New results on the Barlow-Proschan and Natvig measures of component importance in nonrepairable and repairable systems

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Abstract In this paper dynamic and stationary measures of importance of a component in a binary system are considered. To arrive at explicit results we assume the performance processes of the components to be independent and the system to be coherent. Especially, the Barlow-Proschan and the Natvig measures are treated in detail and a series of new results and approaches are given. For the case of components not undergoing repair it is shown that both measures are sensible. Reasonable measures of component importance for repairable systems represent a challenge. A basic idea here is also to take a so-called dual term into account. According to the extended Barlow-Proschan measure a component is important if there are high probabilities both that its failure is the cause of system failure and that its repair is the cause of system repair. Even with this extension results for the stationary Barlow-Proschan measure are not satisfactory. According to the extended Natvig measure a component is important if both by failing it strongly reduces the expected system uptime and by being repaired it strongly reduces the expected system downtime. With this extension the results for the stationary Natvig measure seem very sensible.

Keywords dynamic measures · importance of a system component · nonrepairable systems · repairable systems · stationary measures

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1 Introduction

There seem to be two main reasons for coming up with a measure of importance of system components. Reason 1: it permits the analyst to determine which components merit the most additional research and development to improve overall system reliability at minimum cost or effort. Reason 2: it may suggest the most efficient way to diagnose system failure by generating a repair checklist for an operator to follow.

It should be noted that no measure of importance can be expected to be universally best irrespective of usage purpose. In this paper we will concentrate on what could be considered as allround measures of component importance focusing both on the contributions of the components to system failure and survival or system downtime and uptime. In Natvig and Gåsemyr (2006) two extensions of the Barlow-Proschan measure are considered focusing respectively

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on system failure and survival. Furthermore, in Huseby (2004) two Birnbaum type measures for repairable systems are considered focusing respectively on system downtime and uptime. In addition, we are not considering measures admitting the extreme case of improvement to perfect functioning as for instance the $I_{N_4}^{(i)}$ measure given in Natvig (1985). This is based on the expected increase in system lifetime by replacing the ith component by a perfect one. Examples of importance measure applications in probabilistic risk analysis are given in Cheok et al. (1998) and Borgonovo and Apostolakis (2001). In the present paper we consider importance measures for single components. Some simple joint importance measures for two components in a nonrepairable system are considered in Armstrong (1995) and Wu (2005).

Consider a system consisting of n components. Let $(i = 1, \ldots, n)$:

$$X_i(t) = \begin{cases} 1 & \text{if the } i\text{th component functions at time } t, \\ 0 & \text{if the } i\text{th component is failed at time } t. \end{cases}$$

Assume also that the stochastic processes $\{X_i(t), t \geq 0\}$, $i = 1, \ldots, n$, are mutually independent. For the dynamic approach of the present paper this is a necessary assumption in order to arrive at explicit results. For real life systems modelling of component dependence and common cause failures is of course important, see for instance Gåsemyr and Natvig (1995a, b, 1998, 2005). Introduce $\mathbf{X}(t) = (X_1(t), \ldots, X_n(t))$ and let:

$$\varphi(\mathbf{X}(t)) = \begin{cases} 1 & \text{if the system functions at time } t, \\ 0 & \text{if the system is failed at time } t. \end{cases}$$

The following notation will be used:

$$(\cdot_i, \mathbf{x}) = (x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n).$$

We assume the *structure function* φ to be coherent. Hence, φ is nondecreasing in each argument and each component is relevant, i.e. for each component i there exists a combination of the states of the other components, $(\cdot_i, \mathbf{x})$, such that $\varphi(1_i, \mathbf{x}) = 1$ and $\varphi(0_i, \mathbf{x}) = 0$. For an excellent introduction to coherent structure theory, we refer to Barlow and Proschan (1981).

The paper is organized as follows. In Section 2 the Birnbaum, Barlow-Proschan and Natvig measures of component importance in nonrepairable systems are considered. The Barlow-Proschan and Natvig measures of component importance in repairable systems, with their dual extensions, are respectively treated in Sections 3 and 4. Finally, some concluding remarks are given in Section 5.

2 Measures of component importance in nonrepairable systems

In this section we restrict our attention to the case where the components, and hence the system, cannot be repaired. Let the ith component have an absolutely continuous life distribution $F_i(t)$ with density $f_i(t)$. Then the *reliability* of this component at time t is given by:

$$P[X_i(t) = 1] = 1 - F_i(t) \stackrel{d}{=} \bar{F}_i(t).$$

Introduce $\bar{\mathbf{F}}(t) = (\bar{F}_1(t), \ldots, \bar{F}_n(t))$. Then the reliability of the system at time t is given by:

$$P[\varphi(\mathbf{X}(t)) = 1] = h(\bar{\mathbf{F}}(t)),$$

where h is the system's *reliability function*.

2.1 The Birnbaum measure

Birnbaum (1969) defines the importance of the ith component at time t by:

$$I_B^{(i)}(t) = P[\varphi(1_i, \mathbf{X}(t)) - \varphi(0_i, \mathbf{X}(t)) = 1] = h(1_i, \bar{\mathbf{F}}(t)) - h(0_i, \bar{\mathbf{F}}(t)), \qquad (1)$$

which is the probability that the system is in a state at time t in which the functioning of the ith component is critical, i.e. the system functions if the ith component functions and is failed otherwise. By pivot decomposition it follows that:

$$I_B^{(i)}(t) = \frac{\partial h(\bar{\mathbf{F}}(t))}{\partial \bar{F}_i(t)},$$

indicating that the Birnbaum measure reflects Reason 1. There are two main objections to this measure. Firstly, it gives the importance at fixed points of time leaving for the analyst at the system development phase to determine which points are important. Secondly, the measure does not depend on the reliability of the ith component, whether good or bad, although the ranking of the importances of the components depends on all component reliabilities.

2.2 The Barlow-Proschan measure

These objections cannot be raised to the time-independent Barlow and Proschan (1975) importance of the ith component:

$$I_{B-P}^{(i)} = P[\text{The failure of the } i\text{th component coincides with the failure of the system}].$$

Now obviously:

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) f_i(t) dt = \int_0^\infty [h(1_i, \bar{\mathbf{F}}(t)) - h(0_i, \bar{\mathbf{F}}(t))] f_i(t) dt. \qquad (2)$$

Hence, the Barlow-Proschan measure is a weighted average of the Birnbaum measure, the weight at time t being $f_i(t)$. According to this measure a component is more important the more likely it is to be the direct cause of system failure, i.e. its failure coincides with the failure of the system, indicating that it takes well care of both Reasons 1 and 2.

Since a system failure coincides with the failure of exactly one component, we have:

$$\sum_{i=1}^n I_{B-P}^{(i)} = 1. \qquad (3)$$

This is not true for the Birnbaum measure. The following theorem, with an original proof, generalizing a theorem the proof of which is sketched in Barlow and Proschan (1975), seems to indicate that this is a sensible measure:

Theorem 1. *Let the ith component be in series (parallel) with the rest of the system. Let for an arbitrary component $j \neq i$ $F_i(t) \geq F_j(t) > 0$ ($\bar{F}_i(t) \geq \bar{F}_j(t) > 0$) for all $t \geq 0$. Hence, F_i is less (greater) than F_j in the ordinary stochastic order. Then:*

$$I_{B-P}^{(i)} \geq I_{B-P}^{(j)} + \int_0^\infty \frac{f_j(t)}{\bar{F}_j(t)} h(0_j, \bar{\boldsymbol{F}}(t)) dt$$

$$\left(I_{B-P}^{(i)} \geq I_{B-P}^{(j)} + \int_0^\infty \frac{f_j(t)}{F_j(t)}(1 - h(1_j, \bar{\mathbf{F}}(t)))dt \right).$$

Proof: When the ith component is in series with the rest of the system, we have noting that $h(\bar{\boldsymbol{F}}(t))$ is nonincreasing in t:

$$\begin{aligned}
I_{B-P}^{(i)} &= \int_0^\infty f_i(t)h(1_i, \bar{\mathbf{F}}(t))dt = \int_0^\infty \frac{f_i(t)}{\bar{F}_i(t)} h(\bar{\mathbf{F}}(t))dt \\
&= \big[- \ln(\bar{F}_i(t))h(\bar{\mathbf{F}}(t))\big]_0^\infty + \int_0^\infty \ln(\bar{F}_i(t)) \frac{d}{dt} h(\bar{\mathbf{F}}(t))dt \\
&= \int_0^\infty \ln(\bar{F}_i(t)) \frac{d}{dt} h(\bar{\mathbf{F}}(t))dt \\
&\geq \int_0^\infty \ln(\bar{F}_j(t)) \frac{d}{dt} h(\bar{\mathbf{F}}(t))dt = \int_0^\infty \frac{f_j(t)}{\bar{F}_j(t)} h(\bar{\mathbf{F}}(t))dt \\
&= \int_0^\infty f_j(t) \frac{1}{\bar{F}_j(t)} [\bar{F}_j(t)h(1_j, \bar{\mathbf{F}}(t)) + (1 - \bar{F}_j(t))h(0_j, \bar{\mathbf{F}}(t))]dt \\
&= I_{B-P}^{(j)} + \int_0^\infty \frac{f_j(t)}{\bar{F}_j(t)} h(0_j, \bar{\mathbf{F}}(t))dt.
\end{aligned}$$

Similarly, when the ith component is in parallel with the rest of the system:

$$\begin{aligned}
I_{B-P}^{(i)} &= \int_0^\infty f_i(t)[1 - h(0_i, \bar{\mathbf{F}}(t))]dt \\
&= \int_0^\infty \frac{f_i(t)}{F_i(t)}[1 - h(\bar{\mathbf{F}}(t))]dt = \int_0^\infty (\ln F_i(t)) \frac{d}{dt} h(\bar{\mathbf{F}}(t))dt \\
&\geq \int_0^\infty (\ln F_j(t)) \frac{d}{dt} h(\bar{\mathbf{F}}(t))dt \\
&= \int_0^\infty f_j(t) \frac{1}{F_j(t)} [1 - (1 - F_j(t))h(1_j, \bar{\mathbf{F}}(t)) - F_j(t)h(0_j, \bar{\mathbf{F}}(t))]dt \\
&= I_{B-P}^{(j)} + \int_0^\infty \frac{f_j(t)}{F_j(t)} (1 - h(1_j, \bar{\mathbf{F}}(t)))dt.
\end{aligned}$$

The result in Barlow and Proschan (1975) follows by noting that the second terms on the right hand side of the inequalities in Theorem 1 are nonnegative. Note that these terms are zero when the jth component is respectively in series and in parallel with the rest of the system. This theorem gives lower bounds for how much larger $I_{B-P}^{(i)}$ is than $I_{B-P}^{(j)}$, not just that it is larger. For a very recent book on stochastic orders, we refer to Shaked and Shanthikumar (2007).

As an example consider a three component system where component 1 is in series with a parallel subsystem of components 2 and 3 and where the lifetime of the kth component is exponentially distributed with failure rate λ_k, $k = 1, 2, 3$. Letting $i = 1$ and $j = 2$ and assume $\lambda_1 \geq \lambda_2$, the second term on the right hand side of the first inequality in Theorem 1 equals $\lambda_2/(\lambda_1 + \lambda_3)$. If λ_3 is large compared to λ_2, this term is close to 0. This is natural since then we actually are close to having a series system of components 1 and 2. If on the other hand λ_3 is small compared to λ_2, and λ_1 is close to λ_2, this term is close to 1. This reflects that the importance of component 2 is close to 0 in this case.

2.3 The Natvig measure

Intuitively it seems that components that by failing strongly reduce the expected remaining system lifetime are very important. This seems at least true during the system development phase. This is the motivation for the Natvig (1979) measure. In Natvig (1982) a stochastic representation of this measure was obtained by considering the random variables ($i = 1, \ldots, n$):

$$Z_i = Y_i^1 - Y_i^0, \tag{4}$$

where:

Y_i^0 = remaining system lifetime just *after* the failure of the ith component

Y_i^1 = remaining system lifetime just *after* the failure of the ith component, which, however, immediately undergoes a fictive minimal repair; i.e. it is repaired to have the same distribution of remaining lifetime as it had just before failing.

Thus, Z_i can be interpreted as the increase in system lifetime due to a minimal repair of the ith component at failure. Then the Natvig measure of importance of the ith component is defined as:

$$I_N^{(i)} = \frac{EZ_i}{\sum_{j=1}^n EZ_j}, \tag{5}$$

tacitly assuming $EZ_i < \infty$, $i = 1, \ldots, n$. Obviously:

$$0 \leq I_N^{(i)} \leq 1, \qquad \sum_{i=1}^n I_N^{(i)} = 1. \tag{6}$$

In Natvig (1985) the following surprising relation was proved:

$$EZ_i = \int_0^\infty I_B^{(i)}(t)\bar{F}_i(t)(-\ln \bar{F}_i(t))dt. \tag{7}$$

Hence, as for the Barlow-Proschan measure EZ_i is a weighted average of the Birnbaum measure. The weight at time t, $\bar{F}_i(t)[-\ln \bar{F}_i(t)]$, is the improvement in the reliability of the ith component at time t due to the allowance of one minimal repair at failure. The Natvig measure for dependent components is considered in Norros (1986) using martingale notions and methods.

The following very similar theorem to Theorem 1 is an essential improvement of a theorem in Natvig (1985):

Theorem 2. *Let the ith component be in series (parallel) with the rest of the system. Let for an arbitrary component $j \neq i$ $F_i(t) \geq F_j(t) > 0$ ($\bar{F}_i(t) \geq \bar{F}_j(t) >$*

0) *for all* $t \geq 0$. *Hence,* F_i *is less (greater) than* F_j *in the ordinary stochastic order. Then* $I_N^{(i)} \geq I_N^{(j)}$.

$$EZ_i \geq EZ_j + \int_0^\infty (-\ln \bar{F}_j(t)) h(0_j, \bar{\boldsymbol{F}}(t)) dt$$

$$\left(EZ_i \geq EZ_j + \int_0^\infty [(-\ln \bar{F}_j(t)) \bar{F}_j(t)/F_j(t)](1 - h(1_j, \bar{\boldsymbol{F}}(t))) dt \right).$$

Proof: When the ith component is in series with the rest of the system we have:

$$\begin{aligned} EZ_i &= \int_0^\infty (-\ln \bar{F}_i(t)) \bar{F}_i(t) h(1_i, \bar{\boldsymbol{F}}(t)) dt \\ &= \int_0^\infty (-\ln \bar{F}_i(t)) h(\bar{\boldsymbol{F}}(t)) dt \\ &\geq \int_0^\infty (-\ln \bar{F}_j(t)) h(\bar{\boldsymbol{F}}(t)) dt \\ &= \int_0^\infty (-\ln \bar{F}_j(t)) \bar{F}_j(t) [h(1_j, \bar{\boldsymbol{F}}(t)) - h(0_j, \bar{\boldsymbol{F}}(t))] dt \\ &\quad + \int_0^\infty (-\ln \bar{F}_j(t)) h(0_j, \bar{\boldsymbol{F}}(t)) dt \\ &= EZ_j + \int_0^\infty (-\ln \bar{F}_j(t) h(0_j, \bar{\boldsymbol{F}}(t)) dt. \end{aligned}$$

Similarly, when the ith component is in parallel with the rest of the system, noting that $g(x) = (-\ln x)x/(1-x)$ is nondecreasing in $[0, 1]$:

$$\begin{aligned} EZ_i &= \int_0^\infty (-\ln \bar{F}_i(t)) \bar{F}_i(t)(1 - h(0_i, \bar{\boldsymbol{F}}(t)) dt \\ &= \int_0^\infty [(-\ln \bar{F}_i(t)) \bar{F}_i(t)/F_i(t)](1 - h(\bar{\boldsymbol{F}}(t))) dt \\ &\geq \int_0^\infty [(-\ln \bar{F}_j(t)) \bar{F}_j(t)/F_j(t)][1 - \bar{F}_j(t) h(1_j, \bar{\boldsymbol{F}}(t)) \\ &\quad - F_j(t) h(0_j, \bar{\boldsymbol{F}}(t)) + F_j(t) h(1_j, \bar{\boldsymbol{F}}(t)) - F_j(t) h(1_j, \bar{\boldsymbol{F}}(t))] dt \\ &= EZ_j + \int_0^\infty [(-\ln \bar{F}_j(t)) \bar{F}_j(t)/F_j(t)](1 - h(1_j, \bar{\boldsymbol{F}}(t))) dt. \end{aligned}$$

The first part of the theorem follows by noting that the second terms on the right hand side of the inequalities are nonnegative.

Note again that these second terms are zero when the jth component is respectively in series and in parallel with the rest of the system. This theorem gives lower bounds for how much larger EZ_i is than EZ_j, not just that it is larger.

Consider again the three component system with exponentially distributed lifetimes. The second term on the right hand side of the first inequality in Theorem 2 now equals $\lambda_2/(\lambda_1 + \lambda_3)^2$. Hence, a parallel discussion to the one for the Barlow-Proschan measure can be carried through.

In a way the Natvig measure can be considered as a more complex cousin of the Barlow-Proschan measure. Theorem 3.6 of Natvig (1979) gives an example of a 2 component series system with Weibull distributed lifetimes where the ordering of component importance is different using the Barlow-Proschan and the Natvig measures. A more thorough comparison of the two for different lifetime distributions are given in Natvig (1985).

3 The Barlow-Proschan measure of component importance in repairable systems and its dual extension

In this and the subsequent section we consider the case where the components, and hence the system, can be repaired. Let the ith component have an absolutely continuous repair time distribution with mean ν_i, and let the mean time to failure of the ith component be μ_i, $i = 1, \dots, n$. It is assumed that all lifetimes and repair times are independent. Let $A_i(t)$ be the availability of the ith component at time t, i.e. the probability that the component is functioning at time t. The corresponding stationary availabilities are given by:

$$A_i = \lim_{t\to\infty} A_i(t) = \frac{\mu_i}{\mu_i + \nu_i}\,, \qquad i = 1, \dots, n. \tag{8}$$

Introduce $\mathbf{A}(t) = (A_1(t), \dots, A_n(t))$ and $\mathbf{A} = (A_1, \dots, A_n)$. Now

$$P[\varphi(\mathbf{X}(t)) = 1] = h(\mathbf{A}(t)).$$

3.1 The Barlow-Proschan measure

Let $(i = 1, \dots, n)$:

$N_i(t) =$ the number of failures of the ith component in $[0, t]$

$\tilde{N}_i(t) =$ the number of system failures caused by the ith component in $[0, t]$.

Finally, denote $EN_i(t)$ by $M_i(t)$. In Barlow and Proschan (1975) the following relation is proved somewhat heuristically $(i = 1, \dots, n)$:

$$E\tilde{N}_i(t) = \int_0^t I_B^{(i)}(u)dM_i(u), \tag{9}$$

where:

$$I_B^{(i)}(u) = h(1_i, \mathbf{A}(u)) - h(0_i, \mathbf{A}(u)). \tag{10}$$

However, a rigorous proof can be given. A time dependent Barlow-Proschan measure of the importance of the ith component in the time interval $[0, t]$ in repairable systems is given by:

$$I_{B-P}^{(i)}(t) = \frac{E\tilde{N}_i(t)}{\sum_{j=1}^n E\tilde{N}_j(t)}\,, \tag{11}$$

although not explicitly mentioned in Barlow and Proschan (1975). By a renewal theory argument they arrive at the corresponding stationary measure:

$$I_{B-P}^{(i)} = \lim_{t\to\infty} I_{B-P}^{(i)}(t) = \frac{I_B^{(i)}/(\mu_i + \nu_i)}{\sum_{j=1}^n I_B^{(j)}/(\mu_j + \nu_j)}\,, \tag{12}$$

where:

$$I_B^{(i)} = \lim_{t\to\infty} I_B^{(i)}(t) = h(1_i, \mathbf{A}) - h(0_i, \mathbf{A}). \tag{13}$$

$I^{(i)}_{B-P}$ is the stationary probability that the failure of the ith component is the cause of system failure, given that system failure has occurred.

The following results given in Barlow and Proschan (1975) are straightforward from (12) and (13):

Theorem 3. *For a series system we have:*

$$I^{(i)}_{B-P} = \frac{\mu_i^{-1}}{\sum_{j=1}^n \mu_j^{-1}},$$

whereas for a parallel system we have the dual expression:

$$I^{(i)}_{B-P} = \frac{\nu_i^{-1}}{\sum_{j=1}^n \nu_j^{-1}}.$$

As noted by the authors the former does not depend on component mean repair times, whereas the latter does not depend on component mean times to failure. We feel this is somewhat discomforting.

We have arrived at the following theorem parallel to Theorem 1:

Theorem 4. *Let the ith component be in series (parallel) with the rest of the system. Let for an arbitrary component $j \neq i$ $\mu_i \leq \mu_j$ ($\nu_i \leq \nu_j$). Then $I^{(i)}_{B-P} \geq I^{(j)}_{B-P}$. Furthermore, for the numerator of the measure we have:*

$$\frac{I^{(i)}_B}{\mu_i + \nu_i} \geq \frac{I^{(j)}_B}{\mu_j + \nu_j} + \frac{h(0_j, \boldsymbol{A})}{\mu_i}$$
$$\left(\frac{I^{(i)}_B}{\mu_i + \nu_i} \geq \frac{I^{(j)}_B}{\mu_j + \nu_j} + \frac{1 - h(1_j, \boldsymbol{A})}{\nu_i} \right).$$

Proof: When the ith component is in series with the rest of the system we have:

$$\begin{aligned}\frac{I^{(i)}_B}{\mu_i + \nu_i} &= \frac{h(1_i, \boldsymbol{A})}{\mu_i + \nu_i} = \frac{h(\boldsymbol{A})}{\mu_i} = \frac{1}{\mu_i}[\frac{\mu_j}{\mu_j + \nu_j}(h(1_j, \boldsymbol{A}) - h(0_j, \boldsymbol{A})) + h(0_j, \boldsymbol{A})] \\ &\geq \frac{I^{(j)}_B}{\mu_j + \nu_j} + \frac{h(0_j, \boldsymbol{A})}{\mu_i}.\end{aligned}$$

When the ith component is in parallel with the rest of the system we have:

$$\begin{aligned}\frac{I^{(i)}_B}{\mu_i + \nu_i} &= \frac{1 - h(0_i, \boldsymbol{A})}{\mu_i + \nu_i} = \frac{1 - h(\boldsymbol{A})}{\nu_i} \\ &= \frac{1}{\nu_i}[\frac{\nu_j}{\mu_j + \nu_j}(h(1_j, \boldsymbol{A}) - h(0_j, \boldsymbol{A})) + 1 - h(1_j, \boldsymbol{A})] \\ &\geq \frac{I^{(j)}_B}{\mu_j + \nu_j} + \frac{1 - h(1_j, \boldsymbol{A})}{\nu_i}.\end{aligned}$$

It is still discomforting that the assumption in the first (second) inequality does not depend on the component mean times to repair (failure).

3.2 The dual extension of the Barlow-Proschan measure

As an attempt to improve the Barlow-Proschan measures (11) and (12) for repairable systems it is suggested to take a dual term into account based on the probability that the repair of the ith component is the cause of system repair, given that system repair has occurred. Let $(i = 1, \ldots, n)$:

$V_i(t) =$ the number of repairs of the ith component in $[0, t]$

$\tilde{V}_i(t) =$ the number of system repairs caused by the ith component in $[0, t]$.

Denote $EV_i(t)$ by $R_i(t)$.
Note that:

$$A_i(t) = P[V_i(t) - N_i(t) = 0] = E[V_i(t) - N_i(t) + 1] = R_i(t) - M_i(t) + 1.$$

A more complex expression is given in Aven and Jensen (1999). Parallel to (9) we get $(i = 1, \ldots, n)$:

$$E\tilde{V}_i(t) = \int_0^t I_B^{(i)}(u) dR_i(u). \tag{14}$$

An extended version of (11) is given by:

$$\bar{I}_{B-P}^{(i)}(t) = \frac{E\tilde{N}_i(t) + E\tilde{V}_i(t)}{\sum_{j=1}^{n} [E\tilde{N}_j(t) + E\tilde{V}_j(t)]} = \frac{\int_0^t I_B^{(i)}(u) d(M_i(u) + R_i(u))}{\sum_{j=1}^{n} \int_0^t I_B^{(j)}(u) d(M_j(u) + R_j(u))}. \tag{15}$$

However, since from renewal theory:

$$\lim_{t\to\infty} \frac{M_i(t)}{t} = \lim_{t\to\infty} \frac{R_i(t)}{t} = \frac{1}{\mu_i + \nu_i},$$

it turns out that for the corresponding stationary measure:

$$\bar{I}_{B-P}^{(i)} = \lim_{t\to\infty} \bar{I}_{B-P}^{(i)}(t) = I_{B-P}^{(i)}.$$

Hence, Theorems 3 and 4 are also valid for $\bar{I}_{B-P}^{(i)}$ which is disappointing.

4 The Natvig measure of component importance in repairable systems and its dual extension

We start by introducing some basic random variables $(i = 1, \ldots, n)$:

$T_{ij} =$ the time of the jth failure of the ith component, $j = 1, 2, \ldots$

$S_{ij} =$ the time of the jth repair of the ith component, $j = 0, 1, \ldots$,

where we define $S_{i0} = 0$. Let $(i = 1, \ldots, n$ and $j = 1, 2 \ldots)$:

$U_{ij} = T_{ij} - S_{ij-1} =$ The length of the jth lifetime of the ith component.

$D_{ij} = S_{ij} - T_{ij} =$ The length of the jth repair time of the ith component.

We assume that U_{ij} has an absolutely continuous distribution $F_i(t)$ with density $f_i(t)$ letting $\bar{F}_i(t) \stackrel{d}{=} 1-F_i(t)$. Furthermore, D_{ij} is assumed to have an absolutely continuous distribution $G_i(t)$ with density $g_i(t)$ letting $\bar{G}_i(t) \stackrel{d}{=} 1-G_i(t)$. As for the Barlow-Proschan measure in repairable systems $EU_{ij} = \mu_i$, $ED_{ij} = \nu_i$ and all lifetimes and repair times are assumed independent.

4.1 The Natvig measure

Parallel to the nonrepairable case we argue that components that by failing strongly reduce the expected system uptime should be considered as very important. In order to formalize this, we introduce ($i = 1, \ldots, n$ and $j = 1, 2, \ldots$):

T'_{ij} = the fictive time of the jth failure of the ith component after a fictive minimal repair of the component at T_{ij}.

As for the Barlow-Proschan measure we consider a time interval $[0, t]$ and define:

Y^0_{ij} = system uptime in $[\min(T_{ij}, t), \min(T'_{ij}, t)]$ assuming that the ith component is failed throughout this interval.

Y^1_{ij} = system uptime in $[\min(T_{ij}, t), \min(T'_{ij}, t)]$ assuming that the ith component is functioning throughout this interval as a result of the fictive minimal repair.

In order to arrive at a stochastic representation similar to the nonrepairable case, see (4), we introduce the following random variables ($i = 1, \ldots, n$):

$$Z_{ij} = Y^1_{ij} - Y^0_{ij}, \qquad j = 1, 2, \ldots. \tag{16}$$

Thus, Z_{ij} can be interpreted as the fictive increase in system uptime in the interval $[\min(T_{ij}, t), \min(T'_{ij}, t)]$ as a result of the ith component being functioning instead of failed in this interval. Note that since the minimal repair is fictive, we have chosen to calculate the effect of this repair over the entire interval $[\min(T_{ij}, t), \min(T'_{ij}, t)]$ even though this interval may extend beyond the time of the real repair, S_{ij}.

In order to summarize the effects of all the fictive minimal repairs, we have chosen to simply add up these contributions. Note that the fictive minimal repair periods, i.e., the intervals of the form $[\min(T_{ij}, t), \min(T'_{ij}, t)]$, may sometimes overlap. Thus, at a given point of time we may have contributions from more than one fictive minimal repair. This is efficiently dealt with by the simulation methods presented in Huseby et al. (2008). Taking the expectation, we get:

$$E\Big[\sum_{j=1}^{\infty} I(S_{ij-1} \leq t) Z_{ij}\Big] \stackrel{d}{=} EY_i(t), \tag{17}$$

where I denotes the indicator function. The time dependent Natvig measure of the importance of the ith component in the time interval $[0, t]$ in repairable systems can then be defined as:

$$I_N^{(i)}(t) = \frac{EY_i(t)}{\sum_{j=1}^{n} EY_j(t)}\,. \tag{18}$$

We will now prove the following theorem:

Theorem 5.

$$EY_i(t) = \int_0^t I_B^{(i)}(w)\bar{F}(w)(-\ln \bar{F}_i(w))dw + \int_0^t \int_u^t I_B^{(i)}(w)\bar{F}_i(w-u)(-\ln \bar{F}_i(w-u))dw\, dR_i(u).$$

To prove the theorem in a formal way we must first prove the following lemma.

Lemma 1 *Let $W_1, W_2, \ldots$ be an increasing sequence of positive random variables. Assume that $W_j - W_{j-1}$ are independent with an absolutely continuous distribution $H_j(u)$ and density $h_j(u)$, $j = 1, 2, \ldots$, where $W_0 \stackrel{d}{=} 0$. Let $\rho(u)$ be the jump intensity for the process $N(u) = \sum_{j=1}^{\infty} I(W_j \leq u)$, and let $N = N(t)$. For each $j = 1, 2, \ldots$ let Y_j be a random variable which is independent of $W_1, \ldots, W_{j-1}$ given W_j, and suppose that $E(Y_j|W_j = u)$ does not depend on j. Finally, let $Y = \sum_{j=1}^{N} Y_j$. Then*

$$EY = \int_0^t E(Y_j|W_j = u)\rho(u)du.$$

Proof:

$$\begin{aligned} EY &= E\Big[\sum_{j=1}^{\infty} I(W_j \leq t)Y_j\Big] \\ &= \sum_{j=1}^{\infty} E[E[I(W_j \leq t)Y_j|W_j]] \\ &= \sum_{j=1}^{\infty} E[I(W_j \leq t)E(Y_j|W_j)] \\ &= \sum_{j=1}^{\infty} \int_0^t E(Y_j|W_j = u)(h_1 * h_2 * \cdots * h_j(u))du, \end{aligned}$$

where $*$ denotes convolution. Since $E(Y_j|W_j = u)$ does not depend on j, this equals:

$$\begin{aligned} &= \int_0^t E(Y_j|W_j = u)\sum_{j=1}^{\infty}(h_1 * h_2 * \cdots * h_j(u))du \\ &= \int_0^t E(Y_j|W_j = u)\rho(u)du. \end{aligned}$$

Proof of Theorem 5. We apply Lemma 1 with $(j = 1, 2, \ldots)$:

$$W_j = S_{ij}, \quad Y_j = Z_{ij+1}, \quad N = N(t) = \sum_{j=1}^{\infty} I(S_{ij} \leq t).$$

It will be shown that $E(Z_{ij+1}|S_{ij}=u)$ does not depend on j. Hence from (17), noting that $S_{i0}=0$:

$$\begin{aligned} EY_i(t) &= EZ_{i1} + E\Big[\sum_{j=1}^{\infty} I(S_{ij}\le t)Z_{ij+1}\Big] \\ &= EZ_{i1} + \sum_{j=1}^{N} EY_j = EZ_{i1} + EY \\ &= E(Z_{i1}|S_{i0}=0) + \int_0^t E(Z_{ij+1}|S_{ij}=u)dR_i(u). \end{aligned}$$

Let X_u be the uptime in $[0,u]$ for a system with availability $A(t)$. From Theorem 3.6 of Aven and Jensen (1999) we have:

$$EX_u = \int_0^u A(t)dt.$$

Applying this, we get from (16) for $i=1,\dots,n$ and $j=1,2,\dots$:

$$\begin{aligned} E(Z_{ij+1}|S_{ij}=u) &= E(Y^1_{ij+1}|S_{ij}=u) - E(Y^0_{ij+1}|S_{ij}=u) \\ &= \int_0^{t-u}\int_0^{t-u-z} \Big[h\Big((\bar F_i(z+v)/\bar F_i(z))_i,\, \mathbf{A}(u+z+v)\Big) \\ &\quad - h(0_i,\mathbf{A}(u+z+v))\Big]dv\, f_i(z)dz. \end{aligned}$$

By pivot decomposition this reduces to:

$$\begin{aligned} &\int_0^{t-u}\int_0^{t-u-z} \frac{\bar F_i(z+v)}{\bar F_i(z)} I_B^{(i)}(u+z+v)dv\, f_i(z)dz \\ &= \int_0^{t-u}\int_0^{t-u-v} \frac{\bar F_i(z+v)}{\bar F_i(z)} I_B^{(i)}(u+z+v) f_i(z)dz\, dv \\ &= \int_u^t I_B^{(i)}(w)\bar F_i(w-u)\int_0^{w-u}\frac{f_i(z)}{\bar F_i(z)}dz\, dw \\ &= \int_u^t I_B^{(i)}(w)\bar F_i(w-u)(-\ln \bar F_i(w-u))dw. \end{aligned}$$

Inserting this into the expression for $EY_i(t)$ completes the proof.

In Natvig (1985) it is shown that:

$$\begin{aligned} &P(T'_{ij} - S_{ij-1} > t) \\ &\quad= \bar F_i(t) + \int_0^t f_i(t-u)\frac{\bar F_i(t)}{\bar F_i(t-u)}du \\ &\quad= \bar F_i(t)[1-\ln \bar F_i(t)]. \end{aligned} \tag{19}$$

Hence, applying (19) we get:

$$\begin{aligned} &\int_0^{\infty} \bar F_i(t)(-\ln \bar F_i(t))dt \\ &\quad= \int_0^{\infty} \bar F_i(t)[1-\ln \bar F_i(t)]dt - \int_0^{\infty}\bar F_i(t)dt \\ &\quad= E(T'_{ij}-S_{ij-1}) - E(T_{ij}-S_{ij-1}) \end{aligned} \tag{20}$$

$$= E(T'_{ij} - T_{ij}) \stackrel{d}{=} \mu_i^p.$$

Accordingly, this integral equals the expected prolonged lifetime of the ith component due to a minimal repair.

Now divide the expression for $EY_i(t)$ by t and let $t \to \infty$. Assuming that the first addend vanishes, applying a renewal theory argument as in Barlow and Proschan (1975) we arrive at the following stationary measure corresponding to (18):

$$I_N^{(i)} = \lim_{t\to\infty} I_N^{(i)}(t) = \frac{[I_B^{(i)}/(\mu_i + \nu_i)]\mu_i^p}{\sum_{j=1}^n [I_B^{(j)}/(\mu_j + \nu_j)]\mu_j^p}\,. \tag{21}$$

Parallel to Theorem 3 we arrive at:

Theorem 6. *For a series system:*

$$I_N^{(i)} = \frac{\mu_i^{-1}\mu_i^p}{\sum_{j=1}^n \mu_j^{-1}\mu_j^p}\,,$$

whereas for a parallel system we have:

$$I_N^{(i)} = \frac{\nu_i^{-1}\mu_i^p}{\sum_{j=1}^n \nu_j^{-1}\mu_j^p}\,.$$

Note that for a series system $I_N^{(i)}$ does not depend on the repair time distributions, which is somewhat discomforting. However, for a parallel system as opposed to $I_{B-P}^{(i)}, I_N^{(i)}$ does depend on both the lifetime and repair time distributions.

4.2 The dual extension of the Natvig measure

As for the Barlow-Proschan measure we now also take a dual term into account where components that by being repaired strongly reduce the expected system downtime are considered very important. Introduce ($i = 1, \ldots, n$ and $j = 1, 2, \ldots$):

S'_{ij} = the fictive time of the jth repair of the ith component after a fictive minimal failure of the component at S_{ij}.

Moreover, let:

X_{ij}^0 = system downtime in $[\min(S_{ij}, t), \min(S'_{ij}, t)]$ assuming that the ith component is functioning throughout this interval.

X_{ij}^1 = system downtime in $[\min(S_{ij}, t), \min(S'_{ij}, t)]$ assuming that the ith component is failed throughout this interval as a result of the fictive minimal failure.

We then introduce the following random variables parallel to (16) ($i = 1, \ldots, n$):

$$W_{ij} = X_{ij}^1 - X_{ij}^0, \qquad j = 1, 2, \ldots. \tag{22}$$

In this case W_{ij} can be interpreted as the fictive increase in system downtime in the interval $[\min(S_{ij}, t), \min(S'_{ij}, t)]$ as a result of the ith component being failed instead of functioning in this interval.

Now adding up the contributions from the repairs at S_{ij}, $j = 1, 2, \ldots$, and taking the expectation, we get:

$$E\Big[\sum_{j=1}^{\infty} I(T_{ij} \le t) W_{ij}\Big] \stackrel{d}{=} EX_i(t). \tag{23}$$

Parallel to Theorem 5, we arrive at:

Theorem 7.

$$EX_i(t) = \int_0^t \int_u^t I_B^{(i)}(w)\bar{G}_i(w-u)(-\ln \bar{G}_i(w-u))dw\, dM_i(u).$$

Note that compared to Theorem 5 the first addend vanishes. An extended version of (18) is given by:

$$\bar{I}_N^{(i)}(t) = \frac{EY_i(t) + EX_i(t)}{\sum_{j=1}^n E[Y_j(t) + EX_j(t)]}\,.$$

Completely parallel to (20) we have:

$$\int_0^\infty \bar{G}_i(t)(-\ln \bar{G}_i(t))dt = E(S'_{ij} - S_{ij}) \stackrel{d}{=} \nu_i^p. \tag{24}$$

The corresponding stationary extended measure is now given by:

$$\begin{aligned}\bar{I}_N^{(i)} &= \lim_{t\to\infty} \bar{I}_N^{(i)}(t) \\ &= \frac{[I_B^{(i)}/(\mu_i + \nu_i)](\mu_i^p + \nu_i^p)}{\sum_{j=1}^n [I_B^{(j)}/(\mu_j+\nu_j)](\mu_j^p + \nu_j^p)}\,.\end{aligned} \tag{25}$$

Parallel to Theorem 6 we arrive at:

Theorem 8. *For a series system:*

$$\bar{I}_N^{(i)}(t) = \frac{\mu_i^{-1}(\mu_i^p + \nu_i^p)}{\sum_{j=1}^n \mu_j^{-1}(\mu_j^p + \nu_j^p)}\,,$$

whereas for a parallel system we just get a dual result by replacing the mean times to failure by the mean times to repair.

Note that now both for a series and parallel system $\bar{I}_N^{(i)}$ does depend on both the lifetime and repair time distributions.

Now consider the special case where the lifetime and repair time distributions are Weibull distributed; i.e.

$$\begin{aligned}\bar{F}_i(t) &= e^{-(\lambda_i t)^{\alpha_i}}, \qquad \lambda_i > 0, \;\; \alpha_i > 0 \\ \bar{G}_i(t) &= e^{-(\gamma_i t)^{\beta_i}}, \qquad \gamma_i > 0, \;\; \beta_i > 0\,.\end{aligned}$$

We then have from (20):

$$\begin{aligned}\mu_i^p &= \int_0^\infty \bar{F}_i(w)(-\ln \bar{F}_i(w))dw \\ &= \frac{1}{\alpha_i}\frac{1}{\lambda_i}\int_0^\infty u^{1/\alpha_i+1-1}e^{-u}du = \frac{1}{\alpha_i}\frac{1}{\lambda_i}\Gamma\Big(\frac{1}{\alpha_i}+1\Big) = \frac{\mu_i}{\alpha_i}\,,\end{aligned}$$

and similarly from (24) $\nu_i^p = \nu_i/\beta_i$. Hence, (25) simplifies to:

$$\bar{I}_N^{(i)} = \frac{[I_B^{(i)}/(\mu_i + \nu_i)][\mu_i/\alpha_i + \nu_i/\beta_i]}{\sum_{j=1}^n [I_B^{(j)}/(\mu_j + \nu_j)][\mu_j/\alpha_j + \nu_j/\beta_j]} . \tag{26}$$

Now assume that α_i is increasing and λ_i changing in such a way that μ_i is constant. Hence, according to (8) the availability A_i is unchanged. Then $\bar{I}_N^{(i)}$ is decreasing in α_i. This is natural since a large $\alpha_i > 1$ corresponds to a strongly increasing failure rate and the effect of a minimal repair is small. Hence, according to $\bar{I}_N^{(i)}$ the ith component is of less importance. If on the other hand $\alpha_i < 1$ is small, we have a strongly decreasing failure rate and the effect of a minimal repair is large. Hence, according to $\bar{I}_N^{(i)}$ the ith component is of higher importance. A completely parallel argument is valid for β_i.

By specializing $\alpha_j = \beta_j$, $j = 1, \ldots, n$, we get:

$$\bar{I}_N^{(i)} = \frac{I_B^{(i)}/\alpha_i}{\sum_{j=1}^n I_B^{(j)}/\alpha_j} .$$

By further specializing $\alpha_j = \beta_j = \alpha$, $j = 1, \ldots, n$ this reduces to the standardized stationary Birnbaum measure:

$$\bar{I}_N^{(i)} = \hat{I}_B^{(i)} = \frac{I_B^{(i)}}{\sum_{j=1}^n I_B^{(j)}} .$$

The following result is straightforward:

Theorem 9. *For a series system we have*

$$\hat{I}_B^{(i)} = \frac{1 + \nu_i/\mu_i}{\sum_{j=1}^n (1 + \nu_j/\mu_j)} ,$$

whereas for a parallel system we have the dual expression:

$$\hat{I}_B^{(i)} = \frac{1 + \mu_i/\nu_i}{\sum_{j=1}^n (1 + \mu_j/\nu_j)} .$$

According to this measure the importance of a component in a series system is increasing in mean time to repair and decreasing in mean time to failure, i.e. the poorer the more important. For a parallel system it is the other way round. This seems perfectly sensible. Furthermore, we have the following more satisfactory theorem than Theorem 4:

Theorem 10. *Let the ith component be in series (parallel) with the rest of the system. Let for an arbitrary component $j \neq i$ $A_i \leq A_j$ ($A_i \geq A_j$). Then $\hat{I}_B^{(i)} \geq \hat{I}_B^{(j)}$. Furthermore, for the numerator of the measure we have*

$$I_B^{(i)} \geq I_B^{(j)} + \frac{h(0_j, \boldsymbol{A})}{A_i}$$
$$\left(I_B^{(i)} \geq I_B^{(j)} + \frac{1 - h(1_j, \boldsymbol{A})}{1 - A_i} \right) .$$

The proof is parallel to the one of Theorem 4 and is left to the reader.

5 Concluding remarks

In this paper we have first considered measures of component importance in nonrepairable systems. In this case Theorems 1 and 2 respectively indicate that the Barlow-Proschan and the Natvig measures are sensible. Reasonable measures of component importance for repairable systems represent a challenge. Theorems 3 and 4 covering the stationary Barlow-Proschan measure and its dual extension in this case are not satisfactory.

Theorem 6 covering the stationary Natvig measure for repairable systems is not completely satisfactory since for a series system the measure does not depend on the repair time distributions. However, Theorem 8 covering its dual extension seems very sensible. For Weibull distributed lifetime and repair time distributions the latter measure is given by (26), which has a reasonable performance as a function of the shape parameters. When all shape parameters are equal this measure reduces to the standardized Birnbaum measure which according to Theorems 9 and 10 seems to be sensible. In Natvig et al. (2008) a thorough numerical analysis of the Natvig measures for repairable systems is reported along with an application to an offshore oil and gas production system. The analysis is based on advanced simulation methods presented in Huseby et al. (2008).

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