A FRAMEWORK FOR MULTI-RESERVOIR PRODUCTION OPTIMIZATION

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Abstract

When a large oil or gas field is produced, several reservoirs often share the same processing facility. This facility is typically capable of processing only a limited amount of commodities per unit of time. In order to satisfy these processing limitations, the production needs to be *choked*, i.e., scaled down by a suitable *choke factor*. A *production strategy* is defined as a vector valued function defined for all points of time representing the choke factors applied to reservoirs at any given time. In the present paper we consider the problem of optimizing such production strategies with respect to various types of objective functions. A general framework for handling this problem is developed. A crucial assumption in our approach is that the *potential production rate* from a reservoir can be expressed as a function of the remaining producible volume. The solution to the optimization problem depends on certain key properties, e.g., convexity or concavity, of the objective function and of the potential production rate functions. Using these properties several important special cases can be solved. An *admissible production strategy* is a strategy where the total processing capacity is fully utilized throughout a plateau phase. This phase lasts until the total potential production rate falls below the processing capacity, and after this all the reservoirs are produced without any choking. Under mild restrictions on the objective function the performance of an admissible strategy is uniquely characterized by the state of the reservoirs at the end of the plateau phase. Thus, finding an optimal admissible production strategy, is essentially equivalent to finding the optimal state at the end of the plateau phase. Given the optimal state a backtracking algorithm can then used to derive an optimal production strategy. We will demonstrate this on a specific example.

1   Introduction

Optimization is an important element in the management of large offshore Exploration & Production (E&P) assets, since many investment decisions are irreversible and finance is committed long-term. van den Heever et al. (2001) classify decisions made in reservoir management in two main categories, design decisions and operational decisions. Design decisions comprise selecting the type of platform, the staging of compression and assessing the number of wells to be drilled in a reservoir. These decisions are discrete in nature. In operational decisions production rates from individual reservoirs and wells are assessed. In contrast to design decisions, operational decisions are continuous in nature.

Neiro & Pinto (2004) propose a framework for modelling the entire petroleum supply chain. Ivyer & Grossmann (1998) present a multi-period mixed-integer linear programming formulation for the planning and scheduling of investment and operation in offshore oilfields. In other approaches a case and scenario analysis system is constructed for evaluating uncertainties in the E&P value chain, see Narayanan et al. (2003) for details. In Floris & Peersmann (2000) a

decision scenario analysis framework is presented. Here, scenario and probabilistic analysis is combined with Monte Carlo simulation. Optimization can also be performed using a simulator, where real-time decisions are made subject to production constraints. Davidson & Beckner (2003) and Wang et al. (2002) use this technique. Their decision variables include binary on/off conditions and continuous variables. Uncertainty was not considered in these works.

Many of the contributions listed above focus on the problem of modelling the entire hydrocarbon value chain, where the purpose is to make models for scheduling and planning of hydrocarbon field infrastructures with complex objectives. Since the entire value chain is very complex, many aspects of it needs to be simplified to be able to construct such a comprehensive model. The purpose of the present paper is to focus on the problem of optimizing production in an oil or gas field consisting of many reservoirs, which constitutes an important component in the hydrocarbon value chain. By focusing on only one important component we are able to develop a framework that provides insight into how a large oil or gas field should be produced. The optimization methods developed here can thus be used in the broader context of a total value chain analysis.

To obtain reliable and valid results, having realistic production models is very important. Key properties of the reservoirs are typically assessed by geologists, geophysicists, petroleum engineers and other specialists. This knowledge is then assembled and quantified into a reservoir model. Our analysis starts at the stage where a full-scale reservoir simulation has been performed, and the output from this simulation is given. Simplified production models can then be constructed based on this output. See Haavardsson & Huseby (2007b) for details about this. The present paper will utilize such production models.

We consider a situation where several reservoirs share the same processing facility. Oil, water and gas flow from each reservoir to this facility. The processing facility is only capable of handling limited amounts of the commodities per unit of time. In order to satisfy the resulting constraints, the production needs to be *choked*. In this setting we focus on optimizing the oil production and leave the simultaneous analysis of oil, gas and water production for future work. To avoid issues of dependence between the production profiles of the reservoirs, the production from any reservoir is assumed to be independent of the production from the other reservoirs.

A fundamental model assumption is that the *potential production rate* from a reservoir, can be expressed as a function of the remaining producible volume, or equivalently as a function of the volume produced. Thus, if $Q(t)$ denotes the cumulative production at time $t \geq 0$, and $f(t)$ denotes the potential production rate at the same point of time, we assume that $f(t) = f(Q(t))$. This assumption implies that the potential production rate at a given point of time only depends on the volume produced at that time (or equivalently on the volume left in the reservoir). Thus, if we delay the production from a reservoir, we can still produce the same volume at a later time. We refer to the function f as the *potential production rate function* or *PPR-function* of the reservoir. If a reservoir is produced without any production constraint from time $t = 0$, the cumulative production function will satisfy the following autonomous differential equation:

$$\frac{dQ(t)}{dt} = f(Q(t)), \tag{1.1}$$

with the boundary condition $Q(0) = 0$. The function f would typically be a non-increasing function. In order to ensure a unique solution to (1.1), we will also assume that f is Lipschitz continuous. If $Q = Q(t)$ is the solution to (1.1), we assume that:

$$\lim_{t \to \infty} Q(t) = \int_0^\infty f(Q(u))du = V < \infty. \tag{1.2}$$

That is, the *recoverable volume* from the reservoir, denoted V, is assumed to be *finite*. Note that since f is continuous, (1.2) implies that:

$$\lim_{t\to\infty} f(Q(t)) = f(V) = 0, \qquad (1.3)$$

since otherwise the integral in (1.2) would be divergent.

Due to various kinds of restrictions, including possible time-dependent constraints, the actual production rate will typically be less than or equal to $f(t)$. Still it turns out that the PPR-functions play an important part in the analysis.

The present paper presents the following contributions:

- Section 2 introduces basic concepts and results, including a discussion of objective functions and some mild restrictions we impose on them.

- In Section 3 we turn to the problem of finding the best production strategy. An algorithm for finding the best production strategy and two main results are presented. The first result deals with the solution to the optimization problem if the PPR-functions are convex and the extended version of objective function ϕ is quasi-convex[1], while the second result analogously treats the situation when the PPR-functions are concave and the extended version of objective function ϕ is quasi-concave. A specific type of objective function and an important class of production strategies are presented.

- In Section 4 we consider the case where all the PPR-functions are linear. In this case a specific production strategy is proven to be optimal for a wide class of objective functions. The framework is illustrated on a specific example.

- Section 5 is devoted to generate optimal production strategies using backtracking. Since the performance of an admissible strategy is uniquely characterized by the state of the reservoirs at the end of the plateau phase, the backtracking is initiated using the optimal state at the end of the plateau phase. Given the optimal state a backtracking algorithm can then be used to derive an optimal production strategy.

2   Basic concepts and results

We consider the oil production from n reservoirs that share a processing facility with a constant process capacity $K > 0$, expressed in some suitable unit, e.g., kSm^3 per day. Let $\boldsymbol{Q}(t) = (Q_1(t), \ldots, Q_n(t))$ denote the vector of cumulative production functions for the n reservoirs, and let $\boldsymbol{f}(t) = (f_1(t), \ldots, f_n(t))$ be the corresponding vector of PPR functions. We assume that the PPR functions can be written as:

$$f_i(t) = f_i(Q_i(t)), \qquad t \geq 0,\ i = 1, \ldots n.$$

Note that this assumption implies that the potential production rate of one reservoir does not depend on the volumes produced from the other reservoirs. We will also assume for $i = 1, \ldots n$ that f_i is non-negative and non-increasing as a function of $Q_i(t)$ for all t, and that $\lim_{t\to\infty} Q_i(t) = V_i < \infty$. As already stated, this implies that $\lim_{t\to\infty} f_i(Q_i(t)) = f_i(V_i) = 0$. These assumptions reflect the natural properties that the production rate cannot be negative, that reservoir pressure typically decreases towards zero as more and more oil is produced, and

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[1]For a definition of quasi-convex and quasi-concave functions see Appendix A.2

that the recoverable volume is finite. Finally, to ensure uniqueness of potential production profiles we will also assume that f_i is Lipschitz continuous in Q_i, $i = 1, \ldots, n$.

A *production strategy* is defined as a vector valued function $\boldsymbol{b} = \boldsymbol{b}(t) = (b_1(t), \ldots, b_n(t))$, defined for all $t \geq 0$, where $b_i(t)$ represents the *choke factor* applied to the ith reservoir at time t, $i = 1, \ldots, n$. We refer to the individual b_i-functions as the *choke factor functions* of the production strategy. The *actual production rates* from the reservoirs, after the production is choked is given by:

$$\boldsymbol{q}(t) = (q_1(t), \ldots, q_n(t)),$$

where:

$$q_i(t) = \frac{dQ(t)}{dt} = b_i(t) f_i(Q_i(t)), \qquad i = 1, \ldots, n.$$

We also introduce the total production rate function $q(t) = \sum_{i=1}^{n} q_i(t)$ and the total cumulative production function $Q(t) = \sum_{i=1}^{n} Q_i(t)$. To reflect that $\boldsymbol{q}$, q, $\boldsymbol{Q}$, and Q depend on the chosen production strategy $\boldsymbol{b}$, we sometimes indicate this by writing $\boldsymbol{q}(t) = \boldsymbol{q}(t, \boldsymbol{b})$, $q(t) = q(t, \boldsymbol{b})$, $\boldsymbol{Q}(t) = \boldsymbol{Q}(t, \boldsymbol{b})$, and $Q(t) = Q(t, \boldsymbol{b})$.

To satisfy the physical constraints of the reservoirs and the process facility, we require that:

$$0 \leq q_i(t) \leq f_i(Q_i(t)), \qquad i = 1, \ldots n, \ t \geq 0, \tag{2.1}$$

and that

$$q(t) = \sum_{i=1}^{n} q_i(t) \leq K, \qquad t \geq 0. \tag{2.2}$$

Expressed in terms of the production strategy $\boldsymbol{b}$, this implies that:

$$0 \leq b_i(t) \leq 1, \qquad i = 1, \ldots n, \ t \geq 0, \tag{2.3}$$

and that

$$\sum_{i=1}^{n} b_i(t) f_i(Q_i(t), t) \leq K, \qquad t \geq 0. \tag{2.4}$$

The constraint (2.3) implies that the actual production rate cannot be increased beyond the potential production rate at any given point in time, while the constraint (2.4) states that the actual, total production rate cannot exceed the capacity of the processing facility. Let $\mathcal{B}$ denote the class of production strategies that satisfy the physical constraints (2.3) and (2.4). We refer to production strategies $\boldsymbol{b} \in \mathcal{B}$ as *valid production strategies*.

Intuitively, choosing lower values for the choke factors has the effect that the volumes are produced more slowly. The following fundamental result formalizes this.

Proposition 2.1 *Consider a reservoir with PPR-function $f(t) = f(Q(t))$, and let b^1 and b^2 be two choke factor functions such that $0 \leq b^1(t) \leq b^2(t) \leq 1$ for all $t \geq 0$. Let Q^1 and Q^2 denote the resulting cumulative production functions, and let $q^1(t) = b^1(t) f(Q^1(t))$ and $q^2(t) = b^2(t) f(Q^2(t))$ be the corresponding actual production rates. We assume that $Q^1(0) = Q^2(0) = 0$. Then $Q^1(t) \leq Q^2(t)$ for all $t \geq 0$.*

Proof: The result is essentially a variant of a well-known theorem by Chaplygin (see Dzielinski (2005)). To prove the result we assume for a contradiction that there exists a $t_1 > 0$ such that $Q^1(t_1) > Q^2(t_1)$. We also introduce $t_0 = \sup\{0 \leq t \leq t_1 : Q^1(t) \leq Q^2(t)\}$. Since obviously Q^1 and Q^2 are continuous functions, it follows that $0 \leq t_0 < t_1$, and that $Q^1(t_0) = Q^2(t_0)$ while $Q^1(t) > Q^2(t)$ for all $t \in (t_0, t_1]$. However, since we have assumed that f is non-increasing

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and since $b^1(t) \le b^2(t)$, this implies that $q^1(t) = b^1(t)f(Q^1(t)) \le b^2(t)f(Q^2(t)) = q^2(t)$ for all $t \in (t_0, t_1]$. This contradicts the assumption that $Q^1(t_1) > Q^2(t_1)$. Thus, we conclude that $Q^1(t) \le Q^2(t)$ for all $t \ge 0$ ∎

As a consequence of the above result, we also obtain the following:

Proposition 2.2 *Consider a reservoir with PPR-function $f(t) = f(Q(t))$, and let $\{b^k\}_{k=1}^\infty$ be a monotone (i.e., either nondecreasing or nonincreasing) sequence of choke factor functions. Moreover, let $\{Q(\cdot, b^k)\}_{k=1}^\infty$ be the resulting sequence of cumulative production functions, assuming the boundary condition $Q(0, b^k) = 0$ for all k. Then $\{Q(\cdot, b^k)\}_{k=1}^\infty$ converges pointwise to the cumulative production function $Q(\cdot, b)$ for all $t \ge 0$ where $b = \lim_{k\to\infty} b^k$ is the pointwise limit of the choke factor functions.*

Proof: We first note that since all choke factor functions are bounded, the sequence $\{b^k\}_{k=1}^\infty$ must converge pointwise for all $t \ge 0$. We then let $t \ge 0$ be arbitrary, and consider the sequence $\{Q(t, b^k)\}_{k=1}^\infty$. By Proposition 2.1 it follows that this sequence is monotone. Moreover, the sequence is obviously bounded, and hence convergent as well. Let $Q^* = \lim_{k\to\infty} Q(\cdot, b^k)$ denote the pointwise limit of the cumulative production functions. Thus, it follows that:

$$Q^*(t) = \lim_{k\to\infty} \int_0^t b^k(u)f(Q(u, b^k))du,$$

where the integrand is bounded by the constant $f(0)$. Hence, by Lebesgue's dominated convergence theorem we may interchange the limit and the integral. Moreover, since f is continuous we get that:

$$Q^*(t) = \int_0^t b(u)f(Q^*(u))du.$$

Since f is Lipschitz continuous it follows by the Picard-Lindelöf's theorem that this integral equation has a unique solution with given boundary condition. Thus, we must have $Q^* = Q(\cdot, b)$ as stated ∎

2.1 Objective functions

To evaluate production strategies we introduce an *objective function*, i.e., a mapping $\phi : \mathcal{B} \to \mathbb{R}$ representing some sort of a performance measure. If $b^1, b^2 \in \mathcal{B}$, we prefer b^2 to b^1 if $\phi(b^2) \ge \phi(b^1)$. Moreover, an *optimal production strategy* with respect to ϕ is a production strategy $b^{opt} \in \mathcal{B}$ such that $\phi(b^{opt}) \ge \phi(b)$ for all $b \in \mathcal{B}$.

If $b^1, b^2 \in \mathcal{B}$ are two production strategies such that $Q(t, b^1) \le Q(t, b^2)$ for all $t \ge 0$, one would most likely prefer b^2 to b^1. Thus, a sensible objective function should have the property that $\phi(b^1) \le \phi(b^2)$ whenever $Q(t, b^1) \le Q(t, b^2)$ for all $t \ge 0$. Objective functions satisfying this property will be referred to as *monotone objective functions*. The following result states that monotone objective functions also satisfies a monotonicity with respect to the production strategy.

Proposition 2.3 *Let ϕ be a monotone objective function, and let $b^1, b^2 \in \mathcal{B}$ be such that $b^1(t) \le b^2(t)$ for all $t \ge 0$. Then $\phi(b^1) \le \phi(b^2)$.*

Proof: The result follows directly from Proposition 2.1.

Monotone objective functions will encourage production strategies where the total production rate is sustained at the plateau level K as long as possible. Furthermore, when the plateau

level cannot be sustained, all the reservoirs should be produced without choking. More specifically, if $\sum_{i=1}^{n} f_i(Q_i(t)) \geq K$, one would typically choose $\boldsymbol{b}(t)$ so that $\sum_{i=1}^{n} b_i(t) f_i(Q_i(t)) = K$, while if $\sum_{i=1}^{n} f_i(Q_i(t)) < K$, the obvious choice is to let $\boldsymbol{b}(t) = \boldsymbol{1}$. Production strategies satisfying these conditions are said to be *admissible production strategies*. We let $\mathcal{B}' \subseteq \mathcal{B}$ denote the class of such strategies.

To study this further we introduce the *plateau set*:

$$\Pi_K = \Pi_K(\boldsymbol{b}) = \{t \geq 0 : \sum_{i=1}^{n} f_i(Q_i(t, \boldsymbol{b})) \geq K\}. \tag{2.5}$$

Thus, Π_K is the set of points of time where the total production rate can be sustained at the plateau level given that the production strategy $\boldsymbol{b}$ is used. We also introduce:

$$T_K = T_K(\boldsymbol{b}) = \inf\{t \geq 0 : \sum_{i=1}^{n} f_i(Q_i(t, \boldsymbol{b})) \leq K\}. \tag{2.6}$$

The quantity $T_K(\boldsymbol{b})$ will be referred to as the *plateau length* for the production strategy $\boldsymbol{b}$.

If $\sum_{i=1}^{n} f_i(Q_i(0)) \leq K$, it follows that $T_K = 0$. In this case the optimization problem is trivial since no choking is necessary, and the obvious optimal solution is letting $\boldsymbol{b}(t) = \boldsymbol{1}$ for all t. To avoid this trivial case we henceforth assume that $\sum_{i=1}^{n} f_i(Q_i(0)) = \sum_{i=1}^{n} f_i(0) > K$. It then follows by the continuity and monotonicity of the PPR-functions that Π_K is the non-empty interval $[0, T_K]$. Moreover, in this case T_K can alternatively be expressed as:

$$T_K = T_K(\boldsymbol{b}) = \sup\{t \geq 0 : \sum_{i=1}^{n} f_i(Q_i(t, \boldsymbol{b})) \geq K\}. \tag{2.7}$$

It follows that the production rates of an admissible production strategy satisfy the following constraints:

$$q(t) = \sum_{i=1}^{n} q_i(t) = K, \qquad 0 \leq t \leq T_K, \tag{2.8}$$

and

$$q_i(t) = f_i(Q_i(t)), \qquad t > T_K, \ i = 1, \ldots n. \tag{2.9}$$

The following results states that if the objective function is monotone, an optimal production strategy can always be found within the class of admissible production strategies. Thus, when searching for optimal strategies we can restrict the search to the class $\mathcal{B}'$.

Proposition 2.4 *Let ϕ be a monotone objective function, and let $\boldsymbol{b} \in \mathcal{B}$. Then there exists $\boldsymbol{b}' \in \mathcal{B}'$ such that $\phi(\boldsymbol{b}') \geq \phi(\boldsymbol{b})$.*

Proof: If $\boldsymbol{b} \in \mathcal{B}'$ the result is obvious, so we assume that $\boldsymbol{b} \in \mathcal{B} \setminus \mathcal{B}'$. We can then construct a nondecreasing sequence $\{\boldsymbol{b}^k\}_{k=1}^{\infty}$ of valid production strategies as follows. We start out by defining $\boldsymbol{b}^1 = \boldsymbol{b}$. Thus, $\boldsymbol{b}^1 \in \mathcal{B}$ by assumption. We then assume that we have defined $\boldsymbol{b}^1, \ldots, \boldsymbol{b}^k$ so that $\boldsymbol{b}^j \in \mathcal{B}$ for $j = 1, \ldots, k$, and define $\boldsymbol{b}^{k+1}$ by:

$$\boldsymbol{b}^{k+1}(t) = \alpha_k(t)\boldsymbol{1} + (1 - \alpha_k(t))\boldsymbol{b}^k(t), \qquad t \geq 0, \ k = 1, 2, \ldots, \tag{2.10}$$

where $\alpha_k(t)$ is defined for all $t \geq 0$ and $k = 1, 2, \ldots$ as the largest number in $[0, 1]$ so that:

$$\sum_{i=1}^{n} [\alpha_k(t) + (1 - \alpha_k(t))b_i^k(t)] f_i(Q_i(t, \boldsymbol{b}^k)) \leq K.$$

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Note that if $\boldsymbol{b}^k(t) = \boldsymbol{1}$, we may define $\alpha_k(t)$ arbitrarily.

We observe that since $\boldsymbol{b}^k \in \mathcal{B}$, $\alpha_k(t)$ is well-defined, and that $0 \le b_i^{k+1}(t) \le 1$ for all $t \ge 0$ and $i = 1, \ldots, n$. Moreover, if $t \le T_K(\boldsymbol{b}^k)$ we have:

$$\sum_{i=1}^{n} b_i^{k+1}(t) f_i(Q_i(t, \boldsymbol{b}^k)) = K, \tag{2.11}$$

while if $t > T_K(\boldsymbol{b}^k)$, $b_i^{k+1}(t) = 1$, $i = 1, \ldots, n$.

Since we obviously have that $\boldsymbol{b}^{k+1}(t) \ge \boldsymbol{b}^k(t)$ for all $t \ge 0$, it follows by Proposition 2.1 that $Q_i(t, \boldsymbol{b}^{k+1}) \ge Q_i(t, \boldsymbol{b}^k)$ for all $t \ge 0$ and $i = 1, \ldots, n$. Hence, since the PPR-functions are decreasing, we get for all $t \ge 0$ that:

$$\sum_{i=1}^{n} b_i^{k+1}(t) f_i(Q_i(t, \boldsymbol{b}^{k+1})) \le \sum_{i=1}^{n} b_i^{k+1}(t) f_i(Q_i(t, \boldsymbol{b}^k)) \le K.$$

Hence, $\boldsymbol{b}^{k+1} \in \mathcal{B}$ as well. Thus, it follows by induction that all the production strategies in the sequence are valid.

Since the sequence $\{\boldsymbol{b}^k\}_{k=1}^{\infty}$ is nondecreasing and bounded, it will converge pointwise for each $t \ge 0$, and we let $\boldsymbol{b}'$ denote the limiting production strategy. It is easy to see that $\boldsymbol{b}' \in \mathcal{B}$ as well. Furthermore, using Proposition 2.2 and that the PPR-functions are continuous, it follows that:

$$\lim_{k \to \infty} f_i(Q_i(t, \boldsymbol{b}^k)) = f_i(Q_i(t, \boldsymbol{b}')), \text{ for all } t \ge 0, \text{ and } i = 1, \ldots, n. \tag{2.12}$$

Using once again Proposition 2.1 and that the PPR-functions are decreasing, it is easy to see that $\{T_K(\boldsymbol{b}^k)\}_{k=1}^{\infty}$ is a nonincreasing and hence convergent sequence. Moreover, by (2.12), we get that:

$$\lim_{k \to \infty} T_K(\boldsymbol{b}^k) = T_K(\boldsymbol{b}'). \tag{2.13}$$

If $t \le T_K(\boldsymbol{b}')$, we know that $t \le T_K(\boldsymbol{b}^k)$ for all k as well. Hence, for such t (2.11) holds for all k. By taking the limit in (2.11) we obtain:

$$K = \lim_{k \to \infty} \sum_{i=1}^{n} b_i^{k+1}(t) f_i(Q_i(t, \boldsymbol{b}^k))$$
$$= \sum_{i=1}^{n} b_i'(t) f_i(Q_i(t, \boldsymbol{b}')),$$

where the last equality follows by (2.12).

If $t > T_K(\boldsymbol{b}')$, it follows by (2.13) that there must exist a k_0 such that $t > T_K(\boldsymbol{b}^k)$ for all $k \ge k_0$. Hence, $b_i^k(t) = 1$ for $i = 1, \ldots, n$ and for all $k \ge k_0$, implying that $b_i'(t) = 1$ for $i = 1, \ldots, n$ as well.

Thus, we conclude that $\boldsymbol{b}' \in \mathcal{B}'$, i.e., $\boldsymbol{b}'$ is an admissible production strategy. Since obviously $\boldsymbol{b}'(t) \ge \boldsymbol{b}(t)$ for all $t \ge 0$, it follows by Proposition 2.3 that $\phi(\boldsymbol{b}') \ge \phi(\boldsymbol{b})$, and thus the proof is completed $\blacksquare$

In general the revenue generated by the production may vary between the reservoirs. This may occur if e.g., the quality of the oil, or the average production cost per unit are different from reservoir to reservoir. Such differences should then be reflected in the chosen objective function.

On the other hand, if all the reservoirs are similar, we could restrict ourselves to considering objective functions depending on the production strategy $\boldsymbol{b}$ only through the total production rate function $q(\cdot, \boldsymbol{b})$ (or equivalently through $Q(\cdot, \boldsymbol{b})$). We refer to such objective functions as *symmetric*. Within the class of symmetric objective functions the concept of monotonicity can be simplified as follows:

Proposition 2.5 *Let ϕ be a symmetric objective function. Then ϕ is monotone if and only if for any pair of production strategies, $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ such that $Q(t, \boldsymbol{b}^1) \leq Q(t, \boldsymbol{b}^2)$ for all $t \geq 0$, we have $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^2)$.*

Proof: Assume first that ϕ is monotone, and let $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ be two production strategies such that $Q(t, \boldsymbol{b}^1) \leq Q(t, \boldsymbol{b}^2)$ for all $t \geq 0$. Then we can find a third (possibly invalid) production strategy $\boldsymbol{b}^3$ such that $\boldsymbol{Q}(t, \boldsymbol{b}^1) \leq \boldsymbol{Q}(t, \boldsymbol{b}^3)$ for all $t \geq 0$, and such that $Q(t, \boldsymbol{b}^2) = Q(t, \boldsymbol{b}^3)$ for all $t \geq 0$. Since ϕ is monotone, we have $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^3)$. Moreover, since ϕ is symmetric, we have $\phi(\boldsymbol{b}^3) = \phi(\boldsymbol{b}^2)$. Combining this we get that $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^2)$ as claimed.

Assume then conversely that ϕ is such that for any pair of production strategies, $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ such that $Q(t, \boldsymbol{b}^1) \leq Q(t, \boldsymbol{b}^2)$ for all $t \geq 0$, we have $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^2)$. Then let $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ be two production strategies such that $\boldsymbol{Q}(t, \boldsymbol{b}^1) \leq \boldsymbol{Q}(t, \boldsymbol{b}^2)$ for all $t \geq 0$. Then obviously $Q(t, \boldsymbol{b}^1) \leq Q(t, \boldsymbol{b}^2)$ for all $t \geq 0$ as well, implying that $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^2)$. That is, ϕ is monotone as claimed ∎

Within the class of admissible production strategies any symmetric objective function can be expressed in terms of the system state at the end of the plateau phase. The following result formalizes this:

Proposition 2.6 *Let ϕ be a symmetric objective function, and let $\boldsymbol{b} \in \mathcal{B}'$. Then $\phi(\boldsymbol{b})$ is uniquely determined by $\boldsymbol{Q}(T_K(\boldsymbol{b}))$. Thus, we may write $\phi(\boldsymbol{b}) = \phi(\boldsymbol{Q}(T_K(\boldsymbol{b})))$.*

Proof: Since ϕ is assumed to be symmetric, it depends on $\boldsymbol{b}$ only through q. Furthermore, since $\boldsymbol{b} \in \mathcal{B}'$, we know that $q(t) = K$ whenever $0 \leq t \leq T_K(\boldsymbol{b})$. This implies that:

$$Q(T_K(\boldsymbol{b})) = \sum_{i=1}^{n} Q_i(T_K(\boldsymbol{b})) = KT_K(\boldsymbol{b}).$$

Hence, the plateau length $T_K(\boldsymbol{b})$ can be recovered from $\boldsymbol{Q}(T_K(\boldsymbol{b}))$ as:

$$T_K(\boldsymbol{b}) = K^{-1} \sum_{i=1}^{n} Q_i(T_K(\boldsymbol{b})).$$

If $t > T_K(\boldsymbol{b})$, it follows since $\boldsymbol{b} \in \mathcal{B}'$ that:

$$q(t) = \sum_{i=1}^{n} q_i(t) = \sum_{i=1}^{n} f_i(Q_i(t))$$

By the Picard-Lindelöf's theorem $q_i(t)$ is uniquely determined for all $t > T_K(\boldsymbol{b})$ by its respective differential equation along with the boundary condition given by the value $Q_i(T_K(\boldsymbol{b}))$, $i = 1, \ldots, n$. Thus, $q(t)$ is uniquely determined by $\boldsymbol{Q}(T_K(\boldsymbol{b}))$ for all $t \geq 0$, and hence so is ϕ ∎

3   Optimizing production strategies

We now turn to the problem of finding the *best* production strategy, i.e., the one that maximizes the value of the objective function, ϕ. To simplify this problem, only *monotone, symmetric* objective functions will be discussed. As we shall see, Proposition 2.6 plays a key role when searching for optimal production strategies. In order to explain this, we consider the set of all possible cumulative production vectors for the given field, denoted by $\mathcal{Q}$:

$$\mathcal{Q} = [0, V_1] \times \cdots \times [0, V_n], \tag{3.1}$$

where $V_1, \ldots, V_n$ are the recoverable volumes from the n reservoirs. We then introduce the subsets $\mathcal{M}, \bar{\mathcal{M}} \subseteq \mathcal{Q}$ given respectively by:

$$\mathcal{M} \;=\; \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) \geq K\}, \tag{3.2}$$

$$\bar{\mathcal{M}} \;=\; \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) < K\}. \tag{3.3}$$

We also need the set of boundary points of $\mathcal{M}$ separating $\mathcal{M}$ from $\bar{\mathcal{M}}$, which we denote by $\partial(\mathcal{M})$. Thus, $\boldsymbol{Q} \in \partial(\mathcal{M})$ if and only if every neighborhood of $\boldsymbol{Q}$ intersects both $\mathcal{M}$ and $\bar{\mathcal{M}}$.

Since we have assumed that $\sum_{i=1}^{n} f_i(0) > K > 0$ and $\sum_{i=1}^{n} f_i(V_i) = 0$, both $\mathcal{M}$ and $\bar{\mathcal{M}}$ are non-empty. Moreover, since the PPR-functions are assumed to be continuous, it is easy to see that:

$$\partial(\mathcal{M}) \subseteq \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) = K\}, \tag{3.4}$$

where equality holds if the PPR-functions are *strictly decreasing*.

The following key result shows how the shapes of the sets $\mathcal{M}$ and $\bar{\mathcal{M}}$ depend on the shapes of the PPR-functions.

Proposition 3.1 *Consider a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$.*
(i)   If $f_1, \ldots, f_n$ are convex, the set $\bar{\mathcal{M}}$ is convex.
(ii)   If $f_1, \ldots, f_n$ are concave, the set $\mathcal{M}$ is convex.

Proof: Assume first that the PPR-functions are convex, and let $\boldsymbol{Q}^1 = (Q_1^1, \ldots, Q_n^1)$ and $\boldsymbol{Q}^2 = (Q_1^2, \ldots, Q_n^2)$ be two vectors in $\bar{\mathcal{M}}$. Thus, we have:

$$\sum_{i=1}^{n} f_i(Q_i^j) < K, \qquad j = 1, 2. \tag{3.5}$$

Then let $0 \leq \alpha \leq 1$, and consider the vector $\boldsymbol{Q} = (Q_1, \ldots, Q_n) = \alpha \boldsymbol{Q}^1 + (1-\alpha)\boldsymbol{Q}^2$. Since the PPR-functions are convex, we have:

$$
\begin{aligned}
\sum_{i=1}^{n} f_i(Q_i) \;&=\; \sum_{i=1}^{n} f_i(\alpha Q_i^1 + (1-\alpha)Q_i^2) \\
&\leq\; \alpha \sum_{i=1}^{n} f_i(Q_i^1) + (1-\alpha) \sum_{i=1}^{n} f_i(Q_i^2) < K
\end{aligned}
$$

Thus, we conclude that $\boldsymbol{Q} \in \bar{\mathcal{M}}$ as well. Hence $\bar{\mathcal{M}}$ is convex. The second part of the proposition is proved in a similar way ∎

Note that since convexity is preserved under set closure, we also have the following corollary

Corollary 3.2 *Consider a field with n reservoirs with convex PPR-functions $f_1, \ldots, f_n$. Then the set $\bar{\mathcal{M}} \cup \partial(\mathcal{M})$ is convex.*

Proof: The result follows by realizing that the closure of $\bar{\mathcal{M}}$ is $\bar{\mathcal{M}} \cup \partial(\mathcal{M})$ ∎

By combining (3.3) and (3.4) we get that:

$$\bar{\mathcal{M}} \cup \partial(\mathcal{M}) \subseteq \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) \le K\}, \tag{3.6}$$

where equality holds if the PPR-functions are *strictly decreasing.*

The set $\mathcal{M}$ has the property that the total production rate can be sustained at plateau level as long as $\boldsymbol{Q}(t) \in \mathcal{M}$. More specifically, let $\boldsymbol{b}$ be any production strategy, and consider the points in $\mathcal{Q}$ generated by $\boldsymbol{Q}(t) = \boldsymbol{Q}(t, \boldsymbol{b})$ as t increases. From the boundary conditions we know that $\boldsymbol{Q}(0) = \boldsymbol{0}$. By the continuity of the PPR-functions, $\boldsymbol{Q}(t)$ will move along some path in $\mathcal{M}$ until the boundary $\partial(\mathcal{M})$ is reached.

If $\boldsymbol{b} \in \mathcal{B}$, the resulting path is said to be a *valid path*, while if $\boldsymbol{b} \in \mathcal{B}'$, the path is called an *admissible path*. In general only a subset of $\mathcal{M}$ can be reached by admissible paths. We denote this subset by $\mathcal{M}'$. Moreover, we let $\partial(\mathcal{M}') = \partial(\mathcal{M}) \cap \mathcal{M}'$. We now make the mild but important assumption that $\partial(\mathcal{M}')$ is a $(n-1)$-manifold with boundary denoted by $\partial(\partial(\mathcal{M}'))$. In particular we assume that all points in $\partial(\partial(\mathcal{M}'))$ can be reachable by admissible paths.

For an admissible path the total production rate equals K all the way until the path reaches $\partial(\mathcal{M}')$. Moreover, the plateau length $T_K(\boldsymbol{b})$ is the point of time when the path reaches $\partial(\mathcal{M}')$, implying that:

$$\partial(\mathcal{M}') = \{\boldsymbol{Q}(T_K(\boldsymbol{b})) : \boldsymbol{b} \in \mathcal{B}'\} \tag{3.7}$$

By Proposition 2.6 we know that $\phi(\boldsymbol{b}) = \phi(\boldsymbol{Q}(T_K(\boldsymbol{b})))$ given that $\boldsymbol{b} \in \mathcal{B}'$ and ϕ is symmetric. Hence, the best production strategy can, at least in principle, be found using the following two-stage process:

Algorithm 3.3 *Let ϕ be a monotone, symmetric objective function. Then a production strategy $\boldsymbol{b}$ which is optimal with respect to ϕ can be found as follows:*

STEP 1.   *Find $\boldsymbol{Q}^{opt} \in \partial(\mathcal{M}')$ such that $\phi(\boldsymbol{Q}^{opt}) \ge \phi(\boldsymbol{Q})$ for all $\boldsymbol{Q} \in \partial(\mathcal{M}')$.*

STEP 2.   *Find a production strategy $\boldsymbol{b} \in \mathcal{B}'$ such that $\boldsymbol{Q}(T_K(\boldsymbol{b})) = \boldsymbol{Q}^{opt}$.*

We observe that in the first step of Algorithm 3.3 the objective function ϕ is interpreted simply as a function of the vector $\boldsymbol{Q}$, while in the second step we look for a production strategy $\boldsymbol{b} \in \mathcal{B}'$ generating an admissible path in $\mathcal{M}$ from the origin to the optimal vector $\boldsymbol{Q}^{opt}$.

To solve the optimization problem given in Step 1 of Algorithm 3.3, we assume that it is possible to extend the definition of ϕ to all vectors $\boldsymbol{Q} \in \mathcal{Q}$. Moreover, we assume that the extended version of ϕ is non-decreasing in $\boldsymbol{Q}$. That is, if $\boldsymbol{Q}^1, \boldsymbol{Q}^2 \in \mathcal{Q}$ and $\boldsymbol{Q}^1 \le \boldsymbol{Q}^2$, then $\phi(\boldsymbol{Q}^1) \le \phi(\boldsymbol{Q}^2)$. Having extended ϕ in this way, the problem is now to maximize $\phi(\boldsymbol{Q})$ subject to the constraint that $\boldsymbol{Q} \in \partial(\mathcal{M}')$.

Note that since the PPR-functions are assumed to be non-decreasing, it follows that for any $\boldsymbol{Q} \in \mathcal{M}$, we can always find another vector $\boldsymbol{Q}' \in \partial(\mathcal{M})$ such that $\boldsymbol{Q} \le \boldsymbol{Q}'$. Thus, since ϕ is

assumed to be non-decreasing as well, we have $\phi(\boldsymbol{Q}) \leq \phi(\boldsymbol{Q}')$. In particular, if $\boldsymbol{Q}^* \in \partial(\mathcal{M})$ maximizes ϕ over $\partial(\mathcal{M})$, it follows that $\phi(\boldsymbol{Q}^*) \geq \phi(\boldsymbol{Q})$ for all $\boldsymbol{Q} \in \mathcal{M}$. We also introduce the set $\mathcal{N}$:

$$\mathcal{N} = \{\boldsymbol{Q} \in \mathcal{Q} : \phi(\boldsymbol{Q}) > \phi(\boldsymbol{Q}^*)\}. \tag{3.8}$$

Since $\phi(\boldsymbol{Q}^*) \geq \phi(\boldsymbol{Q})$ for all $\boldsymbol{Q} \in \mathcal{M}$, it follows that $\mathcal{M} \cap \mathcal{N} = \emptyset$.

If $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ as well, then obviously $\boldsymbol{Q}^*$ is a solution to the optimization problem in Step 1 of Algorithm 3.3. Hence, we may let $\boldsymbol{Q}^{opt} = \boldsymbol{Q}^*$. In many cases, however, it may happen that $\boldsymbol{Q}^* \notin \partial(\mathcal{M}')$. In such cases the optimal vector $\boldsymbol{Q}^{opt} \in \partial(\mathcal{M}')$ can typically be found at the boundary, $\partial(\partial(\mathcal{M}'))$.

Using results from Appendix A we are now ready to prove the two main results of this section.

Theorem 3.4 *Consider a field with n reservoirs with convex PPR-functions $f_1, \ldots, f_n$. Furthermore, let ϕ be a symmetric, monotone objective function. Assume also that ϕ, interpreted as a function of $\boldsymbol{Q}$, can be extended to a non-decreasing, quasi-convex[2] function defined on the set $\mathcal{Q}$. Then an optimal vector, denoted $\boldsymbol{Q}^{opt}$, i.e., a vector maximizing $\phi(\boldsymbol{Q})$ subject to $\boldsymbol{Q} \in \partial(\mathcal{M}')$, can always be found within the set $\partial(\partial(\mathcal{M}'))$.*

Proof: Let $\boldsymbol{Q} \in \partial(\mathcal{M}')$ be chosen arbitrarily. Then by Theorem A.4 there exists m vectors $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m \in \partial(\partial(\mathcal{M}'))$ and non-negative numbers $\alpha_1, \ldots, \alpha_m$ such that $\sum_{i=1}^{m} \alpha_i \leq 1$ and such that:

$$\boldsymbol{Q} = \sum_{i=1}^{m} \alpha_i \boldsymbol{Q}_i.$$

We then introduce $\boldsymbol{Q}' = (\sum_{i=1}^{m} \alpha_i)^{-1} \boldsymbol{Q}$. Thus, $\boldsymbol{Q}'$ is a convex combination of $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m$. Moreover, since $\sum_{i=1}^{m} \alpha_i \leq 1$, we have $\boldsymbol{Q} \leq \boldsymbol{Q}'$.

By Corollary 3.2 we know that the set $\bar{\mathcal{M}} \cup \partial(\mathcal{M})$ is convex, so $\boldsymbol{Q}'$ must belong to this set. Hence, since ϕ is assumed to be non-decreasing and quasi-convex, it follows that:

$$\phi(\boldsymbol{Q}) \leq \phi(\boldsymbol{Q}') \leq \max\{\phi(\boldsymbol{Q}_1), \ldots, \phi(\boldsymbol{Q}_m)\}. \tag{3.9}$$

Since $\boldsymbol{Q}$ was chosen arbitrarily, we conclude that for any $\boldsymbol{Q} \in \partial(\mathcal{M}')$, there exists some boundary point $\boldsymbol{Q}^* \in \partial(\partial(\mathcal{M}'))$ such that $\phi(\boldsymbol{Q}) \leq \phi(\boldsymbol{Q}^*)$. Hence, an optimal vector, $\boldsymbol{Q}^{opt}$, can always be found within the set $\partial(\partial(\mathcal{M}'))$ ∎

Note that in the proof of Theorem 3.4 will hold even if the definition of ϕ is extended only to the set $\bar{\mathcal{M}} \cup \partial(\mathcal{M})$, i.e., not to the entire set $\mathcal{Q}$.

Theorem 3.5 *Consider a field with n reservoirs with concave PPR-functions $f_1, \ldots, f_n$. Furthermore, let ϕ be a symmetric, monotone objective function. Assume also that ϕ, interpreted as a function of $\boldsymbol{Q}$, can be extended to a non-decreasing quasi-concave[3] function defined on the set $\mathcal{Q}$. Furthermore, assume that the vector, $\boldsymbol{Q}^*$, maximizes $\phi(\boldsymbol{Q})$ subject to $\boldsymbol{Q} \in \partial(\mathcal{M})$, and that the set $\mathcal{N}$ defined relative to $\boldsymbol{Q}^*$ as in (3.8), is non-empty. Then there exists a hyperplane $H = \{\boldsymbol{Q} : \ell(\boldsymbol{Q}) = c\}$ separating $\mathcal{M}$ and $\mathcal{N}$. Moreover, if ϕ is strictly increasing at $\boldsymbol{Q}^*$, then H supports $\mathcal{M}$ at $\boldsymbol{Q}^*$. Finally, if $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ as well, we may let $\boldsymbol{Q}^{opt} = \boldsymbol{Q}^*$.*

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[2]For a definition of quasi-convex functions see Appendix A.2

[3]For a definition of quasi-concave functions see Appendix A.2

Proof: We first note that since the PPR-functions are assumed to be concave, it follows by Proposition 3.1 that $\mathcal{M}$ is convex. Moreover, since ϕ interpreted as a function of $\boldsymbol{Q}$, is assumed to be quasi-concave, it follows by Proposition A.9 that $\mathcal{N}$ is convex. As already pointed out we obviously have that $\mathcal{M} \cap \mathcal{N} = \emptyset$. Hence, it follows by Theorem A.1 there exists a hyperplane H separating $\mathcal{M}$ and $\mathcal{N}$.

If ϕ is strictly increasing at $\boldsymbol{Q}^*$, it follows that any neighborhood of $\boldsymbol{Q}^*$ must contain a vector $\boldsymbol{Q}$ such that $\phi(\boldsymbol{Q}) > \phi(\boldsymbol{Q}^*)$. Thus, by the definition of $\mathcal{N}$ any such neighborhood must intersect $\mathcal{N}$. Hence, by Proposition A.3 H supports $\mathcal{M}$ at $\boldsymbol{Q}^*$. The final statement that if $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ as well, we may let $\boldsymbol{Q}^{opt} = \boldsymbol{Q}^*$ is obvious from the previous discussion ∎

The two above results indicate how to solve the optimization problem given in Step 1 of Algorithm 3.3 in two important cases. If the PPR-functions are convex and the extended version of objective function ϕ is quasi-convex, the optimal $\boldsymbol{Q}^{opt}$ can be found within the set $\partial(\partial(\mathcal{M}'))$. The extreme points of this set correspond to a certain class of admissible production strategies called *priority strategies* which will be discussed in the next subsection. In certain cases it can be shown that the optimal solution can be found within this class. Since there are only a finite number of priority rules, finding the optimal one is easy, at least in principle. Moreover, given an optimal priority strategy, Step 2 of Algorithm 3.3 is trivial, as the corresponding production strategy $\boldsymbol{b} \in \mathcal{B}'$ is essentially uniquely defined by this rule. We will discuss this further in Section 3.2.

If the PPR-functions are concave and the extended version of objective function ϕ is quasi-concave, Step 1 of Algorithm 3.3 typically involves finding the hyperplane separating $\mathcal{M}$ and $\mathcal{N}$, and thus identify the point $\boldsymbol{Q}^*$ where the hyperplane supports $\mathcal{M}$. Assuming that $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ as well, Step 1 is completed by letting $\boldsymbol{Q}^{opt} = \boldsymbol{Q}^*$. Note, however, that verifying that $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ may in general be a difficult task. Often the easiest way to do this, is by proceeding directly to Step 2, using the $\boldsymbol{Q}^*$ found in Step 1. If we are able to successfully complete Step 2 as well, this implies that $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$.

If the PPR-functions and the extended ϕ-function are differentiable, the standard approach to finding $\boldsymbol{Q}^*$ is by using Lagrange multipliers. An example where this method is used, is given in Section 5.

If the extended ϕ-function is a quasi-linear function of the form $\phi(\boldsymbol{Q}) = h(\ell(\boldsymbol{Q}))$, where h is an increasing function and ℓ is a non-zero linear form, it follows that finding the optimal $\boldsymbol{Q}^*$ is equivalent to maximizing $\ell(\boldsymbol{Q})$ subject to $\boldsymbol{Q} \in \partial(\mathcal{M}')$. If the PPR-functions are piecewise linear and concave, then finding the optimal $\boldsymbol{Q}^*$ can be formulated as a linear programming problem. We will return to this in a future paper.

When $\boldsymbol{Q}^{opt}$ lies in the interior of $\partial(\mathcal{M}')$, there is typically no unique solution to Step 2 of Algorithm 3.3. Typically there will be many admissible paths through $\mathcal{M}$ from $\boldsymbol{0}$ to $\boldsymbol{Q}^{opt}$. When searching for such a path it turns out to be easier to solve the problem backwards, i.e., by starting at $\boldsymbol{Q}^{opt}$ and finding an admissible path back to the origin. The reason for this is that the constraints (2.8) and (2.9) are much easier to satisfy close to the origin where $f_1(Q_1), \ldots, f_n(Q_n)$ are large than at the boundary of M where $f_1(Q_1), \ldots, f_n(Q_n)$ are small. Thus, in order to carry out Step 2 of Algorithm 3.3, we will use a certain *backtracking algorithm* which will be described in Section 5.

3.1 Truncated discounted production

In order to exemplify the results given in the previous subsection, we now consider a more specific type of symmetric monotone objective function, referred to as *truncated discounted*

production, and given by the following expression:

$$\phi_{C,R}(\boldsymbol{b}) = \int_0^\infty I\{q(u) \geq C\}q(u)e^{-Ru}du, \qquad 0 \leq C \leq K, \ R \geq 0. \tag{3.10}$$

The parameter R is interpreted as a *discount rate*, while C defines the *level of truncation*, typically reflecting the minimal acceptable production rate, e.g., the lowest production rate resulting in a non-negative cash-flow.

Since $\phi_{C,R}$ only depends on the production strategy through the total production rate q, it follows that $\phi_{C,R}$ is symmetric. Moreover, the truncation factor $I\{q(u) \geq C\}$ and the discounting factor e^{-Ru} ensure that it is monotone as well.

Different choices of C and R yield different types of objective functions. If we e.g., let $C = 0$ and $R > 0$, the integrand of the objective function is not truncated at any level, so we simply get the *total discounted production*.

On the other hand if we let $C = K$, the production is truncated as soon as it leaves the plateau level. In this case the integrand is positive only when $q(u) = K$. In particular if $\boldsymbol{b} \in \mathcal{B}'$, we know that $q(u) = K$ if and only if $0 \leq u \leq T_K(\boldsymbol{b})$, so in this case (3.10) is reduced to:

$$\phi_{C,R}(\boldsymbol{b}) = \phi_{K,R}(\boldsymbol{b}) = K \int_0^{T_K(\boldsymbol{b})} e^{-Ru}du = KR^{-1}(1 - e^{-RT_K(\boldsymbol{b})}), \tag{3.11}$$

when $R > 0$, while $\phi_{C,0}(\boldsymbol{b}) = \phi_{K,0}(\boldsymbol{b}) = KT_K(\boldsymbol{b})$. Moreover, when $\boldsymbol{b} \in \mathcal{B}'$, we have $q(u) = K$ for all $0 \leq u \leq T_K(\boldsymbol{b})$, so:

$$KT_K(\boldsymbol{b}) = \sum_{i=1}^n Q_i(T_K(\boldsymbol{b})).$$

From this it follows that $\phi_{K,R}$, interpreted as a function of $\boldsymbol{Q}$, can be extended to $\mathcal{Q}$ by letting:

$$\phi_{K,R}(\boldsymbol{Q}) = \begin{cases} KR^{-1}[1 - \exp(-K^{-1}R\ell(\boldsymbol{Q}))] & \text{if } R > 0, \\ \ell(\boldsymbol{Q}) & \text{if } R = 0, \end{cases} \tag{3.12}$$

where we have introduced $\ell(\boldsymbol{Q}) = \sum_{i=1}^n Q_i$. Thus, it follows by Proposition A.10 that $\phi_{K,R}$ is quasi-linear. Moreover, $\boldsymbol{Q}^*$ can be found by maximizing $\ell(\boldsymbol{Q})$ subject to $\boldsymbol{Q} \in \partial(\mathcal{M})$.

Maximizing the plateau production $\ell(\boldsymbol{Q})$ or equivalently the plateau length T_K is often easier than maximizing a general objective function of the form $\phi_{C,R}$. Still the special case where $C = K$ and $R = 0$ and the general case are closely related, and an optimal solution to one of them will often be at least a good approximation to an optimal solution of the others. In Section 4 we shall prove that this in fact holds exactly when the PPR-functions are linear.

3.2 Priority strategies

In this subsection we introduce a specific class of production strategies referred to as *priority strategies*. A priority strategy is characterized by prioritizing the reservoirs according to some suitable criterion. More specifically, we define a priority strategy as follows:

Definition 3.6 *Consider a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$, and let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ be a permutation vector representing the prioritization order of the reservoirs. Then the priority strategy relative to $\boldsymbol{\pi}$ is defined by letting the production rates at time t, $q_1(t), \ldots, q_n(t)$, be given by:*

$$q_{\pi_i}(t) = \min[f_{\pi_i}(Q_{\pi_i}(t)), K - \sum_{j<i} q_{\pi_j}(t)], \qquad i = 1, \ldots, n. \tag{3.13}$$

We observe that when assigning the production rate $q_{\pi_i}(t)$ to reservoir π_i, this is limited by $K - \sum_{j<i} q_{\pi_j}(t)$, i.e., the remaining processing capacity after assigning production rates to all the reservoirs with higher priority. If $f_{\pi_i}(Q_{\pi_i}(t)) \le K - \sum_{j<i} q_{\pi_j}(t)$, reservoir π_i can be produced without any choking, and the remaining processing capacity is passed on to the reservoirs with lower priorities. If on the other hand $f_{\pi_i}(Q_{\pi_i}(t)) > K - \sum_{j<i} q_{\pi_j}(t)$, the production at reservoir π_i is choked so that $q_{\pi_i}(t) = K - \sum_{j<i} q_{\pi_j}(t)$. Thus, all the remaining processing capacity is used on this reservoir, and nothing is passed on to the reservoirs with lower priorities.

The priority strategy can also be expressed in terms of the choke factors at time t, i.e., $b_1(t), \dots, b_n(t)$. Recalling that $q_i(t) = b_i(t) f_i(Q_i(t))$, the choke factors are obtained from (3.13) by dividing both sides of the equation by $f_i(Q_i(t))$. Assuming that $f_{\pi_i}(Q_{\pi_i}(t)) > 0$, we get that:

$$b_{\pi_i}(t) = \min[1, \frac{K - \sum_{j<i} q_{\pi_j}(t)}{f_i(Q_i(t))}], \qquad i = 1, \dots, n. \tag{3.14}$$

If $f_{\pi_i}(Q_{\pi_i}(t)) = 0$, the choke factor $b_{\pi_i}(t)$ can be defined arbitrarily, so as a simple convention we let $b_{\pi_i}(t) = 1$ in this case. In any case we see that the resulting production strategy $\boldsymbol{b}$ is essentially uniquely defined for any priority strategy. In particular, the production strategy corresponding to the priority strategy relative to the permutation $\boldsymbol{\pi}$ is denoted by $\boldsymbol{b^\pi}$. Moreover, the class of all priority strategies is denoted by $\mathcal{B}^{PR}$.

To further explore the properties of priority strategies, we introduce:

$$T_i = T_i(\boldsymbol{b^\pi}) = \inf\{t \ge 0 : \sum_{j=1}^{i} f_{\pi_j}(Q_{\pi_j}(t, \boldsymbol{b^\pi})) \le K\}, \qquad i = 1, \dots, n. \tag{3.15}$$

We also let $T_0 = 0$, and note that we obviously have: $0 = T_0 \le T_1 \le \cdots \le T_n = T_K(\boldsymbol{b^\pi})$. Thus, $T_1, \dots, T_n$ defines an increasing sequence of *subplateau sets*, $[0, T_1], \dots, [0, T_n]$, where the last one is equal to the plateau set Π_K. We will refer to $T_1, \dots, T_n$ as the *subplateau lengths* for the given priority strategy.

We now let $i \in \{1, \dots, n\}$, and assume that $T_{i-1} < t < T_i$. Then the reservoirs $\pi_1, \dots, \pi_{i-1}$ are produced without choking, i.e.:

$$q_{\pi_j}(t) = f_{\pi_j}(Q_{\pi_j}(t)), \qquad j = 1, \dots, i-1. \tag{3.16}$$

Furthermore, the reservoir π_i is produced *with choking* so that:

$$q_{\pi_i}(t) = K - \sum_{j<i} q_{\pi_j}(t) = K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(t)). \tag{3.17}$$

Finally the reservoirs $\pi_{i+1}, \dots, \pi_n$ are not produced at all. Note also that $t = T_i$ is the smallest t where:

$$f_{\pi_i}(Q_{\pi_i}(t)) \le K - \sum_{j<i} q_{\pi_j}(t) = K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(t)). \tag{3.18}$$

Thus, from this point of time the reservoir π_i can be produced without any choking.

Summarizing this we see that for $i = 1, \dots, n$, the production rate, $q_i(t)$ is given by:

$$q_i(t) = \begin{cases} 0 & \text{if } t < T_{i-1}, \\ K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(t)) & \text{if } T_{i-1} \le t < T_i, \\ f_{\pi_i}(Q_{\pi_i}(t)) & \text{if } t \ge T_i. \end{cases} \tag{3.19}$$

The priority strategies have the important property that they generate admissible paths through $\mathcal{M}'$ such that $\boldsymbol{Q}(T_K(\boldsymbol{b^\pi}), \boldsymbol{b^\pi}) \in \partial(\partial(\mathcal{M}'))$. In order to study this further we introduce the set $\mathcal{A} \subseteq \mathcal{Q}$ consisting of the union of all admissible paths. Thus, we have:

$$\mathcal{A} = \{\boldsymbol{Q}(t, \boldsymbol{b}) : t \geq 0, \boldsymbol{b} \in \mathcal{B}'\}.$$

The following lemma shows that the path of a priority strategy follows the boundary of $\mathcal{A}$.

Lemma 3.7 *Consider a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$. Moreover, let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ be a permutation vector, and let $\boldsymbol{b^\pi}$ be the corresponding priority strategy. Then we have:*

$$\boldsymbol{Q}(t, \boldsymbol{b^\pi}) \in \partial(\mathcal{A}) \text{ for all } t \geq 0.$$

Proof: Let $t_1 \geq 0$. We first note that if $Q_{\pi_1}(t_1, \boldsymbol{b^\pi}) = V_{\pi_1}$, then $\boldsymbol{Q}(t_1, \boldsymbol{b^\pi}) \in \partial(\mathcal{Q})$, and hence obviously $\boldsymbol{Q}(t_1, \boldsymbol{b^\pi}) \in \partial(\mathcal{A})$ as well. Thus, in the rest of the proof we can restrict ourselves to the case where $Q_{\pi_1}(t_1, \boldsymbol{b^\pi}) < V_{\pi_1}$. Since reservoir π_1 is given the highest priority, it is easy to see that $Q_{\pi_1}(t, \boldsymbol{b^\pi})$ must be strictly increasing in t for $0 \leq t \leq t_1$.

In order to show that $\boldsymbol{Q}(t_1, \boldsymbol{b^\pi}) \in \partial(\mathcal{A})$, we must show that any neighborhood of $\boldsymbol{Q}(t_1, \boldsymbol{b^\pi})$ contains a point $\boldsymbol{Q}^* \notin \mathcal{A}$. Thus, let N be a neighborhood of $\boldsymbol{Q}(t_1, \boldsymbol{b^\pi})$. Moreover, let $\epsilon > 0$, and consider the point $\boldsymbol{Q}^* = (Q_1^*, \ldots, Q_n^*)$ defined as follows:

$$Q_{\pi_1}^* = Q_{\pi_1}(t_1, \boldsymbol{b^\pi}),$$
$$Q_i^* = Q_i(t_1, \boldsymbol{b^\pi}) - \epsilon, \text{ for all } i \neq \pi_1.$$

By choosing a sufficiently small ϵ, we can ensure that $\boldsymbol{Q}^* \in N$. We then claim that $\boldsymbol{Q}^* \notin \mathcal{A}$. Assume for a contradiction that $\boldsymbol{Q}^* \in \mathcal{A}$. That is, there exists a production strategy $\boldsymbol{b} \in \mathcal{B}'$, and a point of time $t_2 \geq 0$ such that $\boldsymbol{Q}(t_2, \boldsymbol{b}) = \boldsymbol{Q}^*$. Since obviously $b_{\pi_1}(t) \leq b_{\pi_1}^{\boldsymbol{\pi}}(t)$ for all $t \geq 0$, it follows by Proposition 2.1 that $Q_{\pi_1}(t, \boldsymbol{b}) \leq Q_{\pi_1}(t, \boldsymbol{b^\pi})$ for all $t \geq 0$. Hence, since we have assumed that $Q_{\pi_1}(t_2, \boldsymbol{b}) = Q_{\pi_1}(t_1, \boldsymbol{b^\pi})$ and since $Q_{\pi_1}(t, \boldsymbol{b^\pi})$ is strictly increasing in t for $0 \leq t \leq t_1$, this implies that $t_2 \geq t_1$. From this it follows that:

$$Q_{\pi_1}(t_2, \boldsymbol{b}) \leq Q_{\pi_1}(t_2, \boldsymbol{b^\pi}),$$
$$Q_i(t_2, \boldsymbol{b}) < Q_i(t_2, \boldsymbol{b^\pi}), \text{ for all } i \neq \pi_1.$$

Hence:

$$\sum_{i=1}^n Q_i(t_2, \boldsymbol{b}) < \sum_{i=1}^n Q_i(t_2, \boldsymbol{b^\pi}) \leq K t_2.$$

Since $\boldsymbol{b}$ is assumed to be admissible, this implies that $t_2 > T_K(\boldsymbol{b})$. Using Proposition 2.1 again it follows that we also have:

$$Q_{\pi_1}(T_K(\boldsymbol{b}), \boldsymbol{b}) \leq Q_{\pi_1}(T_K(\boldsymbol{b}), \boldsymbol{b^\pi}).$$

Moreover, we also claim that:

$$Q_i(T_K(\boldsymbol{b}), \boldsymbol{b}) < Q_i(T_K(\boldsymbol{b}), \boldsymbol{b^\pi}), \text{ for all } i \neq \pi_1.$$

To explain why this claim is true, we note that if this is *not* the case, by continuity there must exist a point of time $t_0 \in [T_K(\boldsymbol{b}), t_2]$ such that for at least one $i \neq \pi_1$ we have $Q_i(t_0, \boldsymbol{b}) = Q_i(t_0, \boldsymbol{b^\pi})$. However, since $\boldsymbol{b} \in \mathcal{B}'$ and $t_0 \geq T_K(\boldsymbol{b})$, the ith reservoir is produced without any choking throughout the interval $[T_K(\boldsymbol{b}), t_2]$, which contradicts that $Q_i(t_2, \boldsymbol{b}) < Q_i(t_2, \boldsymbol{b^\pi})$.

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Combining all this we get that:

$$\sum_{i=1}^{n} Q_i(T_K(\boldsymbol{b}), \boldsymbol{b}) < \sum_{i=1}^{n} Q_i(T_K(\boldsymbol{b}), \boldsymbol{b^\pi}) \leq K T_K(\boldsymbol{b}),$$

which implies that $\boldsymbol{b}$ cannot be admissible. Hence, we conclude that $\boldsymbol{Q}^* \notin \mathcal{A}$. Thus, we have shown that any neighborhood of $\boldsymbol{Q}(t_1, \boldsymbol{b^\pi})$ contains a point not in $\mathcal{A}$, i.e., $\boldsymbol{Q}(t_1, \boldsymbol{b^\pi}) \in \partial(\mathcal{A})$ ∎

We also need the following lemma which we state without proof:

Lemma 3.8 *Consider a field with n reservoirs. Then we have:*

$$\partial(\partial(\mathcal{M}')) = \partial(\mathcal{A}) \cap \partial(\mathcal{M}). \tag{3.20}$$

Using Lemma 3.7 and Lemma 3.8 we can now show:

Theorem 3.9 *Consider a field with n reservoirs, and let $\boldsymbol{b^\pi}$ be a priority strategy. Then $\boldsymbol{Q}(T_K(\boldsymbol{b^\pi}), \boldsymbol{b^\pi}) \in \partial(\partial(\mathcal{M}'))$.*

Proof: By Lemma 3.7 we have that $\boldsymbol{Q}(T_K(\boldsymbol{b^\pi}), \boldsymbol{b^\pi}) \in \partial(\mathcal{A})$. Moreover, by definition of $T_K(\boldsymbol{b^\pi})$ it follows that $\boldsymbol{Q}(T_K(\boldsymbol{b^\pi}), \boldsymbol{b^\pi}) \in \partial(\mathcal{M})$. Hence, by Lemma 3.8 we must have $\boldsymbol{Q}(T_K(\boldsymbol{b^\pi}), \boldsymbol{b^\pi}) \in \partial(\mathcal{A}) \cap \partial(\mathcal{M}) = \partial(\partial(\mathcal{M}'))$ ∎

When the PPR-functions are convex, and the objective function, ϕ, interpreted as a function of $\boldsymbol{Q}$, is quasi-convex, we know by Theorem 3.4 that an optimal production strategy $\boldsymbol{b}^*$ should be chosen so that $\boldsymbol{Q}(T_K(\boldsymbol{b}^*), \boldsymbol{b}^*) \in \partial(\partial(\mathcal{M}'))$. By Theorem 3.9 we see that priority strategies always satisfies this condition. Thus, priority strategies provide a good starting point for the optimal strategy. We close this section by a result providing a sufficient criterion for when the optimal strategy can be found within the class of priority rules.

Theorem 3.10 *Consider a field with n reservoirs with convex PPR-functions $f_1, \ldots, f_n$. Furthermore, let ϕ be a symmetric, monotone objective function. Assume also that ϕ, interpreted as a function of $\boldsymbol{Q}$, can be extended to a non-decreasing, quasi-convex function defined on the set $\mathcal{Q}$. Finally assume that $\partial(\mathcal{M}')$ is contained in the convex hull of the points $\{\boldsymbol{Q}(T_K(\boldsymbol{b}), \boldsymbol{b}) : \boldsymbol{b} \in \mathcal{B}^{PR}\}$. Then an optimal production strategy can be found within the class $\mathcal{B}^{PR}$.*

Proof: Let $\boldsymbol{Q} \in \partial(\mathcal{M}')$ be chosen arbitrarily. Then by the assumption there exists non-negative numbers $\{\alpha_{\boldsymbol{b}} : \boldsymbol{b} \in \mathcal{B}^{PR}\}$ such that $\sum_{\boldsymbol{b} \in \mathcal{B}^{PR}} \alpha_{\boldsymbol{b}} \leq 1$ and such that:

$$\boldsymbol{Q} = \sum_{\boldsymbol{b} \in \mathcal{B}^{PR}} \alpha_{\boldsymbol{b}} \boldsymbol{Q}(T_K(\boldsymbol{b}), \boldsymbol{b}).$$

From this the result follows by arguments similar to the proof of Theorem 3.4 ∎

4 Optimization with linear PPR-functions

In this section we consider the case where all the PPR-functions are linear. That is, we consider a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$, such that:

$$f_i(Q_i(t)) = D_i(V_i - Q_i(t)), \qquad i = 1, \ldots, n, \tag{4.1}$$

where $V_1, \ldots, V_n$ denotes the recoverable volumes from the n reservoirs, and where we assume that the reservoirs have been indexed so that $0 < D_1 \leq D_2 \leq \cdots \leq D_n$.

We then consider the ith reservoir, and let $T \geq 0$. If this reservoir is produced without any choking, i.e., with a choking factor function $b_i(t) = 1$ for all $t \geq T$, we can solve the differential equation (4.1) for $t \geq T$ given that the cumulative production at time T is $Q_i(T)$, and get:

$$q_i(t) = D_i(V_i - Q_i(T)) \exp(-D_i(t - T)), \qquad t \geq T. \tag{4.2}$$

Moreover, by integrating $q_i(t)$ from T to t we also get:

$$Q_i(t) = V_i(1 - e^{-D_i(t-T)}) + Q_i(T)e^{-D_i(t-T)}, \qquad t \geq T. \tag{4.3}$$

If on the other hand, the reservoir is produced with a choking factor function $b_i(t) \leq 1$ for $t \geq T$ it follows by Proposition 2.1 that $Q_i(t)$ will be less than or equal to the right-hand side of (4.3). These relations will be used in order to prove the following result:

Theorem 4.1 *Consider a field with n reservoirs with linear PPR-functions $f_1, \ldots, f_n$ given by (4.1). Then let $\boldsymbol{b}^1$ denote the priority strategy corresponding to the permutation $\boldsymbol{\pi} = (1, 2, \ldots, n)$, and let $\boldsymbol{b}^2$ be any other valid production strategy. Then $Q(t, \boldsymbol{b}^1) \geq Q(t, \boldsymbol{b}^2)$ for all $t \geq 0$. Thus, $\boldsymbol{b}^1$ is optimal with respect to any monotone, symmetric objective function.*

Proof: We start by introducing the plateau lengths $T_1, \ldots, T_n$ as defined in (3.15). When the priority strategy $\boldsymbol{b}^1$ is used, reservoir 1 is produced at the rate K throughout the interval $[0, T_1]$, the reservoirs 1 and 2 are produced at a total rate K throughout the interval $[0, T_2]$, etc. Moreover, reservoir 1 will be produced without any choking for $t \geq T_1$, reservoir 1 and 2 will be produced without any choking for $t \geq T_2$, etc.

We shall now prove by induction that:

$$\sum_{j=1}^{i} Q_j(t, \boldsymbol{b}^1) \geq \sum_{j=1}^{i} Q_j(t, \boldsymbol{b}^2), \qquad t \geq 0, \ i = 1, \ldots, n. \tag{4.4}$$

Thus, we start out by considering the case where $i = 1$. If $0 \leq t \leq T_1$, then obviously:

$$Q_1(t, \boldsymbol{b}^1) = Kt \geq Q_1(t, \boldsymbol{b}^2).$$

If $t > T_1$, we know that reservoir 1 is produced without any choking when $\boldsymbol{b}^1$ is used. Thus, we have:

$$Q_1(t, \boldsymbol{b}^1) = V_1(1 - e^{-D_1(t-T_1)}) + Q_1(T_1, \boldsymbol{b}^1)e^{-D_1(t-T_1)}.$$

If, on the other hand, $\boldsymbol{b}^2$ is used, we get:

$$Q_1(t, \boldsymbol{b}^2) \leq V_1(1 - e^{-D_1(t-T_1)}) + Q_1(T_1, \boldsymbol{b}^2)e^{-D_1(t-T_1)}.$$

Thus, since $Q_1(T_1, \boldsymbol{b}^1) \geq Q_1(T_1, \boldsymbol{b}^2)$, it follows that $Q_1(t, \boldsymbol{b}^1) \geq Q_1(t, \boldsymbol{b}^2)$ for all $t > T_1$. Hence, we conclude that $Q_1(t, \boldsymbol{b}^1) \geq Q_1(t, \boldsymbol{b}^2)$ for all $t \geq 0$, i.e., (4.4) is proved for $i = 1$.

We then assume that (4.4) is proved for $i = 1, \ldots, (k-1)$, and consider the case where $i = k$. If $0 \leq t \leq T_k$, we have:

$$\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^1) = Kt \geq \sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^2). \tag{4.5}$$

We then consider the case where $t > T_k$. If $\boldsymbol{b}^1$ is used, the reservoirs $1, 2, \ldots, k$ are produced without any choking, thus:

$$\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^1) = \sum_{j=1}^{k} V_j(1 - e^{-D_j(t-T_k)}) + \sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^1)e^{-D_j(t-T_k)}. \tag{4.6}$$

If, on the other hand, $\boldsymbol{b}^2$ is used, we get:

$$\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^2) \leq \sum_{j=1}^{k} V_j(1 - e^{-D_j(t-T_k)}) + \sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^2)e^{-D_j(t-T_k)}. \tag{4.7}$$

By the induction hypothesis and (4.5) we have that:

$$\sum_{j=1}^{i} Q_j(T_k, \boldsymbol{b}^1) \geq \sum_{j=1}^{i} Q_j(T_k, \boldsymbol{b}^2), \qquad i = 1, \ldots, k.$$

Moreover, since $D_1 \leq D_2 \leq \cdots \leq D_k$, we have:

$$e^{-D_1(t-T_k)} \geq \cdots \geq e^{-D_k(t-T_k)}, \text{ for all } t \geq T_k.$$

Then it follows by Lemma B.1 that:

$$\sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^1)e^{-D_j(t-T_k)} \geq \sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^2)e^{-D_j(t-T_k)} \tag{4.8}$$

By combining (4.6), (4.7) and (4.8), for all $t > T_k$ and (4.5) for $0 \leq t \leq T_k$, we get for $t \geq 0$:

$$\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^1) \geq \sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^2).$$

Thus, (4.4) is proved for $i = k$ as well. Hence, the result is proved by induction ∎

Having identified the optimal production strategy in the case of linear PPR-function, we proceed to calculating the resulting production rates and cumulative production functions. Since the optimal solution is a priority strategy, it turns out that it is fairly easy to solve this. We consider once again a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$, of the form given in (4.1). The formulas we are about to present, are valid for any priority strategy, not just the optimal one. Thus, we consider an arbitrary priority strategy $\boldsymbol{b}^{\boldsymbol{\pi}}$ where the permutation vector is $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$.

In order to find the production rates and cumulative production functions, we start out by assuming that the subplateau lengths, $T_1, \ldots, T_n$, are *known*. As in Section 3.2 we also let $T_0 = 0$. Then by combining (3.19) and (4.2) it is easy to see that for $i = 1, \ldots, n$, the production rate, $q_{\pi_i}(t)$ is given by:

$$q_{\pi_i}(t) = \begin{cases} 0 & \text{if } t < T_{i-1}, \\ K - \sum_{j<i} D_{\pi_j}(V_{\pi_j} - Q_{\pi_j}(T_j))e^{-D_{\pi_j}(t-T_j)} & \text{if } t \in [T_{i-1}, T_i), \\ D_{\pi_i}(V_{\pi_i} - Q_{\pi_i}(T_i))e^{-D_{\pi_i}(t-T_i)} & \text{if } t \geq T_i. \end{cases} \tag{4.9}$$

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Moreover, by integrating these production rates we get the following cumulative production functions:

$$Q_{\pi_i}(t) = \begin{cases} 0 & \text{if } t < T_{i-1}, \\ K[t - T_{i-1}] - \sum_{j<i}(V_{\pi_j} - Q_{\pi_j}(T_j))[e^{-D_{\pi_j}(T_{i-1}-T_j)} - e^{-D_{\pi_j}(t-T_j)}] & \text{if } t \in [T_{i-1}, T_i], \\ V_{\pi_i}(1 - e^{-D_{\pi_i}(t-T_i)}) + Q_{\pi_i}(T_i)e^{-D_{\pi_i}(t-T_i)} & \text{if } t > T_i. \end{cases}$$

(4.10)

In order to complete these formulas we need to explain how to determine the subplateau lengths, $T_1, \ldots, T_n$. This will be done as a sequential process where T_1 is determined first. Then T_1 is used to determine T_2, T_1 and T_2 are used to determine T_3, and so on until all the subplateau lengths have been found.

To determine T_1 we first consider the case where $f_{\pi_1}(Q_{\pi_1}(T_0)) \leq K$. In this case it follows by (3.15) that $T_1 = T_0 = 0$, i.e., the first subplateau has zero length. On the other hand, if $f_{\pi_1}(Q_{\pi_1}(T_0)) > K$, T_1 is found as the solution to the equation:

$$f_{\pi_1}(Q_{\pi_1}(t)) = D_{\pi_1}(V_{\pi_1} - Q_{\pi_1}(t)) = K. \tag{4.11}$$

Since obviously $Q_{\pi_1}(t) = Kt$ for all $t \leq T_1$, we get that $T_1 = V_{\pi_1}K^{-1} - D_{\pi_1}^{-1}$ in this case.

We then assume that we have determined $T_1, \ldots, T_{i-1}$, and consider the problem of determining T_i. As for T_1 we first consider the case where $f_{\pi_i}(Q_{\pi_i}(T_{i-1})) \leq K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(T_{i-1}))$. In this case it follows by (3.15) that $T_i = T_{i-1}$, i.e., the ith subplateau has the same length as the $(i-1)$th subplateau. On the other hand, if $f_{\pi_i}(Q_{\pi_i}(T_{i-1})) > K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(T_{i-1}))$, T_i is found as the solution to the equation:

$$f_{\pi_i}(Q_{\pi_i}(t)) = D_{\pi_i}(V_{\pi_i} - Q_{\pi_i}(t)) = K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(t)), \tag{4.12}$$

where $Q_{\pi_i}(t)$ for all $t \in [T_{i-1}, T_i]$ is given by (4.10). In general this equation is easily solvable using standard numerical methods.

4.1 An example with linear PPR-functions

We consider a field with $n = 3$ reservoirs with linear PPR functions, f_1, f_2, f_3 of the form given in (4.1). Moreover, we assume, as above, that the reservoirs are indexed so that $0 < D_1 < D_2 < D_3$. More specifically, we let $D_1 = 0.0003$ $D_2 = 0.0006$, and $D_3 = 0.0010$.

According to Theorem 4.1 the optimal production strategy with respect to any symmetric monotone objective function is the priority strategy corresponding to the permutation $\boldsymbol{\pi} = (1, 2, 3)$. In this example we focus on the objective function $\phi_{K,0}$ defined by letting $C = K$ and $R = 0$ in (3.10). As explained in Section 3.1, the optimal solution maximizes the plateau volume, $\ell(\boldsymbol{Q}) = Q_1 + Q_2 + Q_3$ subject to $\boldsymbol{Q} \in \partial(\mathcal{M}')$.

We observe that the optimal priority strategy does not depend on the producible volumes V_1, V_2, V_3. However, the volumes may still have an impact on the ranking of the different priority rules as well as the differences in performance. To see this we consider two different cases. In the first case we let $V_1 = 15.0$ MSm^3, $V_2 = 10.0$ MSm^3 and $V_3 = 5.0$ MSm^3, while in the second example we let $V_1 = 5.0$ MSm^3, $V_2 = 10.0$ MSm^3 and $V_3 = 15.0$ MSm^3.

In Table 4.1 we have listed the parameter values for the two cases. We have also included columns showing the maximum value of the PPR-functions, i.e., $f_i(0) = D_iV_i$, $i = 1, 2, 3$. In both cases we let $K = 3.0$ kSm^3 per day. We note that in the second case the maximum value

	Case 1			Case 2		
Res.	Producible volume V_i (MSm^3)	Scale parameter D_i	Max rate D_iV_i (kSm^3/d)	Producible volume V_i (MSm^3)	Scale parameter D_i	Max rate D_iV_i (kSm^3/d)
1	15.0	0.0003	4.5	5.0	0.0003	1.5
2	10.0	0.0006	6.0	10.0	0.0006	6.0
3	5.0	0.0010	5.0	15.0	0.0010	15.0

Table 4.1: *Parameter values for the three reservoirs.*

of f_1 is just 1.5 kSm^3 per day. Thus, in this case the first reservoir can never reach the plateau level K alone. Hence, if this reservoir is given the highest priority, the subplateau length T_1 is zero.

By using the formulas (4.9) and (4.10), we may calculate the plateau length $T_K(\boldsymbol{b^\pi})$ for each of the six possible priority strategies. Moreover, we may calculate cumulative production for each of the reservoirs as well as the total cumulative production $KT_K(\boldsymbol{b^\pi})$ at this point of time. The results are shown in Table 4.2.

From the table we see that the priority strategy corresponding to the permutation $(1, 2, 3)$ is indeed optimal in both cases. The second and third best priority strategies correspond to the permutations $(2, 1, 3)$ and $(1, 3, 2)$ respectively. Both these permutations are "neighbors" of the optimal permutation in the sense that they can be obtained from $(1, 2, 3)$ by switching two consecutive entries in the vector. That is, $(2, 1, 3)$ is obtained from $(1, 2, 3)$ by switching the two first entries, while $(1, 3, 2)$ is obtained from $(1, 2, 3)$ by switching the two last entries. We observe, however, that in the first case the the total cumulative productions using the permutations $(2, 1, 3)$ and $(1, 3, 2)$ are very close to each other, while in the second case the permutation $(2, 1, 3)$ produces a result which is closer to the result of the optimal strategy.

Another observation is that the results using the two worst permutations, i.e., $(2, 3, 1)$ and $(3, 2, 1)$ switch places in the two cases. In the first case $(2, 3, 1)$ produces the worst results, while in the second case $(3, 2, 1)$ comes in last.

Summarizing the example, we see that the results confirm that the optimal priority strategy indeed corresponds to the permutation $(1, 2, 3)$ and thus agree with Theorem 4.1. Still we see that the producible volumes also affect the results significantly.

We recall that for any admissible strategy the vector $\boldsymbol{Q}(T_K)$ always belongs to the set $\partial(\mathcal{M}')$. In the linear case $\partial(\mathcal{M}')$ is a part of the hyperplane with equation:

$$\sum_{i=1}^n f_i(Q_i) = \sum_{i=1}^n D_i(V_i - Q_i) = K.$$

Thus, in particular, $\boldsymbol{Q}(T_K, \boldsymbol{b^\pi})$ belongs to this hyperplane for any priority strategy $\boldsymbol{b^\pi}$. In Figure 4.1 and Figure 4.2 we have illustrated the resulting hyperplanes for Case 1 and 2 respectively. Moreover, the plots also show the locations of $\boldsymbol{Q}(T_K, \boldsymbol{b^\pi})$ for each of the six priority strategies. In both cases these six points forms a hexagon. However, as we see, the shapes of these hexagons are quite different. Obviously if the points of two priority strategies are close to each other, then so are their respective plateau productions as well. Thus, in particular the points corresponding to the two best permutations $(1, 2, 3)$ and $(2, 1, 3)$ are much closer together in Case 2 than in

Priority strategy π	Plateau prod. res. 1 $Q_1(T_K)$ MSm³	Plateau prod. res. 2 $Q_2(T_K)$ MSm³	Plateau prod. res. 3 $Q_3(T_K)$ MSm³	Tot. plateau production $\ell(\boldsymbol{Q}(T_K))$ MSm³	Rank
Case 1					
(1, 2, 3)	13.745	9.083	2.927	25.755	1
(2, 1, 3)	11.352	9.897	3.156	24.405	2
(1, 3, 2)	13.551	5.828	4.938	24.317	3
(3, 1, 2)	12.525	6.241	4.998	23.764	4
(3, 2, 1)	6.173	9.424	4.994	20.591	5
(2, 3, 1)	5.810	9.774	4.893	20.477	6
Case 2					
(1, 2, 3)	4.654	9.885	12.173	26.712	1
(2, 1, 3)	4.331	9.932	12.241	26.504	2
(1, 3, 2)	4.585	5.466	14.845	24.896	3
(3, 1, 2)	3.396	5.887	14.949	24.232	4
(2, 3, 1)	0.461	9.883	13.432	23.776	5
(3, 2, 1)	0.655	7.306	14.920	22.880	6

Table 4.2: *Plateau production for the six priority strategies in the two cases.*

Case 1. At the same time their respective plateau productions are closer in Case 2 than Case 1.

A similar but opposite effect holds for the two worst permutations, i.e., $(2, 3, 1)$ and $(3, 2, 1)$. In Case 1 both the points representing these two strategies and the plateau productions are very close to each other. In Case 2 the points are much further apart, and so are the plateau productions.
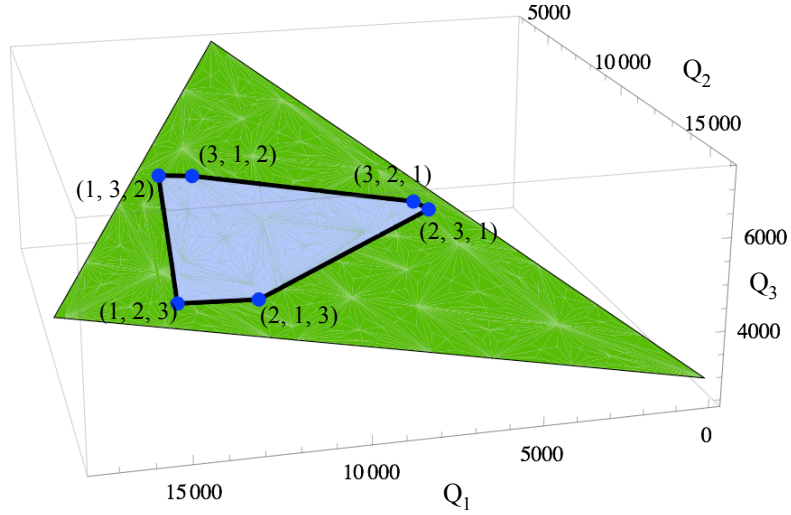
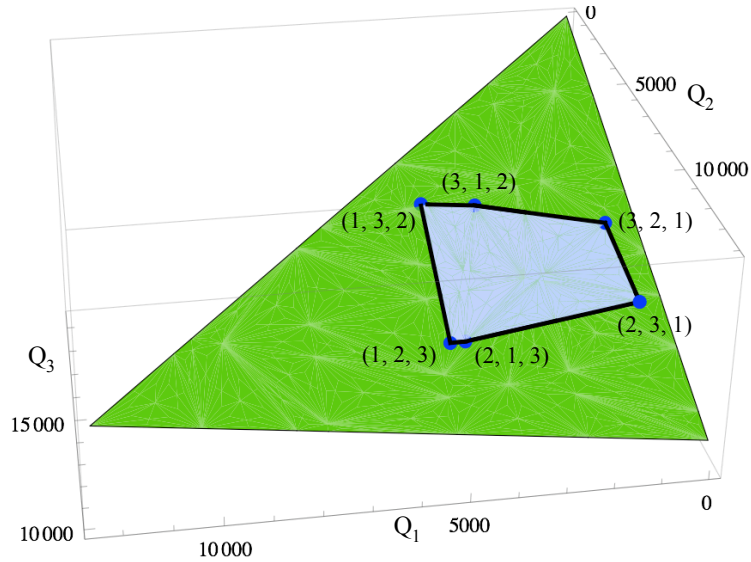Figure 4.1: *The hyperplane containing $\partial(\mathcal{M}')$ in Case 1*

Figure 4.2: *The hyperplane containing $\partial(\mathcal{M}')$ in Case 2*

5 Generating optimal strategies using backtracking

In this section we present a methodology for Step 2 of Algorithm 3.3. Thus, we consider a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$, and assume that Step 1 of this algorithm is completed, where $\boldsymbol{Q}^* = (Q_1^*, \ldots, Q_n^*)$ is the vector maximizing $\phi(\boldsymbol{Q})$ subject to $\boldsymbol{Q} \in \partial(\mathcal{M})$. Moreover, we assume that $T_K^* = K^{-1} \sum_{i=1}^n Q_i^*$ is the point of time when $\boldsymbol{Q}^*$ is reached. The idea is to construct an admissible production strategy generating a path $\{\boldsymbol{Q}(t) : 0 \leq t \leq T_K^*\}$, where $\boldsymbol{Q}(0) = \boldsymbol{0}$ and $\boldsymbol{Q}(T_K^*) = \boldsymbol{Q}^*$. As pointed out in the discussion following Theorem 3.5,

finding such a production strategy implies that $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ as well. Thus, the constructed admissible production strategy is indeed an optimal strategy.

Except in special cases, like e.g., when $\boldsymbol{Q}^*$ corresponds to a priority strategy, there will typically be an infinite number of admissible paths from $\boldsymbol{0}$ to $\boldsymbol{Q}^*$. In order to find *one* such path, we search within the class of piecewise linear paths. More specifically, let $0 = t_0 < t_1 < \cdots < t_N = T_K^*$, and let $\boldsymbol{q}_j = (q_{1,j}, \ldots, q_{n,j})$, $j = 1, \ldots, N$. We then assume that the reservoirs are produced using the following rates:

$$q_i(t) = q_{i,j}, \quad t \in (t_{j-1}, t_j], \; i = 1, \ldots, n, \; j = 1, \ldots, N.$$

Thus, the production rates are constant within each of the N intervals $(t_0, t_1], \ldots, (t_{N-1}, t_N]$. Hence, the cumulative production functions are given by:

$$Q_i(t) = Q_i(t_{j-1}) + q_{i,j}(t - t_{j-1}), \quad t \in (t_{j-1}, t_j], \; i = 1, \ldots, n, \; j = 1, \ldots, N,$$

where we of course assume that $Q_i(t_0) = Q_i(0) = 0$, $i = 1, \ldots, n$. In order to ensure that we have an admissible path, we must have:

$$\sum_{i=1}^{n} q_{i,j} = K, \quad j = 1, \ldots, N, \tag{5.1}$$

and that:

$$0 \le q_{i,j} \le f_i(Q_i(t)), \quad t \in (t_{j-1}, t_j], \; i = 1, \ldots, n, \; j = 1, \ldots, N. \tag{5.2}$$

Since the PPR-functions are assumed to be non-increasing, it follows that the last condition is satisfied if and only if

$$0 \le q_{i,j} \le f_i(Q_i(t_j)), \quad i = 1, \ldots, n, \; j = 1, \ldots, N. \tag{5.3}$$

Finally, we want the path to end up at the optimal point, i.e., we must have $\boldsymbol{Q}(T_K^*) = \boldsymbol{Q}(t_N) = \boldsymbol{Q}^*$. Expressed in terms of $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N$ we get the following condition:

$$\sum_{j=1}^{N} \boldsymbol{q}_j(t_j - t_{j-1}) = \boldsymbol{Q}^*. \tag{5.4}$$

Thus, the problem is reduced to choosing the intervals $(t_0, t_1], \ldots, (t_{N-1}, t_N]$, in particular, the number of intervals N, and finding the vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N$ subject to the conditions (5.1), (5.3) and (5.4).

From a practical point of view it is of interest to keep the number of intervals as small as possible, since this means that the reservoirs can be produced with stable rates. However, if N is too small, it may not be possible to find a piecewise linear admissible path from $\boldsymbol{0}$ to $\boldsymbol{Q}^*$. In order to find a suitable N, we start out by letting N be small, e.g., $N = 1$. If it is possible to find an admissible path from $\boldsymbol{0}$ to $\boldsymbol{Q}^*$ with this N, we are done. If not, we increase N and try once more. This process is repeated until we eventually find an admissible path from $\boldsymbol{0}$ to $\boldsymbol{Q}^*$ given that such a path exists.

For a given N we also need to choose the numbers $t_1, \ldots, t_N$. The easiest choice here would be to distribute these partition points uniformly over the interval $[0, T_K^*]$. Since, however, the condition (5.2) is stricter $f_i(Q_i(t))$ is small, i.e., when t is close to T_K^*, it may be a good idea to distribute the partition points so that we have shorter intervals when t is close to T_K^*, and longer

intervals when t is close to 0. In Appendix C we show how this can be done in a systematic way.

Having chosen N the partition points $t_1, \ldots, t_N$, we now turn to the problem of choosing the vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N$ subject to the conditions (5.1), (5.3) and (5.4). Rather than finding all these vectors at once, it turns out to be easier to determine them one by one, starting backwards with $\boldsymbol{q}_N$. Thus, let:

$$\boldsymbol{Q}^{(k)} = \boldsymbol{Q}^* - \sum_{j>k} \boldsymbol{q}_j(t_j - t_{j-1}), \qquad k = 0, 1, \ldots, N.$$

Thus, in particular $\boldsymbol{Q}^{(0)} = \boldsymbol{0}$, while $\boldsymbol{Q}^{(N)} = \boldsymbol{Q}^*$. As we move backwards from $\boldsymbol{Q}^*$ to $\boldsymbol{0}$, we follow a piecewise linear path through the points $\boldsymbol{Q}^{(N)}, \boldsymbol{Q}^{(N-1)}, \ldots, \boldsymbol{Q}^{(0)}$. At each of these points we are allowed to change direction by choosing the next vector in the set $\{\boldsymbol{q}_N, \boldsymbol{q}_{N-1}, \ldots, \boldsymbol{q}_1\}$. Thus, assume that we have chosen the directions $\boldsymbol{q}_N, \ldots, \boldsymbol{q}_{k+1}$, and that we want to choose $\boldsymbol{q}_k$. At this stage we have constructed an admissible path backwards from $\boldsymbol{Q}^*$ to the point $\boldsymbol{Q}^{(k)}$. Since our goal is to find an admissible path back to $\boldsymbol{0}$, the ideal direction from the point $\boldsymbol{Q}^{(k)}$ is a vector $\boldsymbol{q}_k$ that is parallel to $\boldsymbol{Q}^{(k)}$. If we can find such a vector which at the same time satisfies the conditions (5.1), (5.3) and (5.4), we would be right on track back to $\boldsymbol{0}$. In general,! however, this may not be possible. Thus, we instead look for a vector $\boldsymbol{q}_k$ satisfying the conditions (5.1), (5.3) and (5.4), and such that the angle between $\boldsymbol{q}_k$ and $\boldsymbol{Q}^{(k)}$ is as small as possible. That is, we choose $\boldsymbol{q}_k$ by maximizing the scalar product $\boldsymbol{Q}^{(k)}\boldsymbol{q}_k$ subject to (5.1), (5.3) and (5.4). This optimization problem is a standard linear programming problem which can easily be solved using the well-known Simplex algorithm.

By solving a linear programming problem at each of the points $\boldsymbol{Q}^{(N)}, \boldsymbol{Q}^{(N-1)}, \ldots, \boldsymbol{Q}^{(1)}$, we may be able to construct an admissible path from $\boldsymbol{Q}^*$ back to $\boldsymbol{0}$. If the procedure fails, we increase N, and run the procedure once again, and so forth. In order to avoid an infinite number of runs, however, one would typically specify some suitable maximum number of intervals, denoted by N_{max}. Ideally the process produces an admissible path with $N \leq N_{max}$ intervals. Still it may happen that no such path is found even for a very large value of N_{max}. This obviously happens if $\boldsymbol{Q}^* \notin \partial(\mathcal{M}')$ since by definition no admissible path from $\boldsymbol{0}$ to $\boldsymbol{Q}^*$ exists in this case. Unfortunately, since N_{max} is finite, the process may also fail when $\boldsymbol{Q}^*$ is a point in $\partial(\mathcal{M}')$ very close to or at the boundary of this set. Thus, the algorithm is not guaranteed to work even though there may exist an admissible path from $\boldsymbol{0}$ to $\boldsymbol{Q}^*$. Still in cases where $\boldsymbol{Q}^*$ is located in the central parts of $\partial(\mathcal{M}')$, the algorithm tends to work very well.

5.1 An example with concave PPR-functions

We consider a field with concave PPR-functions $f_1, \ldots, f_{10}$ given by

$$f_i(Q_i(t)) = \sqrt{D_i(V_i - Q_i(t))}, \ i = 1, \ldots, 10, \tag{5.5}$$

where $V_1, \ldots, V_n$ denote the producible volumes from the n reservoirs. Table 5.1 shows the parameter values for the 10 reservoirs. The process capacity constraint $K = 7.5$ kSm^3 per day is used. The max rate for the i-th reservoir is given by $\sqrt{D_i V_i}$ and is obtained by inserting $Q_i(0) = 0$ in (5.5). In this example we use the objective function $\phi_{K,0}$ defined by letting $C = K$ and $R = 0$ in (3.10). By (3.12) it follows that $\phi_{K,0}$, interpreted as a function defined for all $\boldsymbol{Q} \in \mathcal{Q}$ is given by $\phi_{K,0}(\boldsymbol{Q}) = \ell(\boldsymbol{Q}) = \sum_{i=1}^{10} Q_i$. Since the PPR-functions and the extended objective function $\phi_{K,0}(\boldsymbol{Q})$ are differentiable, we may apply Lagrange multipliers in order to

find $\boldsymbol{Q}^*$. Hence, it is easy to show that $\boldsymbol{Q}^*$ is given by

$$\boldsymbol{Q}^* = (Q_1^*, \ldots, Q_n^*) = (V_1 - D_1 \{\frac{K}{\sum_{i=1}^n D_i}\}^2, \ldots, V_n - D_n \{\frac{K}{\sum_{i=1}^n D_i}\}^2).$$

Reservoir	Producible volume V_i (MSm^3)	Scale parameter D_i	Max rate $\sqrt{D_i V_i}$ (kSm^3/d)	Plateau production Q_i^* (MSm^3)
1	4.5	0.001200	3.25	4.204
2	6.5	0.001400	4.20	6.158
3	7.0	0.000600	3.00	6.838
4	10.0	0.000300	2.50	9.921
5	5.0	0.000625	2.50	4.842
6	4.0	0.001100	3.00	3.716
7	6.0	0.001300	4.00	5.664
8	8.0	0.001000	4.00	7.748
9	9.0	0.000500	3.00	8.874
10	5.0	0.002500	5.00	4.370

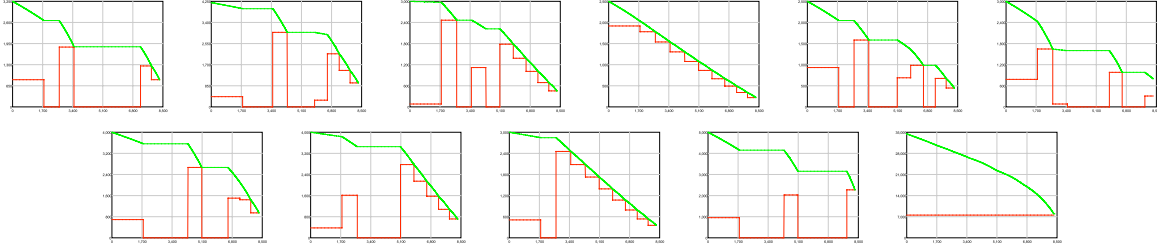Table 5.1: *Parameter values for the 10 reservoirs used in the backtracking example.*

Figure 5.1: *Actual production rates $(q_1(t), \ldots, q_{12}(t))$ (red curves) satisfying the conditions (5.1), (5.3) and (5.4) and PPR-functions $(f_1(Q_1(t)), \ldots, f_{10}(Q_{10}(t)))$ (green curves) for the backtracking example. The last panel shows the total production rate $q(t)$ (red curve) and the sum of the PPR-functions $\sum_{i=1}^{10} f_i(Q_i(t))$ (green curve).*

We then proceed to Step 2 of Algorithm 3.3. To generate a production strategy reaching $\boldsymbol{Q}^*$ we use the approach described in Section 5 where we search for intervals $(t_0, t_1], \ldots, (t_{N-1}, t_N]$ so that the condition expressed in (5.2) is satisfied. For simplicity we distributed the partition points uniformly over the interval $[0, T_K^*]$. Starting out with $N = 1$ and increasing N until an admissible path from $\boldsymbol{0}$ to $\boldsymbol{Q}^*$ was found, it turned out that $N = 12$ periods were needed. Figure 5.1 shows the actual production rates and the PPR-rates of the 10 reservoirs. The total actual production rate and the total PPR-rate are also displayed in Figure 5.1. From Figure 5.1 we see that the conditions (5.1) and (5.2) are satisfied for all $t \geq 0$.

6 Conclusions

In the present paper we have focused on the problem of optimizing the production of an oil or gas field consisting of many reservoirs. We have shown how to construct an optimal production strategy using a procedure described in Algorithm 3.3. The first step of the algorithm involves finding the optimal state of the reservoirs at the end of the plateau phase, i.e, when the path defined by the vector of cumulative productions reaches the set $\partial(\mathcal{M}')$. The second step involves finding an admissible production strategy such that the optimal state is reached.

The key results given in Theorem 3.4 and Theorem 3.5 indicate how to solve the optimization problem given in Step 1 of Algorithm 3.3 in two important cases characterized by the convexity or concavity of the PPR-functions and the quasi-convexity or quasi-concavity of the objective function.

If the optimal state is located at the *boundary* of $\partial(\mathcal{M}')$, the priority strategies play an important part, since these strategies correspond to the extreme points of the boundary of $\partial(\mathcal{M}')$. Searching for an optimal rule within this class is, at least in principle, easy, since there are only a finite number of such strategies. Moreover, having found the best priority strategy, the second step of the algorithm is trivial, since any priority strategy is uniquely defined by the permutation vector representing the ordering of the reservoirs. While there of course are infinitely many other production strategies with cumulative production paths reaching the boundary of $\partial(\mathcal{M}')$, we believe that the priority strategies at least provide a very good approximation to the optimal solution.

In the special case where all the PPR-functions are *linear*, a specific priority strategy is shown to be optimal with respect to any monotone, symmetric objective function.

If the optimal state is located in the *interior* of the set $\partial(\mathcal{M}')$, a backtracking algorithm is proposed for handling Step 2 of Algorithm 3.3. Unless the optimal state is too close to the boundary of $\partial(\mathcal{M}')$ this method produces an admissible production strategy such that the optimal state is reached.

We believe that the general framework developed in this theoretical paper is of fundamental importance in order to gain insight into the general production optimization problem. Still there are many unsolved problems left. In particular, by running Step 1 of Algorithm 3.3 as proposed in the present paper, we only get a *candidate* for the optimal state in the set $\partial(\mathcal{M}')$. Thus, having a precise and easy condition for when this candidate actually belongs to $\partial(\mathcal{M}')$, would be very convenient. Using this we could e.g., avoid running Step 2 of the algorithm in cases where the candidate state does not belong to $\partial(\mathcal{M}')$, in which case we know that no admissible strategy reaching this state can be found. Given such a condition we could also be able to handle combinations of convex and concave PPR-functions.

Furthermore, in order to analyze the robustness of the derived production strategies, it is of interest to incorporate uncertainty into the framework. These issues will addressed in a forth-coming paper, where a certain parametric class of production strategies will be proposed.

In this paper we have focused on single-phase production optimization, i.e., either oil or gas. In real life typically oil, gas and water are produced simultaneously. Thus, extending the framework so that *multi-phase* production optimization can be handled, is of great interest. We will return to this problem in a future research project.

Acknowledgments

A Some results on convexity

A.1 Separating and supporting hyperplanes

Many results in convex optimization theory rest upon the well-known *separating and supporting hyperplane theorems*. For more details see Boyd & Vandenberghe (2004). In the space $\mathbb{R}^n$ a *hyperplane* $H = \{\boldsymbol{x} \in \mathbb{R}^n : \ell(\boldsymbol{x}) = c\}$, where ℓ is a non-zero linear form, divides the space into two closed half-spaces, $H^+ = \{\boldsymbol{x} \in \mathbb{R}^n : \ell(\boldsymbol{x}) \geq c\}$ and $H^- = \{\boldsymbol{x} \in \mathbb{R}^n : \ell(\boldsymbol{x}) \leq c\}$. A hyperplane H is said to *separate* the sets S and T if one of the sets is contained in H^+ while the other is contained in H^-. A hyperplane, H, is said to *support* a set S, if either $S \subseteq H^+$ or $S \subseteq H^-$, and $S \cap H \neq \emptyset$. The separating and supporting hyperplane theorems can now be stated as follows:

Theorem A.1 *Let $S, T \subset \mathbb{R}^n$ be two disjoint convex sets. Then there exists a hyperplane H separating S and T.*

Theorem A.2 *Let $S \subset \mathbb{R}^n$ be a closed convex set, and let $\boldsymbol{x}_0 \in S$ be a point at the boundary of S. Then there exists a hyperplane, H, supporting S such that $\boldsymbol{x}_0 \in H$.*

The following proposition combines Theorem A.1 and Theorem A.2:

Proposition A.3 *Let $S, T \subset \mathbb{R}^n$ be two disjoint convex sets. Moreover, assume that there exists a $\boldsymbol{x}_0 \in S$ such that any neighborhood of $\boldsymbol{x}_0$ intersects T. Then there exists a hyperplane H separating S and T such that H supports S at $\boldsymbol{x}_0$.*

Proof: By Theorem A.1 it follows that there exists a hyperplane $H = \{\boldsymbol{x} \in \mathbb{R}^n : \ell(\boldsymbol{x}) = c\}$ separating S and T. Assume without loss of generality[4] that $S \subseteq H^+$ while $T \subseteq H^-$. To prove that H supports S at $\boldsymbol{x}_0$, it is sufficient to show that $\boldsymbol{x}_0 \in H$, i.e., that $\ell(\boldsymbol{x}_0) = c$. Assume conversely that this is not true, i.e., that $\ell(\boldsymbol{x}_0) > c$. Since the linear form ℓ is continuous, this implies that there exists a neighborhood N of $\boldsymbol{x}_0$ such that $\ell(\boldsymbol{x}) > c$ for all $\boldsymbol{x} \in N$. On the other hand, since $T \subseteq H^-$, we know that $\ell(\boldsymbol{x}) \leq c$ for all $\boldsymbol{x} \in T$. Thus, $N \cap T = \emptyset$. However, this contradicts the assumption that any neighborhood ! of $\boldsymbol{x}_0$ intersects T. Thus, we conclude that $\boldsymbol{x}_0 \in H$ ∎

Using the various sets and notation introduced in Section 3, we can now formulate the following important result:

Theorem A.4 *Consider a field with n reservoirs with convex PPR-functions $f_1, \ldots, f_n$. Moreover, let $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$. Then there exists m (m suitably chosen) vectors $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m \in \partial(\partial(\mathcal{M}'))$ and non-negative numbers $\alpha_1, \ldots, \alpha_m$ such that $\sum_{i=1}^m \alpha_i \leq 1$ and such that:*

$$\boldsymbol{Q}^* = \sum_{i=1}^m \alpha_i \boldsymbol{Q}_i. \tag{A.1}$$

Proof: Let $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$. We start out by noting that the result is trivial if $\boldsymbol{Q}^* \in \partial(\partial(\mathcal{M}'))$. Thus, in the remaining part of the proof we assume that $\boldsymbol{Q}^*$ lies in the interior of $\partial(\mathcal{M}')$, denoted $\partial(\mathcal{M}')^o$.

By Corollary 3.2 we know that $\bar{\mathcal{M}} \cup \partial(\mathcal{M})$ is convex. Hence, by Theorem A.2 there exists a supporting hyperplane $H = \{\boldsymbol{Q} \in \mathbb{R}^n : \ell(\boldsymbol{Q}) = c\}$ such that $\bar{\mathcal{M}} \cup \partial(\mathcal{M}) \subseteq H^+$ and $\boldsymbol{Q}^* \in H$.

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[4]If not, we simply replace H by the equivalent hyperplane representation $H^* = \{\boldsymbol{x} \in \mathbb{R}^n : -\ell(\boldsymbol{x}) = -c\}$, where $H^{*+} = H^-$ and $H^{*-} = H^+$.

In particular, $\partial(\mathcal{M}') \subseteq H^+$, i.e., $\ell(\mathbf{Q}) \geq c$ for all $\mathbf{Q} \in \partial(\mathcal{M}')$. Since we have assumed that $\sum_{i=1}^n f_i(0) > K > 0$, we know that $\mathbf{0} \notin \bar{\mathcal{M}} \cup \partial(\mathcal{M})$. From this it is easy to see that H may be chosen so that $c > 0$.

We then introduce a mapping $\lambda : \partial(\mathcal{M}') \to H$ defined as follows:

$$\lambda(\mathbf{Q}) = c[\ell(\mathbf{Q})]^{-1}\mathbf{Q}, \text{ for all } \mathbf{Q} \in \partial(\mathcal{M}'). \tag{A.2}$$

We observe that since ℓ is linear, we have:

$$\ell(\lambda(\mathbf{Q})) = c[\ell(\mathbf{Q})]^{-1}\ell(\mathbf{Q}) = c,$$

implying that we indeed have $\lambda(\mathbf{Q}) \in H$ for all $\mathbf{Q} \in \partial(\mathcal{M}')$. Note also that for all $\mathbf{Q} \in \partial(\mathcal{M}')$ the scaling factor $c[\ell(\mathbf{Q})]^{-1}$ is always positive (since $c > 0$), and less than or equal to 1 (since $\ell(\mathbf{Q}) \geq c$). In particular, since $\mathbf{Q}^* \in H$, i.e., $\ell(\mathbf{Q}^*) = c$, the scaling factor $c[\ell(\mathbf{Q}^*)]^{-1}$ is 1. Hence, $\lambda(\mathbf{Q}^*) = \mathbf{Q}^*$.

We denote the image of the mapping λ by $\lambda[\partial(\mathcal{M}')]$. We also introduce the boundary and interior of $\lambda[\partial(\mathcal{M}')]$, denoted respectively $\partial(\lambda[\partial(\mathcal{M}')])$ and $\lambda[\partial(\mathcal{M}')]^o$.

Due to the monotonicity of the PPR-functions it is easy to see that λ is a *homeomorphism*. Thus, since $\partial(\mathcal{M}')$ is assumed to be an $(n-1)$-manifold with boundary, then so is the image $\lambda[\partial(\mathcal{M}')]$. Moreover, λ maps the points in $\partial(\partial(\mathcal{M}'))$, over to the set $\partial(\lambda[\partial(\mathcal{M}')])$, and the points in $\partial(\mathcal{M}')^o$, over to the set $\lambda[\partial(\mathcal{M}')]^o$. In particular, since we have assumed that $\mathbf{Q}^* \in \partial(\mathcal{M}')^o$, it follows that $\lambda(\mathbf{Q}^*) = \mathbf{Q}^* \in \lambda[\partial(\mathcal{M}')]^o$.

Since $\lambda[\partial(\mathcal{M}')]$ is an $(n-1)$-manifold with boundary embedded in an $(n-1)$-dimensional hyperplane, this set satisfies the following:

Lemma A.5 *Let $\mathbf{Q}_1, \mathbf{Q}_2 \in H$ be such that $\mathbf{Q}_1 \notin \lambda[\partial(\mathcal{M}')]^o$ while $\mathbf{Q}_2 \in \lambda[\partial(\mathcal{M}')]^o$. Then there exists $\beta \in (0,1]$ such that:*

$$\beta\mathbf{Q}_1 + (1-\beta)\mathbf{Q}_2 \in \partial(\lambda[\partial(\mathcal{M}')]).$$

Furthermore, since $\lambda[\partial(\mathcal{M}')]$ is obviously a bounded set, there exists a convex polytope P, i.e., a convex hull of a finite set of points, such that:

$$\lambda[\partial(\mathcal{M}')] \subseteq P \subset H. \tag{A.3}$$

Let $\mathbf{Q}_1^P, \ldots, \mathbf{Q}_m^P$ be the extreme points of P. Since $\lambda[\partial(\mathcal{M}')]$ is contained in P, any point in $\lambda[\partial(\mathcal{M}')]$ can be written as a convex combination of these extreme points. In particular, since $\mathbf{Q}^* \in \lambda[\partial(\mathcal{M}')]$, there exists non-negative numbers, $\alpha_1^P, \ldots, \alpha_m^P$ such that $\sum_{i=1}^m \alpha_i^P = 1$, and such that:

$$\mathbf{Q}^* = \sum_{i=1}^m \alpha_i^P \mathbf{Q}_i^P. \tag{A.4}$$

By Lemma A.5 we then know that there exists $\beta_1, \ldots, \beta_m \in (0,1]$ such that

$$\mathbf{Q}_i^\partial = \beta_i \mathbf{Q}_i^P + (1-\beta_i)\mathbf{Q}^* \in \partial(\lambda[\partial(\mathcal{M}')]), \qquad i = 1, \ldots, m. \tag{A.5}$$

By solving (A.5) with respect to $\mathbf{Q}_1^P, \ldots, \mathbf{Q}_m^P$ we get:

$$\mathbf{Q}_i^P = \beta_i^{-1}[\mathbf{Q}_i^\partial - (1-\beta_i)\mathbf{Q}^*], \qquad i = 1, \ldots, m. \tag{A.6}$$

Inserting the expressions for $\boldsymbol{Q}_1^P, \ldots, \boldsymbol{Q}_m^P$ given in (A.6) into (A.4), and solving with respect to $\boldsymbol{Q}^*$, yields the following representation:

$$\boldsymbol{Q}^* = \sum_{i=1}^m \alpha_i^\partial \boldsymbol{Q}_i^\partial, \tag{A.7}$$

where the weights $\alpha_1^\partial, \ldots, \alpha_m^\partial$ are given by:

$$\alpha_i^\partial = \frac{\alpha_i^P \beta_i^{-1}}{\sum_{j=1}^m \alpha_j^P \beta_j^{-1}}, \qquad i = 1, \ldots, m. \tag{A.8}$$

Using that $\sum_{i=1}^m \alpha_i^P = 1$, it is easy to show that $\sum_{i=1}^m \alpha_i^\partial = 1$ as well. Hence, (A.7) expresses $\boldsymbol{Q}^*$ as a convex combination of $\boldsymbol{Q}_1^\partial, \ldots, \boldsymbol{Q}_m^\partial$.

Since $\boldsymbol{Q}_1^\partial, \ldots, \boldsymbol{Q}_m^\partial \in \partial(\lambda[\partial(\mathcal{M}')])$, these points are mapped from points in $\partial(\partial(\mathcal{M}'))$. That is, there exists $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m \in \partial(\partial(\mathcal{M}'))$ such that:

$$\boldsymbol{Q}_i^\partial = \lambda(\boldsymbol{Q}_i) = c[\ell(\boldsymbol{Q}_i)]^{-1} \boldsymbol{Q}_i, \qquad i = 1, \ldots, m. \tag{A.9}$$

Inserting the expressions for $\boldsymbol{Q}_1^\partial, \ldots, \boldsymbol{Q}_m^\partial$ given in (A.9) into (A.7), we get the following representation:

$$\boldsymbol{Q}^* = \sum_{i=1}^m \alpha_i \boldsymbol{Q}_i, \tag{A.10}$$

where $\alpha_1, \ldots, \alpha_m$ are given by:

$$\alpha_i = c[\ell(\boldsymbol{Q}_i)]^{-1} \alpha_i^\partial, \qquad i = 1, \ldots, m. \tag{A.11}$$

Finally, since $0 < c[\ell(\boldsymbol{Q}_i)]^{-1} \le 1$, $i = 1, \ldots, m$, we get that $\alpha_i, \ldots, \alpha_m$ are non-negative and that $\sum_{i=1}^m \alpha_i \le \sum_{i=1}^m \alpha_i^\partial = 1$, and this completes the proof of theorem ∎

Note that in the above argument we embed the set $\lambda[\partial(\mathcal{M}')]$ in a convex polytope P with m extreme points, where m is a suitably chosen integer. If $n = 2$, the set $\lambda[\partial(\mathcal{M}')]$ can be embedded within an interval, i.e., a polytope with two extreme points. Similarly, if $n = 3$, the set can be embedded withing a triangle, which is a polytope with three extreme points. In general the set $\lambda[\partial(\mathcal{M}')]$ can always be embedded within an n-dimensional simplex which is a polytope with n extreme points. Thus, we may always choose the polytope P such that $m = n$.

A.2 Quasi-convex functions

In Section 3 we needed the concept of *quasi-convexity*. This is defined as follows (see Boyd & Vandenberghe (2004)):

Definition A.6 *Let $S \subseteq \mathbb{R}^n$ be a convex set. We say that a function $g : S \to \mathbb{R}$ is* quasi-convex *if for any pair of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in S$ and $\lambda \in [0, 1]$ we have:*

$$g(\lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2) \le \max\{g(\boldsymbol{x}_1), g(\boldsymbol{x}_2)\}.$$

Furthermore, the function g is said to be quasi-concave *if $-g$ is quasi-convex. Finally, if g is both quasi-convex and quasi-concave, we say that g is* quasi-linear.

More generally quasi-convexity implies the following:

Proposition A.7 *Let $S \subseteq \mathbb{R}^n$ be a convex set, and let $g : S \to \mathbb{R}$ be a quasi-convex function. Moreover, let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in S$, and let $\lambda_1, \ldots, \lambda_n \in [0,1]$, be such that $\sum_{i=1}^n \lambda_i = 1$. Then:*

$$g(\sum_{i=1}^n \lambda_i \boldsymbol{x}_i) \leq \max\{g(\boldsymbol{x}_1), \ldots, g(\boldsymbol{x}_n)\}. \tag{A.12}$$

Proof: The proof is by induction on n. The result is trivial for $n = 1$, and holds by Definition A.6 for $n = 2$. We then assume, as an induction hypothesis, that the result is proved for $n - 1$ or fewer vectors, and consider the case with n vectors. If $\lambda_n = 1$, the result is again trivial, so we assume that $\lambda_n < 1$, and introduce:

$$\lambda'_i = \lambda_i [\sum_{j=1}^{n-1} \lambda_j]^{-1}, \qquad i = 1, \ldots, (n-1).$$

Since obviously $\sum_{i=1}^{n-1} \lambda'_i = 1$, it follows by the induction hypothesis that:

$$g(\sum_{i=1}^{n-1} \lambda'_i \boldsymbol{x}_i) \leq \max\{g(\boldsymbol{x}_1), \ldots, g(\boldsymbol{x}_{n-1})\}.$$

Hence, we get:

$$\begin{aligned}
g(\sum_{i=1}^n \lambda_i \boldsymbol{x}_i) &= g([\sum_{j=1}^{n-1} \lambda_j][\sum_{i=1}^{n-1} \lambda'_i \boldsymbol{x}_i] + \lambda_n \boldsymbol{x}_n) \\
&\leq \max\{g(\sum_{i=1}^{n-1} \lambda'_i \boldsymbol{x}_i), g(\boldsymbol{x}_n)\} \\
&\leq \max\{\max\{g(\boldsymbol{x}_1), \ldots, g(\boldsymbol{x}_{n-1})\}, g(\boldsymbol{x}_n)\} \\
&= \max\{g(\boldsymbol{x}_1), \ldots, g(\boldsymbol{x}_n)\}.
\end{aligned}$$

Hence, by induction the result is proved ■

The following result provides alternative definitions of quasi-convexity and quasi-concavity (see Boyd & Vandenberghe (2004)):

Proposition A.8 *Let $S \subseteq \mathbb{R}^n$ be a convex set, and let $g : S \to \mathbb{R}$. Then g is quasi-convex if and only if the sets $L_y = \{\boldsymbol{x} \in S : g(\boldsymbol{x}) \leq y\}$ are convex for all y. Similarly, g is quasi-concave if and only if the sets $U_y = \{\boldsymbol{x} \in S : g(\boldsymbol{x}) \geq y\}$ are convex for all y. Finally, g is quasi-linear if and only if L_y and U_y are convex for all y.*

Note that for some y L_y or U_y may be empty. In this setting, however, $\emptyset$ is defined to be convex, so in order to verify quasi-convexity or quasi-concavity, only non-empty sets need to be considered.

Using Proposition A.8 we can also prove the following characterizations:

Proposition A.9 *Let $S \subseteq \mathbb{R}^n$ be a convex set, and let $g : S \to \mathbb{R}$. Then g is quasi-convex if and only if the sets $L_y^o = \{\boldsymbol{x} \in S : g(\boldsymbol{x}) < y\}$ are convex for all y. Similarly, g is quasi-concave if and only if the sets $U_y^o = \{\boldsymbol{x} \in S : g(\boldsymbol{x}) > y\}$ are convex for all y. Finally, g is quasi-linear if and only if L_y^o and U_y^o are convex for all y.*

Proof: Assume first that g is quasi-convex, and consider the set $L_y^o = \{\boldsymbol{x} \in S : g(\boldsymbol{x}) < y\}$. Moreover, let $\boldsymbol{x}^1, \boldsymbol{x}^2 \in L_y^o$, so $g(\boldsymbol{x}^i) < y$, $i = 1, 2$. We then introduce $y' = \max\{g(\boldsymbol{x}^1), g(\boldsymbol{x}^2)\}$, and consider the set $L_{y'}$. Since $y' < y$, it follows that $L_{y'} \subseteq L_y^o$. Moreover, using Proposition A.8 we know that since g is quasi-convex, $L_{y'}$ must be convex. Hence, $\alpha \boldsymbol{x}^1 + (1 - \alpha)\boldsymbol{x}^2 \in L_{y'}$ for all $\alpha \in [0, 1]$. However, since $L_{y'} \subseteq L_y^o$, it follows that $\alpha \boldsymbol{x}^1 + (1 - \alpha)\boldsymbol{x}^2 \in L_y^o$ for all $\alpha \in [0, 1]$ as well. Thus, we conclude that L_y^o is convex.

Assume conversely that the sets $L_y^o = \{\boldsymbol{x} \in S : g(\boldsymbol{x}) < y\}$ are convex for all y, and consider the set $L_y = \{\boldsymbol{x} \in S : g(\boldsymbol{x}) \leq y\}$. We then let $y_n = y + 1/n$, $n = 1, 2, \ldots$. By the assumption the sets $L_{y_1}^o, L_{y_2}^o, \ldots$ are all convex. Moreover,

$$L_y = \bigcap_{n=1}^{\infty} L_{y_n}^o.$$

Hence, since convexity is preserved under intersection (see Boyd & Vandenberghe (2004)), it follows that L_y is convex. Since this holds for all L_y, by Proposition A.8 we get that g is quasi-convex ∎

By combining the above results, we see that a function $g : S \to \mathbb{R}$ is quasi-linear if and only if L_y and its complement are both convex for all y. It is easy to see that this implies that for all y, $\partial(L_y) = S \cap H_y$ where H_y is a hyperplane. The following result provides a sufficient condition for quasi-linearity.

Proposition A.10 *Let $S \subseteq \mathbb{R}^n$ be a convex set, and let $g : S \to \mathbb{R}$. Moreover assume that there exists a non-zero linear form ℓ such that $g(\boldsymbol{x}) = h(\ell(\boldsymbol{x}))$ for all $\boldsymbol{x} \in S$, where $h : \mathbb{R} \to \mathbb{R}$ is either non-decreasing or non-increasing. Then g is quasi-linear.*

Proof: We assume that h is non-decreasing. The proof for the case where h is non-increasing, is completely analogous. We then consider the set $L_y = \{\boldsymbol{x} \in S : h(\ell(\boldsymbol{x})) \leq y\}$ assuming that this is non-empty, and define $z = \sup\{u : h(u) \leq y\}$. Since h is non-decreasing, it follows that $h(u) \leq y$ for all $u < z$, and that $h(u) > y$ for all $u > z$. If $h(z) \leq y$, it follows that $h(u) \leq y$ if and only if $u \leq z$. Hence, in this case $L_y = \{\boldsymbol{x} \in S : \ell(\boldsymbol{x}) \leq z\}$. If $h(z) > y$, we have that $h(u) \leq y$ if and only if $u < z$. Hence, in this case $L_y = \{\boldsymbol{x} \in S : \ell(\boldsymbol{x}) < z\}$. In both cases L_y is convex since ℓ is linear. Using a parallel argument one can show that U_y is convex as well, and thus we conclude that g is quasi-linear ∎

B   A result on dominating sums

Lemma B.1 *Assume that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ are such that:*

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i, \qquad k = 1, \ldots, n. \tag{B.1}$$

Then for any $\boldsymbol{a} \in \mathbb{R}^n$ such that:

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq 0, \tag{B.2}$$

we also have:

$$\sum_{i=1}^{k} x_i a_i \geq \sum_{i=1}^{k} y_i a_i, \qquad k = 1, \ldots, n.$$

Proof: The proof is by induction on k. By B.1 it follows that $x_1 \geq y_1$. Thus, since all the a_is are assumed to be nonnegative, it follows that:

$$x_1 a_1 \geq y_1 a_1.$$

Hence, the result obviously holds for $k = 1$. We then assume that the result is proved for $k \leq m$, and consider the case where $k = m + 1$. We then introduce $b_i = a_i - a_{m+1}$, $i = 1, \ldots, m$. By B.2 it follows that:

$$b_1 \geq b_2 \geq \ldots \geq b_m \geq 0.$$

Hence, by the induction hypothesis we have that:

$$\sum_{i=1}^{m} x_i b_i \geq \sum_{i=1}^{m} y_i b_i.$$

By combining this and B.1 we get that:

$$
\begin{aligned}
\sum_{i=1}^{m+1} x_i a_i &= \sum_{i=1}^{m} x_i(a_i - a_{m+1}) + a_{m+1} \sum_{i=1}^{m+1} x_i \\
&= \sum_{i=1}^{m} x_i b_i + a_{m+1} \sum_{i=1}^{m+1} x_i \\
&\geq \sum_{i=1}^{m} y_i b_i + a_{m+1} \sum_{i=1}^{m+1} y_i \\
&= \sum_{i=1}^{m} y_i(a_i - a_{m+1}) + a_{m+1} \sum_{i=1}^{m+1} y_i \\
&= \sum_{i=1}^{m+1} y_i a_i.
\end{aligned}
$$

Thus, the result holds for $k = m + 1$ as well, and hence for $k = 1, \ldots, n$ by induction.

C   Sequences of partitions

In this section we consider the problem of constructing a sequence of increasingly finer partitions of a given finite interval $[A, B]$ on the real line. We assume that the nth partition in the sequence partitions $[A, B]$ into $n + 1$ intervals, $[t_{0,n}, t_{1,n}], (t_{1,n}, t_{2,n}], \ldots, (t_{n,n}, t_{n+1,n}]$, where $A = t_{0,n} < \cdots < t_{n,n} < t_{n+1,n} = B$. As the number of intervals increases, we want to ensure that all the intervals become shorter and shorter. That is, we want to choose $t_{1,n}, \ldots, t_{n,n}$, $n = 1, 2, \ldots$ such that for any i we have:

$$\lim_{n \to \infty} (t_{i,n} - t_{i-1,n}) = 0.$$

The easiest way to accomplish this is of course to distribute the partition points, $t_{1,n}, \ldots, t_{n,n}$, uniformly over the interval $[A, B]$. That is, we partition $[A, B]$ into intervals of equal lengths, so that:

$$t_{i,n} = i/n, \qquad i = 0, 1, \ldots, (n + 1), \ n = 1, 2, \ldots,$$

from which it follows that for any i

$$\lim_{n\to\infty} (t_{i,n} - t_{i-1,n}) = \lim_{n\to\infty} 1/n = 0,$$

as claimed.

However, what if we want the lengths of the intervals closer to A to converge faster to zero than those closer to B? One possible approach to this is as follows. Let f be a function defined for all non-negative real numbers satisfying the following conditions:

(i) $f(0) = 0$.

(ii) f is strictly increasing.

(iii) f is convex.

(iv) $\lim_{n\to\infty} \frac{f(n)}{f(n+1)} = 1$.

We then define the partition points as:

$$t_{i,n} = A + (B - A)\frac{f(i)}{f(n + 1)}, \qquad i = 0, 1, \ldots, n, (n + 1), \;\; n = 1, 2, \ldots$$

We observe that by (i) and (ii) it follows that $A = t_{0,n} < \cdots < t_{n,n} < t_{n+1,n} = B$. Furthermore, since by (iii) f is assumed to be convex, it follows that:

$$f(i) \leq \frac{f(i + 1) + f(i - 1)}{2}, \qquad i = 1, \ldots n.$$

Hence, by multiplying by 2 and subtracting $f(i) + f(i - 1)$ on both sides, we obtain:

$$f(i) - f(i - 1) \leq f(i + 1) - f(i), \qquad i = 1, \ldots n.$$

Using this it follows that for $i = 1, \ldots, n$ we have:

$$t_{i,n} - t_{i-1,n} = (B - A)\frac{f(i) - f(i - 1)}{f(n + 1)} \leq (B - A)\frac{f(i + 1) - f(i)}{f(n + 1)} = t_{i+1,n} - t_{i,n}.$$

This implies that the intervals closer to A are shorter than those close to B. In particular,

$$0 \leq t_{i,n} - t_{i-1,n} \leq t_{n+1,n} - t_{n,n} = (B - A)\frac{f(n + 1) - f(n)}{f(n + 1)} = (B - A)(1 - \frac{f(n)}{f(n + 1)})$$

Hence, by (iv), we get:

$$0 \leq \lim_{n\to\infty} (t_{i,n} - t_{i-1,n}) \leq \lim_{n\to\infty} (B - A)(1 - \frac{f(n)}{f(n + 1)}) = 0,$$

for $i = 1, \ldots n$.

We observe that if we let f be *linear*, i.e., $f(x) = ax$ where $a > 0$, we get a *uniform* distribution of the partition points. A more satisfactory choice of f satisfying $(i) - (iv)$, is $f(x) = x^2$. It is very easy to verify that this function actually satisfies the requirements.

A similar method can be used if we want the lengths of the intervals closer to B to converge faster to zero than those closer to A. In this case we may also use a function f satisfying the requirements $(i) - (iv)$. However, now we define the partition points as:

$$t_{i,n} = B + (A - B)\frac{f(n + 1 - i)}{f(n + 1)}, \qquad i = 0, 1, \ldots, n, (n + 1), \;\; n = 1, 2, \ldots$$

Then using a parallel argument as above it is easy to verify that the resulting partitions behave the way we want them to.

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