trading strategy β_p, its time-t cash flow is given by

$$x_p(t; \beta_p) = c(t) + [\beta_p(t-1) - \beta_p(t)]s(t).$$

If the firm declines to invest in the project and follows trading strategy β_p^-, its time-t cash flow is given by

$$x_p^-(t; \beta_p^-) = [\beta_p^-(t-1) - \beta_p^-(t)]s(t).$$

The breakeven buying price is defined as the lump-sum time-0 payment v_b that makes the maximum expected utility with the project equal to the maximum expected utility without the project:

$$\max_{\beta_p} E[U(x_p(0; \beta_p) - v_b, x_p(1; \beta_p), \ldots, x_p(T; \beta_p))]$$

$$= \max_{\beta_p^-} E[U(x_p^-(0, \beta_p^-), x_p^-(1; \beta_p^-), \ldots, x_p^-(T; \beta_p^-))]$$

where E denotes expectations taken with respect to the firm's subjective probability distribution. The breakeven selling price is defined analogously as the v_s such that

$$\max_{\beta_p} E[U(x_p(0; \beta_p), x_p(1; \beta_p), \ldots, x_p(T; \beta_p))]$$

$$= \max_{\beta_p^-} E[U(x_p^-(0; \beta_p^-) + v_s,$$

$$x_p^-(1; \beta_p^-), \ldots, x_p^-(T; \beta_p^-))].$$

To be certain that the breakeven buying and selling prices are well-defined, we will assume that, for any project c, each of these maxima are finite and are obtained by some trading strategy.

A full decision tree for the simple capital budgeting example is shown in Figure 4. Here, in addition to choosing among the three alternatives, the firm may buy or sell shares of the risk-free and twin securities. Though our results hold for arbitrary utility functions, to make the example concrete suppose the firm's time and risk preferences are captured by the utility function

$$U(x_0, x_1) = -\exp(-x_0/200) - \exp(-x_1/220)$$

where x_0 and x_1 are the time-0 and time-1 net cash flows. This utility function is strictly concave in (x_0, x_1), and continuous and strictly increasing in x_0 and x_1. In the neighborhood of $(x_0, x_1) = (0, 0)$, the firm is indifferent between $1.00 at time 0 and $1.10 at time 1, implying a marginal time preference captured by a 10 percent discount rate. The utilities and expected utilities shown in Figure 4 were computed using this utility function and assuming that the license to Defer may be obtained at zero cost. Given these assumptions, the optimal strategy is to Defer, borrow $34.24 and buy 0.90 shares of the twin security. With other costs for the license, we would need to redo the analysis and solve for a new optimal strategy.

To determine the value of the Invest Now and Defer alternatives, we first need to define what we mean by value. In this example, given that the firm is considering investing in the plant, the appropriate definition of value would be the breakeven buying price—the price at which the firm is indifferent between "buying" the project and declining. Explicitly solving the problem of

Figure 4 for varying buying prices, we find breakeven buying prices of −$4.00 and $25.07 for the Invest Now and Defer alternatives—*exactly the values given by the option pricing analysis*.

3.5. Consistency and Separation

The preceding consistency result—that the values given by the decision tree analysis are the same as those given by the option pricing analysis—generalizes beyond the specifics of the example. The intuition behind this result is similar to the no-arbitrage argument underlying the definition of value in the option pricing approach. If the market is complete, one can construct a portfolio whose payoffs exactly replicate the payoffs of the project. If the project costs more than this portfolio, then it is obviously unattractive: the firm can obtain the same payoffs more cheaply by buying the replicating portfolio. If the project costs less than this portfolio, then it is obviously attractive: the firm can do the project and sell the replicating portfolio and thereby lock in a certain profit. Thus the breakeven buying price must be equal to the value of the replicating portfolio. A similar argument shows that the breakeven selling price must also be equal to the value of the replicating portfolio. We formalize this result as follows.

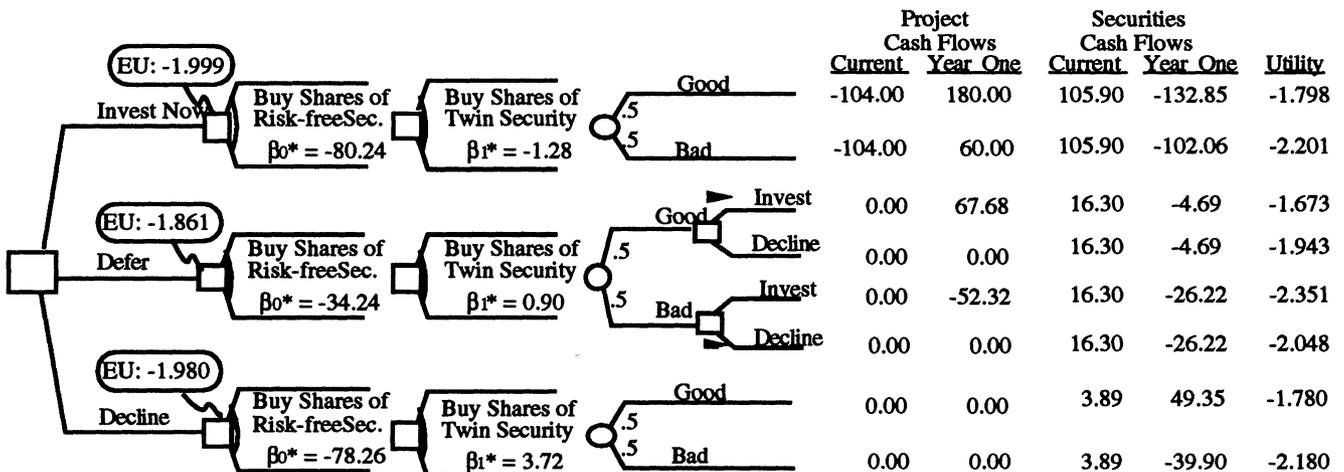CONSISTENCY THEOREM (COMPLETE MARKETS). *If the securities market is complete, then the firm's breakeven buying and selling prices for any project are both equal to the option-pricing value.*

Furthermore, given a project with some managerial flexibility, if the firm chooses a project management strategy to maximize the project's value, the option pricing and decision analysis approaches must also give the same optimal project management strategies. In this sense, the two approaches are consistent.

On the other hand, if we look at the "inputs" and "outputs" of the two analyses, we see that the approaches are quite different. Both methods require the firm to specify state-contingent cash flows for the project and state-contingent values for the securities for all times, all possible states of the world, and all project management strategies under consideration. While this is all that is required in the option pricing approach, the decision analysis approach also requires the firm to specify probabilities and a utility function describing its preferences for cash flows over time. In return for these additional inputs, we get an additional output, the optimal strategy for investing in securities. If the firm is not particularly interested in this additional output, then clearly option pricing methods provide a simpler and more direct way to compute the project's value and determine the optimal project management strategy.

Even if the firm is interested in computing the optimal securities investment as well as the project value and

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Figure 4   Full Decision Tree for the Simple Capital Budgeting Example

	Project Cash Flows		Securities Cash Flows		
	Current	Year One	Current	Year One	Utility
Invest Now (EU: -1.999) — Good	-104.00	180.00	105.90	-132.85	-1.798
Bad	-104.00	60.00	105.90	-102.06	-2.201
Defer (EU: -1.861) — Good/Invest	0.00	67.68	16.30	-4.69	-1.673
Good/Decline	0.00	0.00	16.30	-4.69	-1.943
Bad/Invest	0.00	-52.32	16.30	-26.22	-2.351
Bad/Decline	0.00	0.00	16.30	-26.22	-2.048
Decline (EU: -1.980) — Good	0.00	0.00	3.89	49.35	-1.780
Bad	0.00	0.00	3.89	-39.90	-2.180

management strategy, the options analysis suggests a useful decomposition of this *grand problem* into two simpler subproblems: the *investment problem* and the *financing problem*. In the investment problem, we use option pricing methods to determine the project value v^*, the optimal project management strategy α^* and the replicating trading strategy β_r^*. In the financing problem, we ignore the project cash flows and determine the firm's optimal securities investment β_f^* given that its initial wealth is increased by v^*. The optimal "grand" securities investment, taking into account the project cash flows, is then given by $\beta_g^* = \beta_f^* - \beta_r^*$. In doing the project and selling the replicating portfolio, the firm perfectly hedges the project's risk and reduces its net effect on cash flows to a lump-sum time-0 receipt of v^*. The cash flows generated by $\beta_g^* = \beta_f^* - \beta_r^*$ are thus exactly the cash flows in the financing problem.

This separation result can be viewed as an extension of Fisher's separation theorem to complete markets under uncertainty and can be stated as follows (Fisher 1930; see Hirshleifer 1970 for a discussion of the extension to complete markets under uncertainty).

SEPARATION THEOREM (COMPLETE MARKETS). *If the securities market is complete, given a project c_α, let*

$$v^* = \max_{\alpha} \mathrm{E}_\pi\left[\sum_{t=0}^{T} \frac{c_\alpha(t)}{(1+r_f)^t}\right], \quad (\textit{Investment Problem})$$

let α^* denote a maximizing project management strategy, and let β_r^* be a replicating trading strategy for c_{α^*}. Let β_f^* denote a trading strategy that maximizes,

$$\max_{\beta_f} \mathrm{E}[U(x_f(0; \beta_f) + v^*, x_f(1; \beta_f), \ldots, x_f(T; \beta_f))]$$

(*Financing Problem*)

where

$$x_f(t; \beta_f) = [\beta_f(t-1) - \beta_f(t)]\mathbf{s}(t).$$

Then α^* and $\beta_g^* = \beta_f^* - \beta_r^*$ solve

$$\max_{\alpha,\beta_g} \mathrm{E}[U(x_g(0; \alpha, \beta_g), x_g(1; \alpha, \beta_g), \ldots, x_g(T; \alpha, \beta_g))]$$
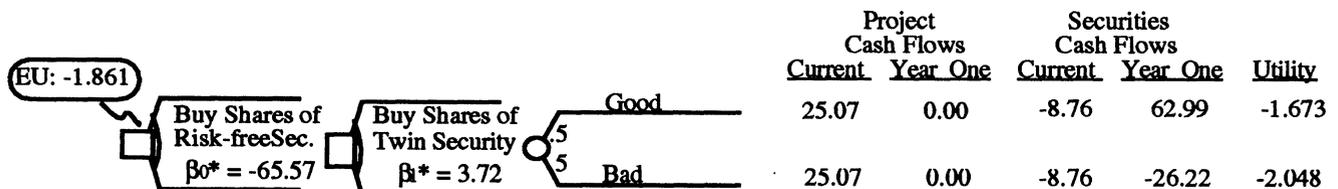
(*Grand Problem*)

where

$$x_g(t; \alpha, \beta_g) = c_\alpha(t) + [\beta_g(t-1) - \beta_g(t)]\mathbf{s}(t).$$

In the simple capital budgeting example, the investment problem is depicted in Figure 3; the optimal strategy is to Defer and has a value of $25.07. The replicating trading strategy for the Defer alternative consists of borrowing $31.33 and buying 2.82 shares of the twin security. The financing problem is shown in Figure 5; here we see that the optimal financing strategy is to buy 3.72 shares of the twin security and borrow $82.76 dollars. Subtracting off the replicating portfolio to find the solution to the grand decision problem, we have an optimal grand strategy of buying 0.90 ($=3.72 - 2.82$) shares of the twin security and borrowing $34.24 ($=\$65.57 - \$31.33$). These are exactly the amounts borrowed and shares purchased in the solution to the grand problem shown in Figure 4.

The same separation principle applies when valuing several "nonoverlapping" projects. (Two projects are nonoverlapping if the choice of strategy for one project does not affect the other's available strategies or state-contingent payoffs; overlapping projects need to be analyzed as a single project.) One can solve investment problems for several different projects and then sum their values and solve a single financing problem. The investment problems may be solved by one group (Capital Budgeting) and the financing problem by another group (Treasury); coordination requires only that

Figure 5   Investment Problem for the Simple Capital Budgeting Example

				Project Cash Flows		Securities Cash Flows		
				Current	Year One	Current	Year One	Utility
EU: -1.861	Buy Shares of Risk-freeSec. $\beta_0^* = -65.57$	Buy Shares of Twin Security $\beta_1^* = 3.72$	Good	25.07	0.00	-8.76	62.99	-1.673
			Bad	25.07	0.00	-8.76	-26.22	-2.048

Capital Budgeting tell Treasury the aggregate NPV. The optimal grand securities investment is then given by the solution to the financing problem less the sum of the project replicating portfolios.

4. Incomplete Markets: An Expanded Capital Budgeting Problem

The consistency and separation results of the previous section depend critically on the assumption that the market is complete in that every risk can be perfectly hedged by trading securities. While this assumption may be quite reasonable when valuing options on stocks or other derivative securities, most "real" risky projects can at best be partially hedged by trading securities. In this section, we describe how the results of the previous section generalize to the case where markets are incomplete. We illustrate the incomplete markets case using an expanded version of the simple capital budgeting problem. Again, we consider a naive decision tree analysis, an option pricing analysis, and a full decision tree analysis.

4.1. Naive Decision Tree Analysis

A decision tree for the expanded capital budgeting problem is shown in Figure 6. Here, in addition to being uncertain about the level of demand, the firm is uncertain about the efficiency of the plant being constructed. Suppose the plant can be either "efficient" or "inefficient" and that the true efficiency is not revealed until after the plant is built. If demand is good, there is a .50 probability that the plant will be efficient; if demand is bad, there is a .75 probability that the plant will be efficient. The payoffs in these cases are shown in Figure 6. (These values and probabilities were selected so the expected payoffs for each level of demand are the same as in the original problem.) As before, we assume the firm can borrow and lend as well as buy and sell shares of the "twin security" whose payoffs are tied to the level of demand, but we assume that there is no such security for plant efficiency.

Again, "the problem with the [naive] decision-tree approach is that we do not know the appropriate discount rate" (Copeland et al. 1990, pp. 350–351). One might argue that since the plant's efficiency risk is firm-

Figure 6 Naive Decision Tree for the Expanded Capital Budgeting Example
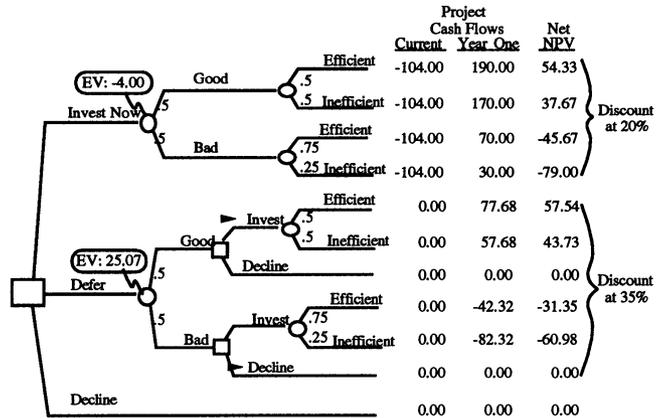
specific, it should not affect the risk-adjusted discount rate. Thus, suppose we use a 20 percent risk-adjusted discount rate for the Invest Now alternative and, having learned from our previous mistake, use a 35 percent discount rate for the Defer alternative. As shown in Figure 6, this gives values of −$4.00, $25.07 and $0 for the Invest Now, Defer and Decline alternatives, the same as before.

4.2. Option Pricing Analysis

How can we use option pricing methods in this expanded problem? Without a market equivalent for the efficiency uncertainty, we cannot construct a perfect replicating trading strategy or identify a unique risk-neutral probability distribution and thus we cannot determine a unique option-pricing value for the project.[7] We can, however, extend the basic ideas underlying the option pricing analysis to determine bounds on the project's value.

To compute these bounds, we introduce *dominating* and *dominated trading strategies* as an extension of our earlier notion of replicating trading strategies. Given a

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[7] We also have some semantic problems defining exactly what is meant by the value of a non-traded project. Earlier the option-pricing value of a project was defined as the price the project would have if it were traded in an arbitrage-free market and was computed from the current prices of traded securities. This definition does not work well in general because the introduction of the project into the market may create new investment opportunities and change the prices of the traded securities.

project c, a trading strategy β dominates the project if the future cash flows generated by β are always greater than or equal to those of the project; i.e., $[\beta(t-1) - \beta(t)]s(t) \geq c(t)$ for all $t > 0$ in all possible states of the world. A trading strategy is dominated by the project if the future cash flows generated by the trading strategy are always less than or equal to those of the project. If the project is traded in a market that does not allow arbitrage, its current value must be less than or equal to the current market value of every dominating trading strategy and greater than or equal to the current market value of every dominated trading strategy. Thus the option pricing approach gives upper and lower bounds, $\bar{v}$ and $\underline{v}$, on the project's value:

$$\bar{v} = c(0) + \min_{\beta} \{\beta(0)s(0) : [\beta(t-1) - \beta(t)]s(t)$$

$$\geq c(t) \text{ for all } t > 0\},$$

$$\underline{v} = c(0) + \max_{\beta} \{\beta(0)s(0) : [\beta(t-1) - \beta(t)]s(t)$$

$$\leq c(t) \text{ for all } t > 0\}.$$

In the special case where there is a replicating trading strategy (for example, when markets are complete), these bounds collapse to determine a unique project value.

Alternatively, rather than considering dominating and dominated portfolios, we compute bounds by considering the set of risk-neutral distributions that are consistent with market information. In this approach, the option-pricing bounds are given by

$$\bar{v} = \sup_{\pi \in \Pi} \mathrm{E}_{\pi}\left[\sum_{t=0}^{T} \frac{c(t)}{(1+r_f)^t}\right] \quad \text{and}$$

$$\underline{v} = \inf_{\pi \in \Pi} \mathrm{E}_{\pi}\left[\sum_{t=0}^{T} \frac{c(t)}{(1+r_f)^t}\right],$$

where Π denotes the set of risk-neutral distributions satisfying Equation (2). (See Harrison and Kreps 1979 for proof and discussion.)

The usefulness of these bounds depends on how much of the project risk can be hedged by trading securities. In our expanded capital budgeting example, we can determine unique risk-neutral probabilities for the level of demand (.4 for the good state and .6 for the bad state, as before) but cannot place any bounds

on the risk-neutral probabilities for the plant's efficiency. In this case, the upper bounds are given by assuming the plant is certainly efficient and the lower bounds are given by assuming the plant is certainly inefficient. For the Invest Now alternative, the upper bound is $5.26 (=0.4($190)/1.08 + 0.6($70)/1.08 − $104) and the lower bound is −$24.37 (=0.4($170)/1.08 + 0.6($30)/ 1.08 − $104); thus the Decline alternative is no longer necessarily preferred to the Invest Now alternative. Similarly, the upper and lower bounds for the Defer alternative are $28.77 and $21.36. If the license to defer costs less than $16.10 (=$21.36 − $5.26), the optimal strategy would be to Defer. But if the license costs $25, the optimal strategy is unclear and all three strategies are potentially optimal.

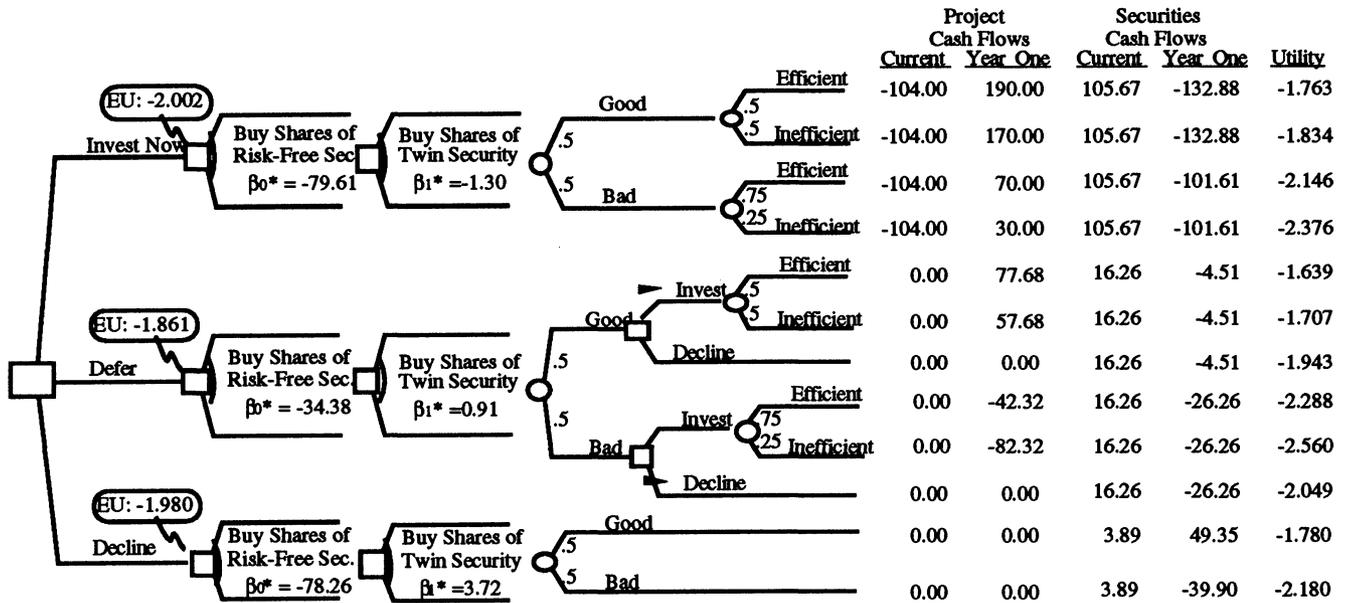### 4.3. Full Decision Tree Analysis

The full analysis of this expanded capital budgeting problem is analogous to the full analysis of the original version and is shown in Figure 7. (We use the same utility function as before.) If the license were free, the optimal strategy would be to Defer, buy .91 shares of the twin security, and borrow $34.38. The breakeven buying prices are −$4.47 and $24.98 for the Invest Now and the Defer alternatives, respectively. These values are slightly less than the previous values (−$4.00 and $25.07) reflecting the additional risk premium for the efficiency uncertainty. The optimal securities investments are similarly slightly changed.

Maintaining our previous definitions of breakeven buying and selling prices, straightforward arbitrage arguments lead to the following generalization of the consistency theorem for complete markets.

CONSISTENCY THEOREM (INCOMPLETE MARKETS). *The firm's breakeven buying and selling prices for any project may differ, but both lie between the bounds given by the option pricing analysis.*

Thus we see that when option pricing methods give a unique project value and optimal strategy (for example, when markets are complete), a full decision tree analysis will give the same value and optimal strategy. When option pricing methods give bounds on the project value and identify a class of potentially optimal strategies, a full decision tree analysis will give a value that lies between these bounds and an optimal strategy in this set.

Figure 7   Full Decision Tree for the Expanded Capital Budgeting Example

	Project Cash Flows		Securities Cash Flows		
	Current	Year One	Current	Year One	Utility
Efficient	-104.00	190.00	105.67	-132.88	-1.763
Inefficient	-104.00	170.00	105.67	-132.88	-1.834
Efficient	-104.00	70.00	105.67	-101.61	-2.146
Inefficient	-104.00	30.00	105.67	-101.61	-2.376
Efficient	0.00	77.68	16.26	-4.51	-1.639
Inefficient	0.00	57.68	16.26	-4.51	-1.707
Decline	0.00	0.00	16.26	-4.51	-1.943
Efficient	0.00	-42.32	16.26	-26.26	-2.288
Inefficient	0.00	-82.32	16.26	-26.26	-2.560
Decline	0.00	0.00	16.26	-26.26	-2.049
Good	0.00	0.00	3.89	49.35	-1.780
Bad	0.00	0.00	3.89	-39.90	-2.180

When markets are incomplete, it is impossible to state a separation theorem of the form given for complete markets. Because we can no longer replicate project cash flows by trading securities, the investment and financing problems must generally be solved jointly as the firm's preferences for project cash flows will be linked to the performance of its securities investments.

5. Partially Complete Markets with Restricted Preferences

If we are willing to restrict the firm's preferences and assume the market is "partially complete," we can develop a new procedure for valuing projects that integrates decision analysis and option pricing methods and satisfies consistency and separation properties analogous to those obtained with complete markets. We first describe the restrictions on preferences and markets and then describe the integrated valuation procedure and our consistency and separation results.

5.1. Preference Restrictions

We will make two assumptions about the firm's preferences:

1) **Additive Independence:** The firm's preferences for risky cash flow streams $(x(0), x(1), \ldots, x(T))$ de-

pend only the marginal distributions for period cash flows and not on the joint distribution.

2) **Δ-Property:** If the firm is indifferent between receiving a time-t gamble $\tilde{x}(t)$ and a certain amount $CE_t[\tilde{x}(t)]$, then for any constant Δ, the firm is also indifferent between $\tilde{x}(t) + \Delta$ and $CE_t[\tilde{x}(t)] + \Delta$.

The additive independence condition implies that the firm's utility function can be written as

$$U(x(0), x(1), \ldots, x(T)) = \sum_{t=0}^{T} k_t u_t(x(t))$$

where u_t is a utility function for time-t cash flows alone (see Keeney and Raiffa 1976, p. 295 or Fishburn 1970, p. 149). The Δ-property then implies the preferences for time-t cash flows exhibit constant absolute risk aversion, the u_t can be written

$$u_t(x(t)) = -\exp(-x(t)/\rho_t),$$

and the certainty equivalent of a time-t gamble can be written

$$CE_t[\tilde{x}(t)] = -\rho_t \ln(E[\exp(-\tilde{x}(t)/\rho_t)]).$$

By our assumptions that preferences must be strictly increasing and concave, the *utility weights* k_t and *period risk tolerances* ρ_t must be positive. The utility function

that we used in our example is of this form and has utility weights $k_0 = k_1 = 1$ and risk tolerances $\rho_0 = \$200$ and $\rho_1 = \$220$.

To demonstrate how we will use these preference restrictions, suppose the firm is offered an "incremental gamble" $\tilde{x}(0)$ that involves only time-0 cash flows and is resolved immediately. Given this incremental gamble and a project c, the firm's problem is to choose a trading strategy β_p that solves

$$\max_{\beta_p} E[U(x_p(0; \beta_p) + \tilde{x}(0), x_p(1; \beta_p), \ldots, x_p(T; \beta_p))]$$

(4)

where

$$x_p(t; \beta_p) = c(t) + [\beta_p(t-1) - \beta_p(t)]s(t).$$

Let β_p^* denote the optimal solution to (4) in the case where $\tilde{x}(0) = 0$. The following proposition describes how these incremental risks are valued and how the firm spreads these risks over time.

PROPOSITION 1. *If the firm's preferences satisfy additive independence and the Δ-property, then, for any $\tilde{x}(0)$,*

a) the firm is indifferent between receiving the gamble and a certain amount

$$ECE_0[\tilde{x}(0)] = -R_0 \ln(E[\exp(-\tilde{x}(0)/R_0)])$$

(5)

where

$$R_0 = \sum_{t=0}^{T} \frac{\rho_t}{(1+r)^t};$$

b) the optimal solution to (4) is $\beta_p^* + \beta_b$ where $\beta_b(t) = (\beta_{b0}(t), 0, \ldots, 0)$ and

$$\beta_{b0}(t) = \sum_{\tau=t+1}^{T} \frac{\rho_\tau \tilde{x}(0)}{R_0(1+r_f)^\tau};$$

(6)

the optimal cash flows given by this trading strategy are

$$x_p(t; \beta_p^*) + \frac{\rho_t}{R_0} \tilde{x}(0).$$

(7)

The first part of this result says that the incremental gamble $\tilde{x}(0)$ is valued as if the firm has an exponential utility function with an *effective risk tolerance R_0* equal to the discounted sum of its period risk tolerances; the amounts $ECE_0[\tilde{x}(0)]$ are referred to as *effective certainty equivalents*. This summing of risk tolerances can be in-

terpreted as a result of risk-sharing: though the gamble is resolved at time-0 and involves only time-0 cash flows, the firm may share this risk with future periods by adjusting its securities investment in response to the outcome of the gamble. In the second part, we see that the optimal adjustment involves only the risk-free security, and that, with this adjustment, each period bears a share of $\tilde{x}(0)$ in proportion to its discounted period risk tolerance.[8]

The result of Proposition 1 will be key to understanding our valuation procedure for partially complete markets. The part of a project's payoff that cannot be hedged by trading (its "private" risk) will be valued by computing effective certainty equivalents and the optimal trading strategy will include a rebalancing term, like β_b, that shares the unhedgable risks across time.

5.2. Partially Complete Markets

To implement our procedure for valuing projects in incomplete markets, we must refine the basic framework introduced in §2 to distinguish between *market* uncertainties and *private* uncertainties. Intuitively, market uncertainties are those that can be perfectly hedged by trading securities and private uncertainties are project-specific uncertainties that cannot be hedged. In our example, the level of demand is a market uncertainty and the plant's efficiency is a private uncertainty. In a more realistic problem like valuing oil reserves, the spot price for oil could be modeled as a market uncertainty and reserve-specific uncertainties such as reservoir size, drilling costs, etc. could be modeled as private uncertainties.

To formalize the distinction between market and private uncertainties, we assume that the state of the world γ can be written as a vector of market and private states of the world, $\gamma = (\gamma^m, \gamma^p)$, so the time-$t$ state of information ω_t can be written as a vector of market and private states of information $\omega_t = (\omega_t^m, \omega_t^p)$. We say that

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[8] This result is analogous to a result of Wilson's (1968) "theory of syndicates." Wilson shows that a syndicate whose members all have exponential utilities and agree on probabilities will, as a group, behave as if they have an exponential utility function with a risk tolerance equal to the sum of the member's risk tolerances and will share risks in proportion to their risk tolerances. Here risks are shared across time using the risk-free security and risk tolerances are discounted to reflect interest earned on the risk-free security.

the market is *partially complete* if the following conditions are satisfied:

(1) security prices depend only on the market states and thus can be written as a function of the market state of information as $\mathbf{s}(t, \omega_t^m)$;

(2) the market is complete with respect to market uncertainties (i.e., the security price process $\mathbf{s}(t, \omega_t^m)$ "spans" the space of cash flows dependent only on market states ω_t^m as in the definition of completeness in §2).

(3) private events convey no information about future market events (i.e., given ω_{t-1}^m, the firm believes that ω_t^m and ω_{t-1}^p are independent events).

Conditions (1) and (2) imply that we can determine unique risk-neutral probabilities for the market states ω_t^m. Note that while condition (3) says that private uncertainties must be independent of future market events, contemporaneous market and private uncertainties may be dependent (as in the example).

5.3. Integrated Valuation Procedure

Given these restrictions on the firm's preferences and partial completeness of the market, we can integrate decision analysis and option pricing methods into a new procedure for valuing projects separately from the financing problem. The basic idea is to decompose the project cash flows into its market and private components. We then use market information to value the market risks and use subjective beliefs and preferences to value the private risks. By substituting effective certainty equivalents for private risks, we reduce a problem with incomplete markets to an equivalent one in which markets are complete.

Our development of the integrated valuation procedure parallels the development of the complete markets procedures in §2. We begin by defining *certainty-equivalent replicating trading strategies* that are like replicating trading strategies in complete markets and which match the project's effective certainty equivalent in each market state. Since the project c generates no cash flows after time T, the final replicating portfolio $\beta_r(T, \omega_T)$ is equal to 0 for all ω_T. For earlier times, given $\beta_r(t, \omega_t)$ for all ω_t, we find $\beta_r(t-1, \omega_{t-1})$ by solving

$$\beta_r(t-1, \omega_{t-1})\mathbf{s}(t, \omega_t^m)$$

$$= \mathrm{ECE}_t[c(t, \omega_t) + \beta_r(t, \omega_t)\mathbf{s}(t, \omega_t^m)|\omega_t^m, \omega_{t-1}] \quad (8)$$

where, as in Equation (5), the effective certainty equivalent $\mathrm{ECE}_t[-]$ is calculated using an exponential utility function with an effective risk-tolerance R_t that is the NPV of the future period risk tolerances:

$$\mathrm{ECE}_t[\tilde{x}(t)|\omega_t^m, \omega_{t-1}]$$

$$= -R_t \ln(\mathrm{E}[\exp(-\tilde{x}(t)/R_t)|\omega_t^m, \omega_{t-1}]),$$

and

$$R_t = \sum_{\tau=t}^{T} \frac{\rho_t}{(1 + r)^{\tau-t}}.$$

Like Equation (1), for each ω_{t-1} Equation (8) defines a set of M equations in $N + 1$ unknowns where M is the number of states ω_t^m included in ω_{t-1}^m. Because of our assumption about the completeness of the market with respect to market uncertainties, for any project c, (8) can be solved for $\beta_r(t-1, \omega_{t-1})$.

As in complete markets, we define the value of a project to be the current value of its certainty-equivalent replicating trading strategy plus any time-0 project cash flows: $c(0) + \beta_r(0)\mathbf{s}(0)$. Although we no longer have an "objective" no-arbitrage basis for this definition of value (it depends on the firm's beliefs and preferences through the definition of the certainty-equivalent replicating trading strategy), we will show that, as in the case of complete markets, these values are equal to the breakeven buying and selling prices given by a full decision tree analysis.

We can illustrate this integrated procedure by finding the certainty-equivalent replicating portfolio for the Defer alternative in our extended capital budgeting example. Since time-1 is the last period of the model, the time-1 effective risk tolerance is $220. In the good state, the time-1 certainty equivalent is

$$\$67.45 = -\$220 \ln(0.5 \exp(-\$77.68/\$220)$$
$$+ 0.5 \exp(-\$57.68/\$220))$$

and, in the bad state, the certainty equivalent is $0.00. Letting β_0 and β_1 denote the number of shares of the risk-free and twin securities purchased, we construct a certainty-equivalent replicating portfolio for the Defer alternative by applying equation (8) and equating the portfolio values and time-1 certainty equivalents in the good and bad states:

Good State: $\beta_0(\$1.08) + \beta_1(\$36) = \$67.45$

Bad State: $\beta_0(\$1.08) + \beta_1(\$12) = \$0.00$.

Solving these two equations gives $\beta_0 = -31.23$ and $\beta_1 = 2.81$, so the replicating portfolio consists of borrowing $31.23 and buying 2.81 shares of the twin security. The current market value of the certainty equivalent replicating portfolio—and hence the value of the project—is 2.81 ($20) − $31.23 = $24.98, exactly the breakeven buying price we found in the full decision tree analysis of the previous section.

5.4. Integrated Rollback Procedure

Before stating our consistency and separation results, we first describe an *integrated rollback procedure* that provides a straightforward and intuitive method for computing project values and generalizes the risk-neutral valuation procedure for complete markets. As illustrated in Figure 8, this integrated rollback procedure uses risk-neutral probabilities for market uncertainties and the firm's probabilities for private uncertainties.

To apply this procedure, first compute NPVs for all endpoints using the risk-free rate. Then, as with the usual rollback procedure, start at the right side of the tree and work toward the left:

(1) upon encountering a node representing a private uncertainty, replace the node with the certainty equivalent given by using the firm's probabilities and an exponential utility function with a risk tolerance equal to the present value of the time-t effective risk-tolerance, $R_t/(1 + r_f)^t$;[9]

(2) upon encountering a node corresponding to a market uncertainty, replace the node with the expected value computed using the risk-neutral probabilities; and

(3) upon encountering a decision node, choose the branch with the maximum value and replace the node with this value.

The first case reduces incompleteness from the problem by replacing private risks with their effective certainty equivalents. The other two cases implement the risk-neutral valuation procedure for complete markets.

[9] The formula for this is like the formula for $\text{ECE}[\tilde{x}(t)|\omega_t^m, \omega_{t-1}]$ (see Equation 8) except, because we have discounted cash flows to present (time-0) values, we must similarly discount the time-t effective risk tolerance R_t.

Figure 8 Integrated Decision Tree for the Expanded Capital Budgeting Problem
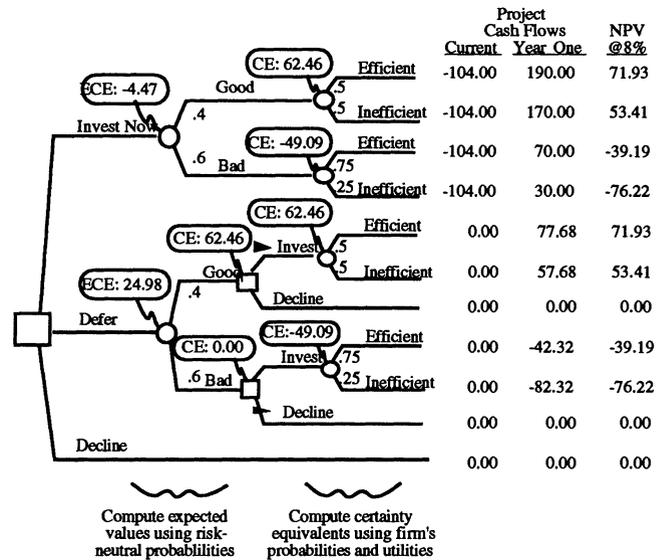
We can illustrate this procedure by computing the value of the Invest Now alternative using the decision tree of Figure 8. Rolling across the efficiency uncertainty, we find certainty equivalents of

$$\$62.46 = -\$220/1.08$$
$$\times \ln(0.5 \exp(-\$71.93/(\$220/1.08)))$$
$$+ 0.5 \exp(-\$53.41/(\$220/1.08))$$

for the good state and −$49.09 for the bad state; these values are slightly less than the expected values of 62.67 and −$48.44. Rolling across the market uncertainty, we find an expected certainty equivalent of −$4.47 (=0.4($62.46) + 0.6(−$49.09)), exactly the same as the breakeven buying price computed using the full decision tree analysis. Similarly, applying the integrated roll-back procedure with the Defer alternative, we find a value of $24.98 as given by explicitly constructing the certainty-equivalent replicating portfolio. A proof showing that this result holds in general is given in the appendix.

To better understand our integrated valuation procedure, it helps to consider the two extreme cases of market completeness. If the securities market is complete, then the firm's subjective probabilities do not appear anywhere in the tree, the firm's risk tolerances are

not used, and the integrated rollback procedure reduces to the risk-neutral procedure used with complete markets. In the other extreme where the only security available is the risk-free security, the firm's probabilities and risk tolerances are used at every node. Even in this case, we still discount all cash flows using the risk-free rate: thus given these restrictions on preferences, the ability to borrow and lend at the risk-free rate is sufficient to "smooth" time preferences so that they can be captured by NPVs computed using a single discount rate. Intuitively, the firm borrows and lends to bring its marginal expected utilities for cash flows at different times into equilibrium with the market's risk-free rate so that incremental cash flows are valued as if time preferences were represented by NPVs computed using the risk-free rate. Consequently, the only parameters of the firm's utility function used in the integrated valuation procedure are the risk tolerances; we need not assess the utility weights k_t until we solve the financing problem.[10]

5.5. Consistency and Separation

Having reduced a problem with incomplete markets to one in which markets are complete, the consistency and separation results for complete markets can now be generalized. Maintaining our earlier definitions of breakeven buying and selling prices, the consistency theorem carries over directly.

CONSISTENCY THEOREM (PARTIALLY COMPLETE MARKETS WITH RESTRICTED PREFERENCES). *If the securities market is partially complete and the firm's preferences satisfy additive independence and the Δ-property, then the firm's breakeven buying and selling prices for any project are both equal to the value given by the integrated valuation procedure.*

The separation theorem also generalizes but with some modifications. As in the complete markets case, we solve the investment problem to determine the optimal project management strategy α^*, its value v^* and its certainty-equivalent replicating trading strategy β_r^* and then solve the financing problem to determine an optimal securities investment β_f^*. With complete mar-

kets, the optimal grand trading strategy β_g^*, taking into account both project and securities cash flows, is given by $\beta_f^* - \beta_r^*$: the project cash flows are perfectly hedged by the replicating portfolio and the effect of the project on cash flows is reduced to a lump-sum time-0 receipt of the project's value v^*. With partially complete markets, the certainty-equivalent replicating portfolios do not perfectly hedge project values and, if the project's value in some period exceeds (or falls short of) its certainty equivalent, the firm may want to rebalance its securities investments to shift some of this windfall (or shortfall) to future time periods.

If the firm's preferences satisfy the preference restrictions discussed in §5.1, this portfolio rebalancing takes a particularly simple form and involves only an adjustment in the holdings of the risk-free security. For a project c with certainty-equivalent replicating trading strategy β_r, let $x_w(t)$ denote the "windfalls" corresponding to the difference between the value of the project at time t, $c(t) + \beta_r(t)\mathbf{s}(t)$, and the time-$t$ value of the replicating portfolio constructed in the previous period, $\beta_r(t-1)\mathbf{s}(t)$:

$$x_w(t) = c(t) + [\beta_r(t) - \beta_r(t-1)]\mathbf{s}(t).$$

As was the case with incremental risks in §5.1, when these cash flow windfalls are allocated optimally, each period bears a share of the risk in proportion to its discounted risk tolerance.[11] To achieve this sharing of windfalls across periods, at time t, the firm must purchase an additional

$$\beta_{b0}(t) = \left(\sum_{\tau=t+1}^{T} \frac{\rho_\tau}{(1+r_f)^\tau} \right)\left(\sum_{\tau=1}^{t} \frac{x_w(\tau)}{R_\tau} \right)$$
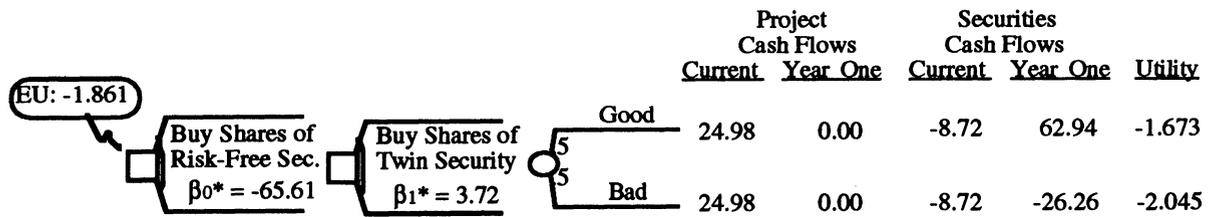
shares of the risk-free security (compare with equation 6), giving a *rebalancing trading strategy* $\beta_b(t) = (\beta_{b0}(t), 0, \ldots, 0)$. Like the replicating trading strategy, the rebalancing trading strategy does not depend on the firm's probabilities for market states and may be determined independently of the solution to the financing problem.

With these definitions, we can state our separation for partially complete markets as follows.

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[10] Actually we do not need to assess the utility weights k_t until we determine the optimal position in the risk-free security, as the utility weights do not affect the firm's optimal investment in risky securities.

[11] This allocation of windfalls is analogous to the sharing of incremental risks in Equation (7) and is displayed more explicitly in Lemma 1 in the appendix; see also the discussion following Lemma 3.

Figure 9   Investment Problem for the Expanded Capital Budgeting Example

		Project Cash Flows		Securities Cash Flows		Utility
		Current	Year One	Current	Year One	
Good		24.98	0.00	-8.72	62.94	-1.673
Bad		24.98	0.00	-8.72	-26.26	-2.045

EU: -1.861

Buy Shares of Risk-Free Sec. $\beta_0^* = -65.61$

Buy Shares of Twin Security $\beta_1^* = 3.72$

SEPARATION THEOREM (PARTIALLY COMPLETE MARKETS WITH RESTRICTED PREFERENCES). *Suppose the securities market is partially complete and the firm's preferences satisfy additive independence and* Δ-*property. Given a project* c_α, *let* α^* *and* v^* *denote a maximizing project management strategy and maximizing value as given by the integrated valuation procedure (i.e., a solution to the investment problem), let* β_r^* *be a certainty-equivalent replicating trading strategy for* c_{α^*}, *and let* β_b^* *be the corresponding rebalancing trading strategy. Let* β_f^* *denote a trading strategy that solves,*[12]

$$\max_{\beta_f} E[U(x_f(0; \beta_f) + v^*, x_f(1; \beta_f), \dots, x_f(T; \beta_f))]$$

(*Financing Problem*)

where

$$x_f(t; \beta_f) = [\beta_f(t-1) - \beta_f(t)]s(t).$$

Then α^* *and* $\beta_g^* = \beta_f^* - \beta_r^* + \beta_b^*$ *solve*

$$\max_{\alpha, \beta_g} E[U(x_g(0; \alpha, \beta_g), x_g(1; \alpha, \beta_g), \dots, x_g(T; \alpha, \beta_g))]$$

(*Grand Problem*)

where

$$x_g(t; \alpha, \beta_g) = c_\alpha(t) + [\beta_g(t-1) - \beta_g(t)]s(t).$$

There are two key differences between this separation theorem and the complete markets result: (1) we solve the investment problem using our integrated procedure rather than option pricing methods and (2) the optimal

grand trading strategy is given by $\beta_f^* - \beta_r^* + \beta_b^*$ rather than $\beta_f^* - \beta_r^*$. In the special case of complete markets, this result reduces to the complete markets result: our integrated procedure reduces to the option pricing procedure and, as project risks are perfectly hedged, there are no windfalls and there is no need for rebalancing (i.e., $\beta_b^* = 0$).

We can illustrate the separation theorem using the expanded capital budgeting example: the investment problem is depicted in Figure 8, the financing problem in Figure 9, and the grand problem in Figure 7. Notice that, as required by the theorem, the securities investment given as a solution to the grand problem (buy 0.91 shares of the twin security and borrow $34.38) is exactly the difference between the solution to the financing problem (buy 3.72 shares and borrow $65.61) and the certainty-equivalent replicating portfolio (buy 2.81 shares and borrow $31.23); the rebalancing strategy is zero in this case as all of the private uncertainties are resolved in the final time period and $R_T = 0$.

Finally, we note that our preference restrictions are necessary for separation to hold with incomplete markets. If the market is incomplete, then for some projects, the firm will wind up holding some "residual" risk. In order for separation to hold, the value of this residual risk must be independent of the outcomes of securities investments and the firm's probabilities for market states. Additive independence is required to ensure that the value of a residual risk in one period is independent of the securities payoffs and probabilities in other periods (see Keeney and Raiffa 1976, p. 242). The Δ-property (i.e., constant absolute risk aversion) is required to ensure that the value of a time-t residual risk, computed conditional on the time-t market state, does not vary with changes of wealth due to securities payoffs in the same period.

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[12] Note that because future securities prices are assumed to independent of the private states, the firm need not consider private states when solving the investment problem: there will be an optimal β_f that is independent of the private states.

5.6. Corporate Applications

Throughout this paper, we have attributed beliefs and preferences to the firm as if it were privately owned and operated by a single owner/manager, or equivalently, as if its owners/managers were of one mind. While this is consistent with the decision analysis approach in corporate settings, finance theorists and practitioners view this personification of the firm skeptically (Jensen and Meckling 1976) and argue that a firm should make decisions consistent with the beliefs and preferences of its diverse owners.

When markets are complete, investment decisions can be made solely on the basis of market information and all owners, regardless of their beliefs and preferences, will agree on appropriate project values and management strategies. The financing decisions require the use of subjective beliefs and preferences, but since (regardless of market completeness) owners can replicate or negate these financing decisions through their own securities transactions, the firm's financing decisions are "irrelevant" (in the sense of Modigliani and Miller 1958) from the owners' perspective. The group's grand decision problem can be decomposed along the lines of the separation result: the firm can make investment decisions and manage its risks by shorting the appropriate replicating portfolio (and subsequently rebalancing, if necessary). The owners separately make their own financing decisions.[13]

When markets are incomplete, investment decisions require the use of subjective preferences and beliefs. In our framework, we ask owners and managers to agree upon risk tolerances and a mechanism for assigning probabilities to private uncertainties. We can appeal to Wilson's "theory of syndicates" for some results along these lines (Wilson 1968). Suppose the owners' preferences satisfy additive independence and the Δ-property. Then to achieve Pareto optimality, the owners should place bets among themselves on the private states of the world (essentially completing the market among themselves). After placing these bets, they

should own shares of the firm in proportion to their risk tolerances and the firm's risk tolerance should be equal to the sum of the owners' risk tolerances. If the owners do not agree upon probabilities (e.g., through delegation to an expert manager), then the firm should "sample" the owners' probabilities in proportion to their shares (see Wilson for a precise definition of "sampling"). In this case, the firm's investment decisions are unanimously supported by every owner.

For a large publicly owned firm, this sampling of owners' beliefs and preferences is impractical. Taking Wilson's argument to the limit and summing risk tolerances over all market participants, one could argue that a large publicly owned firm should be essentially risk neutral. Insofar as beliefs are concerned, most investors have little information about the opportunities facing the company and, if sampled, would defer to the manager's judgment. In practice, we believe that publicly owned firms operate rather autonomously with managers using their own judgment and, to some extent, their own preferences. Investors choose companies in the same way they choose mutual funds: they look for managers that they believe have good judgment and risk preferences consistent with their own. If managers exhibit poor judgment or are excessively risk-averse, the company's stock may sell at a discount and be a target for takeover by a group of investors who fancy themselves more sagacious or who are less risk averse.

6. Conclusions

We have two fundamental conclusions. Our first conclusion, highlighted by the consistency theorems for complete and incomplete markets, is that option pricing and decision analysis methods are fully compatible. The problems attributed to decision analysis by Copeland et al. (1990) and others are the result of using risk-adjusted discount rates to capture time and risk preferences as well as market opportunities to borrow and trade. If time and risk preferences are captured using a utility function and the market opportunities are explicitly modeled, decision analysis and option pricing analysis will give consistent results. When both methods give unique values and strategies, they give the same values and strategies. When option pricing methods give bounds on the project value and identify a set of po-

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[13] Technically either the firm or its owners could manage project risks as both have equal access to the securities market. But since project risk management requires intimate knowledge of project cash flows, it seems more natural for the firm to do it. If the firm shorts the replicating portfolio, the owners only need to know the value of the project to solve their financing problems.

tentially optimal strategies, a correct decision tree analysis will give a value that lies between these bounds and an optimal strategy that is a member of this potentially optimal set.

Our second conclusion, highlighted by the separation theorems, is that decision analysis and option pricing methods can be profitably integrated. This integration allows us to extend the option pricing methods to incomplete markets and simplify the analysis of projects that can be partially hedged by trading securities. In practice, we often analyze projects without explicitly modeling opportunities for borrowing or trading securities. Our results suggest that when doing this, we should: (1) use risk-neutral probabilities rather than subjective probabilities for risks that can be hedged by trading securities; (2) compute NPVs using the risk-free rate; (3) use exponential utility functions to capture risk preferences; and (4), when "rolling back" a decision tree to compute project values and strategies, assign risk premiums only to private risks. Finally, in doing these project analyses, it is important to remember that the investment problem is only part of the grand problem and that, if the firm undertakes a project, it should manage risks by shorting the appropriate replicating portfolio and subsequently rebalancing if necessary.[14]

Appendix: Proofs

Proofs for §3: Complete Markets

CONSISTENCY THEOREM. We focus on proving that the breakeven selling price is equal to the value given by the option pricing analysis; a similar argument holds for the breakeven buying price. Let β_r be a replicating trading strategy for the project c and let β_p^- be a trading strategy for the "without project" maximization problem (the one on the right side of the equation defining breakeven selling price). Taking $\beta_p = \beta_p^- - \beta_r$, the cash flows associated with the "with project" maximization problem become

$$x_p(t; \beta_p) = [\beta_p^-(t-1) - \beta_p^-(t)]\mathbf{s}(t)$$
$$- [\beta_r(t-1) - \beta_r(t)]\mathbf{s}(t) + c(t).$$

For all $t > 0$, the last two terms cancel because of the definition of the replicating trading strategy, so $x_p(t; \beta_p) = x_p^-(t; \beta_p^-)$. For $t = 0$, we find

$$x_p(0; \beta_p) = x_p^-(0; \beta_p^-) + \beta_r(0)\mathbf{s}(0) + c(0).$$

Therefore with $v_s = \beta_r(0)\mathbf{s}(0) + c(0)$ and $\beta_p = \beta_p^- - \beta_r$, the "with project" problem is identical to the "without project" problem and, thus, the breakeven selling price must be equal to $\beta_r(0)\mathbf{s}(0) + c(0)$, the value given by the option pricing analysis. $\square$

SEPARATION THEOREM. The fact that the same project management strategy α^* solves the grand and investment problems follows from the consistency theorem for complete markets. Fixing $\alpha = \alpha^*$ and taking $\beta_g = \beta_f - \beta_r^*$, the cash flows associated with the grand problem become

$$x_g(t; \alpha^*, \beta_g) = [\beta_f(t-1) - \beta_f(t)]\mathbf{s}(t)$$
$$- [\beta_r^*(t-1) - \beta_r^*(t)]\mathbf{s}(t) + c_{\alpha^*}(t).$$

For all $t > 0$, the last two terms cancel because of the definition of the replicating trading strategy, so $x_g(t; \alpha^*, \beta_g) = x_f(t; \beta_f)$. For $t = 0$, we find $x_g(0; \alpha^*, \beta_g) = x_f(0; \beta_f) + v^*$. Thus, the grand problem reduces to the financing problem and $\beta_g^* = \beta_f^* - \beta_r^*$. $\square$

Proof for §4: Incomplete Markets

CONSISTENCY THEOREM. We focus on the breakeven buying price and show that it lies between the bounds given by the option pricing analysis; a similar argument holds for breakeven selling prices. Suppose the price of the project were greater than the upper bound given by the options analysis, then there exists a trading strategy that dominates the project and costs less than the project. In this case, the project is obviously unattractive as the company can do better investing in securities. Similarly, if the project's price is less than the lower bound, then the project is obviously attractive as the company can do the project and sell a dominated portfolio that costs more than the project and lock in a certain profit. $\square$

Proofs for §5: Partially Complete Markets with Restricted Preferences

PROPOSITION 1. Part (a) of the proposition follows from the consistency theorem (established below) and part (b) follows from the separation theorem. As Proposition 1 is stated for expository reasons and is not used elsewhere, we will not explicitly prove it here.

Integrated Rollback Procedure. We need to prove that the value given by the integrated rollback procedure is equal to $c(0) + \beta_r(0)\mathbf{s}(0)$ where β_r is project c's certainty-equivalent replicating portfolio. Let $v(t)$ denote the value given by the starting the integrated rollback procedure at time t in state of information ω_t (including only cash flows received in time t and after). We will prove, using backward induction, that

$$v(t) = c(t) + \beta_r(t)\mathbf{s}(t). \quad (A1)$$

At time $t = T$, since the project yields a certain cash flow $c(T)$ and generates no future cash flows, since $\beta_r(T) = \mathbf{0}$, (A1) holds with $v(T) = c(T)$. Now suppose that (A1) holds for some time t and all possible time-t states ω_t; to complete the inductive proof we need to show that (A1) holds for $t - 1$ as well. We show this by establishing the following sequence of equalities:

$c(t-1) + \beta_r(t-1)\mathbf{s}(t-1)$

$$= c(t-1) + E_\pi\left[\frac{1}{(1+r_f)}\beta_r(t-1)\mathbf{s}(t)|\omega_{t-1}\right]$$

$$= c(t-1) + E_\pi\left[ECE_t\left[\frac{1}{(1+r_f)}(c(t)+\beta_r(t)\mathbf{s}(t))|\omega_t^m, \omega_{t-1}^p\right]\Big|\omega_{t-1}\right]$$

$$= c(t-1) + E_\pi\left[ECE_t\left[\frac{1}{(1+r_f)}v(t)|\omega_t^m, \omega_{t-1}^p\right]\Big|\omega_{t-1}\right]$$

$$= v(t-1).$$

The first equality follows from our assumptions that securities prices depend only on market states (condition 1) and that the securities market "spans" these marketed states (condition 2), implying the existence of unique risk-neutral probabilities for the market states ω_t^m conditional on ω_{t-1}. The second equality follows from our definition of the certainty-equivalent replicating strategy. The third equality follows from our induction hypothesis and the fourth equality defines our integrated rollback procedure; we discount cash flows at the risk-free rate, compute effective certainty equivalents over the private uncertainties and use risk-neutral probabilities to take expectations over the market uncertainties. □

We prove the consistency and separation theorems after establishing a series of lemmas. In the statements of the lemmas and their proofs, we consider a fixed project c with value v and replicating and rebalancing strategies β_r and β_b. The windfalls x_w are then defined as in §5.5. For any β_f and β_g, we define the grand and financing cash flows as in the grand and financing problem in the separation theorem:

$$x_g(t; \beta_g) = c(t) + [\beta_g(t-1) - \beta_g(t)]\mathbf{s}(t), \quad \text{and}$$

$$x_f(t; \beta_f) = [\beta_f(t-1) - \beta_f(t)]\mathbf{s}(t). \tag{A2}$$

(Note that here the project strategy α is fixed and we suppress α in the definition of x_g.)

LEMMA 1. *For any β_f and $\beta_g = \beta_f - \beta_r + \beta_b$,*

$$x_g(t; \beta_g) = x_f(t; \beta_f) + \rho_t \sum_{\tau=1}^{t} \frac{x_w(\tau)}{R_\tau} \quad \text{for } t > 0 \quad \text{and} \tag{A3a}$$

$$x_g(t; \beta_g) = x_f(t; \beta_f) + v \quad \text{for } t = 0. \tag{A3b}$$

PROOF. Substituting $\beta_g = \beta_f - \beta_r + \beta_b$ into the definition of $x_g(t; \beta_g)$, we have

$$x_g(t; \beta_g) = c(t) + (\beta_f(t-1) - \beta_f(t))\mathbf{s}(t)$$
$$- (\beta_r(t-1) - \beta_r(t))\mathbf{s}(t) + (\beta_b(t-1) - \beta_b(t))\mathbf{s}(t). \tag{A4}$$

Equation (A3b) then follows from noting $\beta_b(0) = 0$ and $v = c(0) + \beta_r(0)\mathbf{s}(0)$. To establish (A3a), note that

$[\beta_b(t-1) - \beta_b(t)]\mathbf{s}(t)$

$$= \left(\frac{\rho_t}{(1+r_f)^t}\sum_{\tau=1}^{t}\frac{x_w(\tau)}{R_\tau} - \frac{x_w(t)}{R_t}\sum_{\tau=t}^{T}\frac{\rho_\tau}{(1+r_f)^\tau}\right)(1+r_f)^t$$

$$= \rho_t\sum_{\tau=1}^{t}\frac{x_w(\tau)}{R_\tau} - c(t) + (\beta_r(t-1) - \beta_r(t))\mathbf{s}(t).$$

The first equality follows the definition of the rebalancing portfolio and cancelling common terms. The second equality follows from the definitions of R_t and $x_w(t)$. Substituting this expression back into equation (A4) yields equation (A3a). □

LEMMA 2. *For any β_f and $\beta_g = \beta_f - \beta_r + \beta_b$,*

$$CE_t\left[\rho_t\frac{x_w(t)}{R_t}\Big|\omega_t^m, \omega_{t-1}^p\right] = 0.$$

(Note that $\rho_t(x_w(t)/R_t)$ is the share of $x_w(t)$ that is absorbed in period t.)

PROOF. Using the definition of CE_t, this result is equivalent to

$$E\left[\exp\left(-\frac{x_w(t)}{R_t}\right)\Big|\omega_t^m, \omega_{t-1}^p\right] = 1. \tag{A5}$$

Using the definition of x_w, we can rewrite the left side of (A5) as

$$\exp\left(\frac{\beta_r(t-1)\mathbf{s}(t)}{R_t}\right)E\left[\exp\left(-\frac{c(t) + \beta_r(t)\mathbf{s}(t)}{R_t}\right)\Big|\omega_t^m, \omega_{t-1}^p\right]$$

which, by the definition of the replicating portfolio, is equal to one. □

LEMMA 3. *For any β_f and $\beta_g = \beta_f - \beta_r + \beta_b$,*

$$CE_t[x_g(t; \beta_g)] = CE_t[x_f(t; \beta_f)] \quad \text{for } t > 0.$$

PROOF. For $t > 0$, we have

$$CE_t[x_g(t; \beta_g))] = CE_t\left[x_f(t; \beta_f) + \rho_t\sum_{\tau=1}^{t}\frac{x_w(\tau)}{R_\tau}\right]$$

$$= CE_t\left[x_f(t; \beta_f) + \rho_t\sum_{\tau=1}^{t-1}\frac{x_w(\tau)}{R_\tau}\right.$$
$$\left. + CE_t\left[\rho_t\frac{x_w(t)}{R_t}\Big|\omega_t^m, \omega_{t-1}^p\right]\right]$$

$$= CE_t\left[x_f(t; \beta_f) + \rho_t\sum_{\tau=1}^{t-1}\frac{x_w(\tau)}{R_\tau}\right]. \tag{A6}$$

The first equality follows from equation (A3a) of Lemma 1. The second equality is derived by computing certainty equivalents iteratively (conditioning on the time-t market state ω_t^m and the time-$(t-1)$ market state ω_{t-1}^p and taking certainty equivalents over the time t private state ω_t^p) and then applying the Δ-property, noting that β_f and $x_f(t; \beta_f)$ are independent of all private states, and, for $\tau < t$, $x_w(\tau)$ is independent of ω_t^p. The next equality follows from Lemma 2. At this point, we have eliminated the $\tau = t$ term from the summation in (A6). Repeating this process and taking expections over the earlier private

state ω_{t-1}^p, ω_{t-2}^p, ..., ω_1^p (all conditioned on the time-t market state ω_t^m) we can eliminate the other terms to obtain the desired result. □

The first lemma shows that if the firm manages project risks according to β_r and β_b, the effect of the project on the grand cash flows is reduced to a lump-sum receipt of the project's value v at time 0 plus a residual cash flow stream in which each period's windfalls are shared with future periods in proportion to the discounted period risk tolerances (compare with Equation 7). In the third lemma, we see that given current information, the certainty equivalent of each period's residual cash flow is zero. Thus, if the firm manages project risks according to β_r and β_b, the net effect of a project on the firm's cash flows is reduced to a lump-sum receipt of v plus a residual cash flow stream that has zero value.

Given these lemmas, we establish the consistency and separation theorems after first establishing the separation theorem for a fixed project.

SEPARATION THEOREM FOR A FIXED PROJECT. Let c be a fixed project with value v, and with replicating and rebalancing trading strategies β_r and β_b, respectively. Let β_f^* be an optimal solution to the financing problem. To establish the separation theorem for a fixed project, we need to show that $\beta_g = \beta_f^* - \beta_r + \beta_b$ is optimal for the grand problem (with fixed project):

$$\max_{\beta_g} E[U(x_g(0; \beta_g), x_g(1; \beta_g), \ldots, x_g(T; \beta_g))]$$

where x_g is defined as in (A2).

Because of our assumption that U is strictly concave, the first-order conditions for optimality are both necessary and sufficient for optimality of β_g^*. These first-order conditions may be written as

$$\frac{\partial}{\partial \beta_g(t, \omega_t)} E[U(x_g(0; \beta_g), x_g(1; \beta_g), \ldots, x_g(T; \beta_g)) | \omega_t] = 0,$$

for all t and ω_t. Using the additive form of the firm's utility function, the first-order conditions are equivalent to

$$k_t \mathbf{s}(t) u_t'(x_g(t; \beta_g)) = k_{t+1} E[\mathbf{s}(t+1) u_{t+1}'(x_g(t+1; \beta_g)) | \omega_t] \quad \text{(A7)}$$

where u_t' denotes the derivative of the utility function u_t for time-t cash flows. This equation can be interpreted as follows. The left side of (A7) is an $n + 1$ vector whose entries represent the marginal utility of the cash flows at time-t (in state ω_t) required to purchase one share of each security. The right side of (A7) is the corresponding vector whose entries represent the expected marginal utilities of the cash flows provided by these shares at time $t + 1$. With an optimal trading strategy, the two vectors of marginal utilities will be equal.

Focusing on the case where $t > 0$ (the proof for $t = 0$ is similar), taking $\beta_g = \beta_f^* - \beta_r + \beta_b$, rewriting (A7) using Equation (A3a) and dropping common constant factors, we see that (A7) holds if and only if:

$$k_t \mathbf{s}(t) u_t'(x_f(t; \beta_f^*))$$

$$= k_{t+1} E\left[\mathbf{s}(t+1) u_{t+1}'(x_f(t+1; \beta_f^*)) \exp\left(-\frac{x_w(t+1)}{R_{t+1}}\right) \bigg| \omega_t\right]. \quad \text{(A8)}$$

Noting that $\mathbf{s}(t+1)$ and $x_f(t+1; \beta_f^*)$ are both independent of ω_{t+1}^p and taking expectations iteratively, first over ω_{t+1}^p and then over ω_{t+1}^m, we can rewrite the right side of (A8) as:

$$E\left[\mathbf{s}(t+1) u_{t+1}'(x_f(t+1; \beta_f^*)) E\left[\exp\left(-\frac{x_w(t+1)}{R_{t+1}}\right) \bigg| \omega_{t+1}^m, \omega_t^p\right] \bigg| \omega_t\right].$$

Applying Equation (A6), established in the proof of Lemma 2, this becomes

$$E[\mathbf{s}(t+1) u_{t+1}'(x_f(t+1; \beta_f^*)) | \omega_t].$$

Noting that this expression is independent of ω_t^p and substituting back into (A8), we find that (A7) is satisfied if

$$k_t \mathbf{s}(t) u_t'(x_f(t; \beta_f^*))$$

$$= k_{t+1} E[\mathbf{s}(t+1) u_{t+1}'(x_f(t+1; \beta_f^*)) | \omega_t^m].$$

These are precisely the first-order necessary and sufficient conditions for the financing problem (they are analogous to the conditions of Equation A7). Thus if β_f^* is optimal for the financing problem, then $\beta_g = \beta_f^* - \beta_r + \beta_b$ is optimal for the grand problem. □

CONSISTENCY THEOREM. We first show the breakeven selling price for any project c is equal to the value v given by our integrated valuation procedure. Examining the equality defining the breakeven selling price, we see that the problem on the left side of the equality is equivalent to the grand problem (for a fixed project) and, if v is the breakeven selling price, the right side is equivalent to the financing problem. Thus we can establish the consistency theorem by showing that maximal expected utilities in the grand and financing problems are equal. Taking β_f^* to be the solution to the financing problem, by the separation theorem for a fixed project, the solution to the grand problem is given by $\beta_g^* = \beta_f^* - \beta_r + \beta_b$. Applying Equation (A2b) of Lemma 1 and Lemma 2, we see that for $\beta_g^* = \beta_f^* - \beta_r + \beta_b$, the certainty equivalents of the grand and financing cash flows are the same in each period. Given additive independence of preferences, this then implies the overall maximal expected utilities for the grand and financing problems must be equal as well, and the breakeven selling price is equal to v.

To show that the breakeven buying price for any project c is equal to the value v given by our integrated valuation procedure, let c'' be a new project identical to c except for the time-0 cash flows where $c''(0) = c(0) - v$. From the definition of our integrated valuation procedure, the value of c'' is equal to 0. By the previous argument for breakeven selling prices, then the firm is indifferent between receiving c'' and a lump-sum time-0 payment equal to 0. Thus v is the breakeven buying price for project c. □

PROOF OF SEPARATION THEOREM. The separation theorem for a fixed project was established earlier. We complete the proof by showing that the project management strategy α^* that solves the investment problem also solves the grand problem. In the consistency theorem, we saw that for any fixed project c the financing and grand problems have equal maximal expected utilities. Since the firm's preferences are strictly increasing in time-0 cash flows, the project management strategy that maximizes v, maximizes the expected utility in the financing problem and, hence, in the grand problem. □

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