Signed Domination of Oriented Matroid Systems

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Abstract. The domination function has played an important part in reliability theory. While most of the work in this field has been restricted to various types of network system models, many of the results can be generalized to much wider families of systems associated with matroids. Previous papers have explored the relation between undirected network systems and matroids. In this paper the main focus is on directed network systems and *oriented matroids*. Classical results for directed network systems include the fact that the signed domination is either $+1$ or -1 if the network is acyclic, and zero otherwise. It turns out that these results can be generalized to systems derived from oriented matroids. Several classes of such systems will be discussed.

Keywords. Reliability, directed networks, oriented matroids

Introduction

The domination function has played an important part in reliability theory. Classical references on this topic are [11], and [12]. More recent work in this area related to the present paper includes [4] and [5]. Most of the work in the field has been restricted to various types of network system models. However, many of the results can be generalized to much wider families of systems associated with matroids. Previous papers, e.g., [6], [7], [8], and [10] have explored the relation between undirected network systems and matroids. In this paper we focus on directed network systems and oriented matroids.

1. Basic Concepts

We start out by reviewing the basic concepts of reliability theory (see [1]). A *binary monotone system* is an ordered pair (E, ϕ) where $E = \{1, \ldots, n\}$ is a nonempty finite set, and ϕ is a binary nondecreasing function defined for all binary vectors $\boldsymbol{X} = (X_1, \ldots, X_n)$. The elements of E are interpreted as *components* of some technological *system*. Each component can be either *functioning* or *failed*. The vector $\boldsymbol{X}$ is referred to as the component state vector. That is, for all $i \in E$, $X_i = 1$ if the ith component is functioning and zero otherwise. The function ϕ is called the *structure function* of the

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system and represents the system state as a function of the component states. That is, $\phi = \phi(\boldsymbol{X}) = 1$ if the system is functioning and zero otherwise. A *minimal path set* of a binary monotone system (E, ϕ), is a minimal subset $P \subseteq E$ such that if $X_i = 1$ for all $i \in P$, and zero otherwise, then $\phi(\boldsymbol{X}) = 1$.

It is well-known (see [1]) that the structure function of a binary monotone system is always *multilinear*. That is, it can be written in the following form:

$$\phi(\boldsymbol{X}) = \sum_{A \subseteq E} \delta(A) \prod_{i \in A} X_i$$

The function δ, defined for all subsets $A \subseteq E$, is called the *signed domination function* of the system. The system reliability can also be expressed in terms of the signed domination function as:

$$\Pr(\phi(\boldsymbol{X}) = 1) = E[\phi(\boldsymbol{X})] = \sum_{A \subseteq E} \delta(A) E[\prod_{i \in A} X_i] \tag{1}$$

Thus, we see that both the structure function and the system reliability is uniquely determined by the signed domination function. Since the number of terms in the right-hand sum in (1) is 2^n, this formula may be very slow to compute. Fortunately, however, many systems have signed domination functions where $\delta(A)$ is zero for a large number of sets. This may simplify the calculations considerably.

The formula (1) is of particular interest in the study of directed network systems. Such a system is illustrated in Figure 1. The components of the system are the edges, labeled $1, 2, \ldots, 7$. The system is said to be functioning if there exists a directed path of functioning edges from the source s to the terminal t. If (E, ϕ) is a directed network

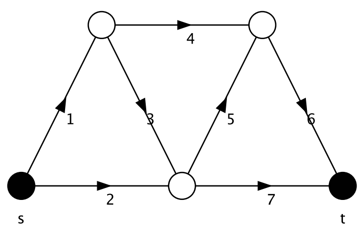

Figure 1. *An acyclic directed network*

system, and $A \subseteq E$, then $v(A)$ denotes the number of nodes being adjacent to at least one edge in A. A key result for directed network systems is the following classical theorem (see [11]):

Theorem 1 *If (E, ϕ) is a directed network system, then the signed domination function satisfies the following:*

$$\delta(A) = (-1)^{|A| - v(A) + 1},$$

if A is an acyclic union *of minimal path sets (i.e., a union of minimal path sets which does not contain any directed circuit of the network). Otherwise $\delta(A) = 0$.*

The main purpose of this paper is to explore the possibility of generalizing the results for directed network systems. It turns out that this can be done within the framework of oriented matroids.

2. Oriented Matroid Systems

A *signed set* is a set M along with a mapping $\sigma_M : M \to \{+, -\}$, called the *sign mapping* of the set. With a slight abuse of notation, M refers both to the signed set itself as well as the underlying unsigned set of elements. The sign mapping σ_M defines a partition of M into two subsets, $M^+ = \{e \in M : \sigma_M(e) = +\}$ and $M^- = \{e \in M : \sigma_M(e) = -\}$. M^+ and M^- are referred to as the *positive* and *negative* elements of M respectively. If M is a signed set with $M^+ = \{e_1, \dots, e_i\}$ and $M^- = \{f_1, \dots, f_j\}$, we indicate this by writing M as $\{e_1, \dots, e_i, \bar{f}_1, \dots, \bar{f}_j\}$. If $M = M^+$, M is called a *positive* set, while if $M = M^-$, M is called a *negative* set. $-M$ denotes the signed set obtained from M by reversing the signs of all the elements, i.e., $\sigma_{-M}(e) = -\sigma_M(e)$ for all $e \in M$. If $\mathcal{M}$ is a family of signed sets, the family of sign mappings, $\{\sigma_M : M \in \mathcal{M}\}$, is called the *sign signature* of $\mathcal{M}$.

Signed sets can be used to describe paths in directed networks by letting the positive elements represent edges directed the *same way* as the path, while negative elements represent edges directed the *opposite way* of the path. As an example consider once again the directed network system shown in Figure 1. The signed minimal path sets from the source s to the terminal t are:

$$
\begin{aligned}
&P_1 = \{1, 4, 6\}, \qquad P_2 = \{1, 4, \bar{5}, 7\}, \qquad P_3 = \{1, 3, 5, 6\}, \\
&P_4 = \{1, 3, 7\}, \qquad P_5 = \{2, 5, 6\}, \qquad P_6 = \{2, \bar{3}, 4, 6\}, \qquad P_7 = \{2, 7\},
\end{aligned}
$$

while the *positive* minimal path sets between s an t are P_1, P_3, P_4, P_5, P_7. We now proceed by adding an "artificial" edge x from t to s, and thus turning all the paths into circuits. See Figure 2. Let $\mathcal{M}$ denote the family of all signed circuits in the extended network. We also introduce the following families of sets:

$$
\begin{aligned}
&\bar{\mathcal{P}} = \{(M \setminus x) : M \in \mathcal{M},\ x \in M^+\}, \\
&\mathcal{P} = \{(M \setminus x) : M \in \mathcal{M},\ x \in M^+ \text{ and } (M \setminus x)^- = \emptyset\}.
\end{aligned}
$$

It is easy to see that $\bar{\mathcal{P}}$ is the family of *all* signed minimal path sets from the source s to the terminal t, while $\mathcal{P}$ is the family of the positive such sets. Given the element x, $\bar{\mathcal{P}}$ and $\mathcal{P}$ can be derived from $\mathcal{M}$ without any knowledge of the node structure of the network. Thus, all relevant information about the system is stored within $\mathcal{M}$.

The family of signed circuits of a directed graph satisfies certain properties which can be formalized within the theory of *oriented matroids*. An oriented matroid is defined as follows (see [3]):

Definition 2 *An* oriented matroid *is an ordered pair $(F, \mathcal{M})$ where F is a nonempty finite set, and $\mathcal{M}$ is a family of signed subsets of F, called* signed circuits *satisfying the following properties:*

(O1) *$\emptyset$ is not a signed circuit.*

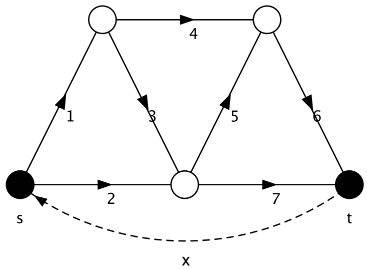

Figure 2. *A 2-terminal directed network system with an artificial edge, x*

(O2) *If M is a signed circuit, then so is $-M$.*
(O3) *For all $M_1, M_2 \in \mathcal{M}$ such that $M_1 \subseteq M_2$, we either have $M_1 = M_2$ or $M_1 = -M_2$.*
(O4) *If M_1 and M_2 are signed circuits such that $M_1 \neq -M_2$, and $e \in M_1^+ \cap M_2^-$, then there exists a third signed circuit M_3 with $M_3^+ \subseteq (M_1^+ \cup M_2^+) \setminus e$ and $M_3^- \subseteq (M_1^- \cup M_2^-) \setminus e$.*

If $(F, \mathcal{M})$ is an oriented matroid, the elements of F may sometimes be interpreted as vectors in a linear space, in which case the circuits correspond to *minimal linearly dependent sets*. An *independent set* of an oriented matroid is defined as a set which does not contain any circuit. If $(F, \mathcal{M})$ is an oriented matroid, the *rank function* of the matroid, denoted $\rho(A)$, is defined for all $A \subseteq E$ as the cardinality of the largest independent subset of A.

Definition 3 *Let $(E \cup x, \mathcal{M})$ be an oriented matroid, and let (E, ϕ) be a binary monotone system with minimal path set family $\mathcal{P}$ given by:*

$$\mathcal{P} = \{(M \setminus x) : M \in \mathcal{M},\ x \in M^+ \text{ and } (M \setminus x)^- = \emptyset\} \tag{2}$$

We then say that (E, ϕ) is the oriented matroid system derived from the oriented matroid $(E \cup x, \mathcal{M})$ with respect to x, and write this as $(E \cup x, \mathcal{M}) \to (E, \phi)$. If $(E \cup x, \mathcal{M}) \to (E, \phi)$, a subset $A \subseteq E$ is said to be cyclic *if there exists a positive circuit $M \in \mathcal{M}$ such that $M \subseteq A$. If no such circuit exists, A is said to be* acyclic. *In particular the system (E, ϕ) is said to be cyclic (acyclic) if E is cyclic (acyclic).*

The class of oriented matroid systems generalizes the class of 2-terminal directed network systems. Moreover, Theorem 1 can be generalized to the class of oriented matroid systems:

Theorem 4 *If $(E \cup x, \mathcal{M}) \to (E, \phi)$, then:*

$$\delta(A) = (-1)^{|A| - \rho(A \cup x)},$$

if A is an acyclic union of minimal path sets (i.e., a union of minimal path sets which does not contain any positive circuit of $\mathcal{M}$). Otherwise $\delta(A) = 0$.

Proof: See [9] ■

3. Oriented Matrix Systems

In order to introduce the class of oriented matrix systems, we start out by letting (E, ϕ) be a binary monotone system where $E = \{1, \ldots, n\}$. If the component state vector is $\boldsymbol{X}$, we introduce the set $A = A(\boldsymbol{X}) = \{i : X_i = 1\}$. For each $i \in E$ we associate a vector denoted $\boldsymbol{v}_i$ belonging to some vector space over an ordered field, say e.g., $\mathbb{R}$. We also introduce a "target" vector $\boldsymbol{u}$ belonging to the same vector space. We then define $\phi(\boldsymbol{X})$ to be 1 if there exists $\{\lambda_i \geq 0 : i \in A(\boldsymbol{X})\}$ so that:

$$\sum_{i \in A} \lambda_i \boldsymbol{v}_i = \boldsymbol{u}, \tag{3}$$

and zero otherwise. Thus, the system is functioning if and only if the convex cone spanned by the vectors $\{\boldsymbol{v}_i : i \in A\}$ contains the target vector. We refer to such a system as an *oriented matrix system*. It can be shown that such a system is in fact a special case of an oriented matroid system. We denote the corresponding matroid by $(E \cup x, \mathcal{M})$. To the artificial component x we associate the vector $\boldsymbol{v}_x = -\boldsymbol{u}$. The family of signed circuits $\mathcal{M}$ consists of the sets $M \subseteq (E \cup x)$ such that $\{\boldsymbol{v}_i : i \in M\}$ is a minimal linearly dependent set of vectors. Thus, if $M \in \mathcal{M}$, there exists a set of non-zero constants $\{\lambda_i : i \in M\}$ such that:

$$\sum_{i \in M} \lambda_i \boldsymbol{v}_i = \boldsymbol{0}. \tag{4}$$

Moreover, given $\{\lambda_i : i \in M\}$, the sign map of M is defined so that $M^+ = \{i : \lambda_i > 0\}$, while $M^- = \{i : \lambda_i < 0\}$.

Finally, the rank function of $(E \cup x, \mathcal{M})$, denoted ρ, reduces to "ordinary" matrix rank. That is, if $A \subseteq (E \cup x)$, then $\rho(A)$ is equal to the rank of the matrix with columns $\{\boldsymbol{v}_i : i \in A\}$.

We observe that if $M \in \mathcal{M}$, $x \in M^+$ and $(M \setminus x)^- = \emptyset$, we have:

$$\sum_{i \in M \setminus x} \frac{\lambda_i}{\lambda_x} \boldsymbol{v}_i = -\boldsymbol{v}_x = \boldsymbol{u}. \tag{5}$$

Thus, $(M \setminus x)$ is indeed a minimal path set of (E, ϕ).

Since (E, ϕ) is an oriented matroid system, it follows by Theorem 4 that $\delta(A) = (-1)^{|A| - \rho(A \cup x)}$ if A is an acyclic union of minimal path sets and zero otherwise.

The class of oriented matrix systems can be viewed as a generalization of the class of 2-terminal directed network systems. In particular, if (E, ϕ) is a 2-terminal directed network system, the associated vectors correspond to the columns of the node-arc incidence matrix of the network graph, including the artificial edge x from the terminal back to the source. (See Figure 2).

We recall that for an oriented matroid system (E, ϕ) a subset $A \subseteq E$ is acyclic if A does not contain any positive circuits. Thus, in an oriented matrix system (E, ϕ) with associated vectors $\{\boldsymbol{v}_i : i \in E\}$, $A \subseteq E$ is is cyclic if there exists a set of nonnegative numbers $\{\lambda_i : i \in A\}$ where $\lambda_j > 0$ for at least one $j \in A$, and such that:

$$\sum_{i\in A} \lambda_i \boldsymbol{v}_i = \mathbf{0}. \tag{6}$$

Note that if (6) holds for the set of nonnegative numbers $\{\lambda_i : i \in A\}$ and $c > 0$, then (6) also holds for $\{c\lambda_i : i \in A\}$. Thus, since not all the λ_is are zero, we may scale them so they add up to 1, in which case the left-hand side of (6) becomes a convex combination of the $\boldsymbol{v}_i$s. Hence, A is cyclic if and only if $\mathbf{0}$ is contained in the convex hull of $\{\boldsymbol{v}_i : i \in A\}$. If not, A is acyclic.

4. Oriented k-out-of-n systems

Let (E, ϕ) be a binary montone system where $|E| = n$, and assume that $\phi(\boldsymbol{X}) = 1$ if $|A(\boldsymbol{X})| \geq k$ and zero otherwise. Then the system is said to be a k-out-of-n system. That is, the system is functioning if and only if at least k of the n components are functioning. Thus, the minimal path sets of a k-out-of-n system are all sets $P \subseteq E$ such that $|P| = k$. The class of k-out-of-n systems has been studied extensively in the reliability litterature. See e.g., [1]. An efficient algorithm for calculating the reliability of k-out-of-n systems is given in [2].

In [6] it is shown that k-out-of-n systems can be associated with matroids in the same way as undirected network systems. It turns out that it is possible to derive oriented matroid systems from the class of k-out-of-n systems as well. Thus, we let $E = \{1, \ldots, n\}$ be a set of components and let k be an integer such that $1 \leq k \leq n$. We then consider what is known as a "uniform" oriented matroid $(E \cup x, \mathcal{M})$ with rank k. See [3]. That is, $\mathcal{M}$ is given as.

$$\mathcal{M} = \{M \subseteq (E \cup x) : |M| = k + 1\}, \tag{7}$$

and equipped with a suitable sign signature. Note that since all the circuits of $(E \cup x, \mathcal{M})$ contains $k + 1$ elements, it follows that the largest independent subsets of $E \cup x$ contain k elements. Thus, by definition of the rank we indeed have that $\rho(E \cup x) = k$.

Then let $(E, \bar{\phi})$ be the binary monotone system with minimal path sets $\bar{\mathcal{P}} = \{(M \setminus x) : x \in M^+\}$. Hence, $\bar{\mathcal{P}}$ consists of all subsets of E with cardinality k, so $(E, \bar{\phi})$ is a k-out-of-n system.

Now, consider instead the system (E, ϕ) with minimal path sets $\mathcal{P} = \{P \in \bar{\mathcal{P}} : P^- = \emptyset\}$. Thus, only the positive sets of $\bar{\mathcal{P}}$ are included in $\mathcal{P}$. By definition (E, ϕ) is an oriented matroid system, and we then refer to this system as an *oriented* k-out-of-n system. Note that the exact form of (E, ϕ) depends on the sign signature of $(E \cup x, \mathcal{M})$. Thus, in general there will be many different types of oriented k-out-of-n systems. Some of these are acyclic, while others are cyclic. In the case where (E, ϕ) is acyclic, i.e., where E does not contain any positive circuits, it follows by Theorem 4 that:

$$\delta(E) = (-1)^{|E| - \rho(E \cup x)} = (-1)^{n-k}, \tag{8}$$

while in the cyclic case $\delta(E) = 0$.

Example 5 *Let (E, ϕ) be an oriented matrix system where $E = \{1, \ldots, 5\}$. Assume that the associated vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_5$ all have the same length and are located in the first*

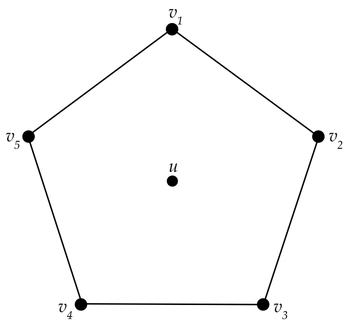

Figure 3. *Vectors in $\mathbb{R}^3$ forming a regular pentagon, and projected into a plane orthogonal to the center point*

octant of $\mathbb{R}^3$ forming a regular pentagon. Furthermore, assume that the target vector $\boldsymbol{u}$ is located at the center of this pentagon. The system is illustrated Figure 3, where we have projected all the points into a plane orthogonal to the center point of the pentagon. As usual we denote the corresponding matroid by $(E \cup x, \mathcal{M})$, and let $\boldsymbol{v}_x = -\boldsymbol{u}$.

By the choice of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_5, \boldsymbol{v}_x$ it is clear that any set of three of these vectors forms a basis for $\mathbb{R}^3$. Since $\mathcal{M}$ by definition consists of the sets $M \subseteq (E \cup x)$ such that $\{\boldsymbol{v}_i : i \in M\}$ is a minimal linearly dependent set of vectors, it follows that we in this case have $\mathcal{M} = \{M \subseteq (E \cup x) : |M| = 4\}$. Thus, $(E \cup x, \mathcal{M})$ is a uniform oriented matroid, and we have:

$$\rho(E \cup x) = \operatorname{rank}[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_5, \boldsymbol{v}_x] = 3. \tag{9}$$

Hence, by the definition of oriented k-out-of-n systems it is evident that (E, ϕ) is an oriented 3-out-of-5 system. On the other hand (E, ϕ) is by definition also an oriented matrix system. Thus, if $A(\boldsymbol{X}) \subseteq E$ is the set of functioning components, it follows that $\phi(\boldsymbol{X}) = 1$ if and only if the target vector $\boldsymbol{u}$ is contained in the convex cone spanned by the vectors $\{\boldsymbol{v}_i : i \in A(\boldsymbol{X})\}$. Considering the projection in Figure 3 this is equivalent to the projection of $\boldsymbol{u}$ being contained in the polygon spanned by the projections of the vectors $\{\boldsymbol{v}_i : i \in A(\boldsymbol{X})\}$. For this to hold we must have $|A(\boldsymbol{X})| \geq 3$. Moreover, if $|A(\boldsymbol{X})| = 3$, the projections cannot be consecutive points in the pentagon. Thus, e.g., the triangle corresponding to the set $\{1, 2, 4\}$ contains the projection of the target, so $\phi(1, 1, 0, 1, 0) = 1$. On the other hand the triangle corresponding to the set $\{1, 2, 3\}$ does not contain the projection of the target, so $\phi(1, 1, 1, 0, 0) = 0$. From this we get that the minimal path sets of the system are $\mathcal{P} = \{P_1, \ldots, P_5\}$ where $P_1 = \{1, 2, 4\}$, $P_2 = \{2, 3, 5\}$, $P_3 = \{1, 3, 4\}$, $P_4 = \{2, 4, 5\}$, and $P_5 = \{1, 3, 5\}$.

Since all the associated vectors are located in the first octant of $\mathbb{R}^3$, the convex hull of these vectors cannot contain **0**. *Thus, (E, ϕ) is acyclic. Hence, by Theorem 4 it follows that $\delta(E) = (-1)^{|E| - \rho(E \cup x)} = (-1)^{5-3} = 1$.*

5. Discussion

In the present paper we have introduced the class of oriented matroid systems, and shown how the classical domination results for directed network systems can be extended to this class. Since 2-terminal directed network systems are special cases of oriented matroid systems, the domination results for such network systems are covered completely by our results. In [6] and [7] it was shown that multi-terminal undirected network systems can be handled in a unified way using matroid theory. Thus, a natural conjecture would be that similar unifying results can be obtained in the directed case. Preliminary investigations of this, however, indicates that the problem is much more difficult than in the undirected case, and that certain restrictions will apply.

Acknowledgments

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