Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

Buffered environmental contours

Kristina Rognlien Dahl and Arne Bang Huseby

Department of Mathematics, University of Oslo

The main idea

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

Introduce a new concept for verification of the safety of mechanical structures: Buffered environmental contours.

Generalization of environmental contours which provides more information about tail behavior and the level of failure.

Why? Enables safer and more controlled classification of mechanical structures.

The performance function

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

Define a performance function

$$g(\mathrm{x}, \mathrm{V})$$

depending on some design variables $\mathrm{x} = (x_1, x_2, \ldots, x_m)'$ and some environmental quantities $\mathrm{V} = (V_1, V_2, \ldots, V_n)' \in \mathcal{V}$, where $\mathcal{V} \subseteq \mathbb{R}^n$.

$g(\mathrm{x}, \mathrm{V})$ is called the state of the structure.

If $g(\mathrm{x}, \mathrm{V}) > 0$, the structure is failed, while if $g(\mathrm{x}, \mathrm{V}) \leq 0$, the structure is functioning.

The failure region, the failure probability and the reliability

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

For a given x the set $\mathcal{F}(x) = \{v \in \mathcal{V} : g(x, v) > 0\}$ is called the failure region of the structure.

The failure probability, $p_f(x)$, is the probability that the structure is failed:

$$p_f(x) := P(g(x, V) > 0).$$

The reliability, $R(x)$, of the system is the probability that the system is functioning:

$$R(x) := 1 - p_f(x)$$

The buffered failure probability

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

Recall: The α-quantile, $q_\alpha(\mathrm{x})$, of the distribution of the random variable $g(\mathrm{x}, \mathrm{V})$ is the value of the inverse of its cdf at α.

The α-superquantile of $g(\mathrm{x}, \mathrm{V})$, $\bar{q}_\alpha(\mathrm{x})$ is

$$\bar{q}_\alpha(\mathrm{x}) = E[g(\mathrm{x}, \mathrm{V}) | g(\mathrm{x}, \mathrm{V}) > q_\alpha(\mathrm{x})].$$

Let F denote the cdf of $g(\mathrm{x}, \mathrm{V})$. Then, Rockafellar and Royset [4] define the buffered failure probability, $\bar{p}_f(\mathrm{x})$, as

$$\bar{p}_f(\mathrm{x}) = P(g(\mathrm{x}, \mathrm{V}) > q_\alpha(\mathrm{x})) = 1 - F(q_\alpha(\mathrm{x})),$$

where α is chosen so that $\bar{q}_\alpha(\mathrm{x}) = 0$.

Example: Calculating the buffered failure probability for Gaussian with mean −2.5, standard deviation 1.5

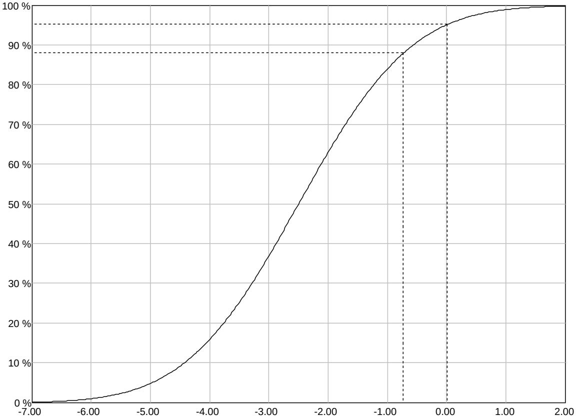

Figure: Buffered failure probability calculation where: $p_f(x) = 0.048$, $q_\alpha(x) = -0.743$, $\alpha = F(q_\alpha(x)) = 0.879$, and $\bar{p}_f(x) = 1 - \alpha = 0.121$.

Properties of the buffered failure probability

For an α corresponding to a buffered failure probability, one can show by contradiction that $q_\alpha(\mathrm{x}) \leq 0$.

It follows that $\alpha = F(q_\alpha(\mathrm{x})) \leq F(0)$. Thus,

$$\bar{p}_f(\mathrm{x}) = 1 - \alpha \geq 1 - F(0) = p_f(\mathrm{x}).$$

Hence, the buffered failure probability is more conservative than the failure probability.

Rockafellar and Royset [4] list several advantages of the buffered failure probability compared to the failure probability:

- Computational efficiency
- Better suited for design optimization algorithms
- Contains more information about the tail behavior of the distribution of $g(\mathrm{x}, \mathrm{V})$.

The purpose of environmental contours

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

Used in early design stage (when $\mathcal{F}$ is unknown) to determine which designs are safe.

At this early stage it is often not possible to express a precise functional relationship between x and the performance of the structure: Skip x, so $g(\mathsf{V})$, $\mathcal{F}$, $p_f(\mathcal{F}) = P(\mathsf{V} \in \mathcal{F})$.

$\mathcal{F}$ is unknown, but can argue that $\mathcal{F}$ belongs to a family, $\mathcal{E}$, of failure regions.

Goal: Make sure that $P(\mathsf{V} \in \mathcal{F})$ is acceptable for all $\mathcal{F} \in \mathcal{E}$.

The purpose of environmental contours

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

How? Introduce $\mathcal{B} \subseteq \mathbb{R}^n$ chosen such that for any failure region $\mathcal{F}$ which does not overlap with $\mathcal{B}$, the failure probability $P(\mathrm{V} \in \mathcal{F})$ is small.

Also,

$$\mathcal{F} \cap \mathcal{B} \subseteq \partial\mathcal{B} \text{ for all } \mathcal{F} \in \mathcal{E},$$

where $\partial\mathcal{B}$ is the boundary of $\mathcal{B}$.

This boundary is then referred to as an environmental contour.

Environmental contours have been studied by Winterstein et al. [6], Baarholm et al. [1] and Fontaine et al. [2].

Example: An environmental contour

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

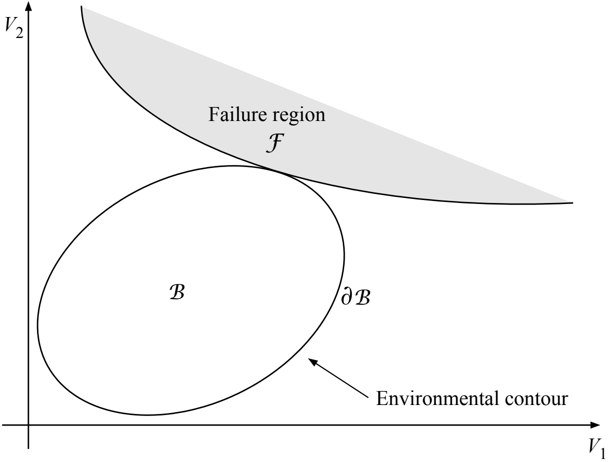

Figure: An environmental contour $\partial\mathcal{B}$ and a failure region $\mathcal{F}$.

The exceedence probability

Define the exceedence probability of $\mathcal{B}$ wrt. $\mathcal{E}$ as:

$$P_e(\mathcal{B}, \mathcal{E}) := \sup\{p_f(\mathcal{F}) : \mathcal{F} \in \mathcal{E}\}.$$

For a given target probability P_e the objective is to choose an environmental contour $\partial\mathcal{B}$ such that:

$$P_e(\mathcal{B}, \mathcal{E}) = P_e.$$

Construction of contours via Monte Carlo

We consider a method for constructing environmental contours via Monte Carlo introduced in Huseby et al. [3].

Let $\mathcal{U}$ be the set of all unit vectors in $\mathbb{R}^n$, and let $\mathrm{u} \in \mathcal{U}$.

Introduce a function $C(\mathrm{u})$:

$$C(\mathrm{u}) := \inf\{C : P(\mathrm{u}'\mathrm{V} > C) \leq P_e\}$$

So, $C(\mathrm{u})$ is the $(1 - P_e)$-quantile of the distribution of $\mathrm{u}'\mathrm{V}$.

Monte Carlo estimation of $C(\mathrm{u})$

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

$C(\mathrm{u})$ can be estimated by using Monte Carlo simulation:

- Let $\mathrm{V}_1, \dots, \mathrm{V}_N$ be a random sample from the distribution of V.
- Choose $\mathrm{u} \in \mathcal{U}$, and let $Y_r(\mathrm{u}) = \mathrm{u}'\mathrm{V}_r$, $r = 1, \dots, N$.
- Sort these in ascending order: $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(N)}$.
- Since $C(\mathrm{u})$ is the $(1 - P_e)$-quantile in the distribution, a natural estimator is:

$$\hat{C}(\mathrm{u}) = Y_{(k)}, \text{ where } k \text{ is chosen s.t. } \frac{k}{N} \approx 1 - P_e.$$

Definition of the environmental contour

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

For each $u \in \mathcal{U}$, introduce the halfspaces:

$$\Pi^-(u) = \{v : u'v \leq C(u)\},$$
$$\Pi^+(u) = \{v : u'v > C(u)\}.$$

Define the environmental contour as the boundary $\partial\mathcal{B}$ of the convex set $\mathcal{B}$ given by:

$$\mathcal{B} := \bigcap_{u \in \mathcal{U}} \Pi^-(u)$$

Can show that the exceedence probability of $\mathcal{B}$ with respect to $\mathcal{E}$ is given by:

$$P_e(\mathcal{B}, \mathcal{E}) = P_e,$$

The purpose of buffered environmental contours

Introduce a new concept called buffered environmental contours.

Combines the ideas behind buffered failure probabilities and environmental contours.

For a given performance function g its failure probability, p_f, can be computed based on the failure region of g alone.

In contrast, computing the buffered failure probability, $\bar{p}_f$, requires more detailed information about the distribution of g.

Indicate this by expressing $\bar{p}_f$ as a function of g, denoted $\bar{p}_f(g)$.

Construction of buffered contours via Monte Carlo

Main idea

Structural design

Buffered failure probability

Environmental contours

Buffered environmental contours

Numerical example: Waves

Let P_e be a given target probability. To introduce buffering, we let:

$$\bar{C}(\mathrm{u}) := \mathrm{E}[\mathrm{u}'\mathrm{V} | \mathrm{u}'\mathrm{V} > C(\mathrm{u})].$$

$\bar{C}(\mathrm{u})$ can be estimated by using Monte Carlo simulation:

- Let $\mathrm{V}_1, \ldots, \mathrm{V}_N$ be a random sample from the distribution of V.
- Choose $\mathrm{u} \in \mathcal{U}$ and let $Y_r(\mathrm{u}) = \mathrm{u}'\mathrm{V}_r$, $r = 1, \ldots, N$.
- Sort these in ascending order: $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(N)}$.
- As before, estimate $C(\mathrm{u})$ by

$$\hat{C}(\mathrm{u}) = Y_{(k)}, \text{ where } k \text{ is chosen s.t. } \frac{k}{N} \approx 1 - P_e.$$

Construction of buffered contours via Monte Carlo

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

- Estimate $\bar{C}(\mathrm{u})$ by computing the average value of the sampled values which are greater than $Y_{(k)}$:

$$\hat{\bar{C}}(\mathrm{u}) = \frac{1}{N-k}\sum_{r>k} Y_{(r)}.$$

For each $\mathrm{u} \in \mathcal{U}$, introduce the halfspaces:

$$\bar{\Pi}^{-}(\mathrm{u}) = \{\mathrm{v} : \mathrm{u}'\mathrm{v} \leq \bar{C}(\mathrm{u})\},$$
$$\bar{\Pi}^{+}(\mathrm{u}) = \{\mathrm{v} : \mathrm{u}'\mathrm{v} > \bar{C}(\mathrm{u})\},$$

Finally, define the buffered environmental contour as the boundary $\partial\bar{\mathcal{B}}$ of the convex set $\bar{\mathcal{B}}$ given by:

$$\bar{\mathcal{B}} := \bigcap_{\mathrm{u}\in\mathcal{U}} \bar{\Pi}^{-}(\mathrm{u})$$

The buffered environmental contour is more conservative

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

It follows from the previous definitions that $\mathcal{B} \subset \bar{\mathcal{B}}$.

Thus, given that the same target probability P_e is used to construct both contours, the buffered environmental contour is more conservative than the classical environmental contour.

Next step: Identify a family $\mathcal{G}$ of performance functions defined relative to the set $\bar{\mathcal{B}}$ such that $\bar{p}_f(g) \leq P_e$ for all $g \in \mathcal{G}$.

A set of safe performance functions

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

Need more control over the distributions of the performance functions. Therefore, introduce a performance function $\Gamma(\mathrm{u}, \cdot)$ given by:

$$\Gamma(\mathrm{u}, \mathrm{V}) = \mathrm{u}'\mathrm{V} - \bar{C}(\mathrm{u})$$

Can prove that $\bar{p}_f(\Gamma(\mathrm{u}, \cdot)) = P_e$ for all $\mathrm{u} \in \mathcal{U}$, so the performance function $\Gamma(\mathrm{u}, \cdot)$ has the desired buffered failure probability P_e for all u.

Let $\mathcal{G}$ be the family of all performance functions g for which there exists a $\mathrm{u} \in \mathcal{U}$ such that $g(\mathrm{v}) \leq \Gamma(\mathrm{u}, \mathrm{v})$ for all $\mathrm{v} \in \mathcal{V}$ (so the $\Gamma(\mathrm{u}, \cdot)$-functions are maximal elements in this family).

Theorem

For all $g \in \mathcal{G}$ we have $\bar{p}_f(g) \leq P_e$.

The buffered exceedence probability

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

We introduce the buffered exceedence probability of $\bar{\mathcal{B}}$ with respect to $\mathcal{G}$ defined as:

$$\bar{P}_e(\bar{\mathcal{B}}, \mathcal{G}) := \sup\{\bar{p}_f(g) : g \in \mathcal{G}\}.$$

By the definition of $\mathcal{G}$ it follows that $\Gamma(\mathrm{u}, \cdot) \in \mathcal{G}$ for all $\mathrm{u} \in \mathcal{U}$. Hence:

$$\begin{aligned}\bar{P}_e(\bar{\mathcal{B}}, \mathcal{G}) &= \sup\{\bar{p}_f(g) : g \in \mathcal{G}\} \\ &= \sup\{\bar{p}_f(\Gamma(\mathrm{u}, \cdot)) : \mathrm{u} \in \mathcal{U}\} = P_e,\end{aligned}$$

Thus, we conclude that the contour $\partial\bar{\mathcal{B}}$ has the correct buffered exceedence probability with respect to $\mathcal{G}$.

Waves in North West Australia

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

We illustrate the proposed method with a numerical example introduced in Vanem and Bitner-Gregersen [5]:

Consider joint long-term models for significant wave height, H, and wave period, T.

The joint model (conditioned on the value of significant wave height):

$$f_{T,H}(t,h) = f_H(h) f_{T|H}(t|h)$$

Simultaneous distributions have been fitted to data assuming:

- A three-parameter Weibull distribution for the significant wave height, H,
- A lognormal conditional distribution for the wave period, T.

Two different types of waves: Swell and wind sea

The parameters are fitted based on a data set from North West Australia.

Consider data for two different cases: swell and wind sea.

(a) Swell

(b) Wind sea

Environmental contours for swell and wind sea

We use a return period of 25 years. The models are fitted using sea states representing periods of 1 hour $\implies$ The desired exceedence probability is:

$$P_e = \frac{1}{25 \cdot 365.25 \cdot 24} = 4.5631 \cdot 10^{-6}.$$

The environmental contours are estimated via the methods in Huseby et al. [3].

For the buffered environmental contours, $\hat{C}(u)$ is replaced by $\hat{\bar{C}}(u)$ for all $u \in \mathcal{U}$.

Swell

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

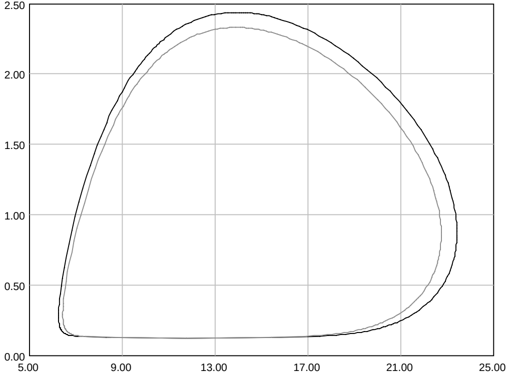

Figure: Buffered environmental contour (black) and classical environmental contour (gray) for North West Australia Swell with return period 25 years.

Wind sea

Main idea
Structural design
Buffered failure probability
Environmental contours
Buffered environmental contours
Numerical example: Waves

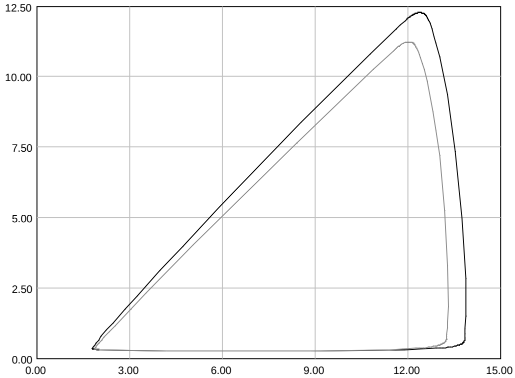

Figure: Buffered environmental contour (black) and classical environmental contour (gray) for North West Australia Wind Sea with return period 25 years.

Conclusions

We have introduced buffered environmental contours.

For the same target probability P_e, the buffered environmental contour is more conservative than the classical contours.

For manageable damages, a higher target probability might be OK $\implies$ In real-life applications a buffered environmental contour may not be so conservative.

Some key references

Baarholm, G., S. Haver and O. Okland,
Combining contours and significant wave height and peak period with platform response distributions for predicting design response,
Marine Structures, 23, 147-163, 2010.

Fontaine, E., P. Orsero, A. Ledoux, R. Nerzic, M. Prevesto and V. Quiniou,
Reliability analysis and response based design of a moored FPSO in West Africa,
Structural Safety. 41. 82-96, 2013.

Huseby, A. B., E. Vanem and B. Natvig,
A new approach ot environmental contours for ocean engineering applications based on direct Monte Carlo simulations,
Ocean Engineering, 60, 125-135, 2013.

Rockafellar, R. T. and Royset, J. O.,
On buffered failure probability in design and optimization of structures,
Reliability Engineering and System Safety, 95, pp. 499-510, 2010.

Vanem, E. and E. Bitner-Gregersen,
Alternative environmental contours for marine structural design - A comparison study,
Journal of Offshore Mechanics and Arctic Engineering, 137, 051601-1-051601-8, 2015.

Winterstein, S., T. Ude, C. Cornell, P. Bjerager and S. Haver,
Environmental parameters for extreme response: Inverse form omission factors,
Proc. 6th International Conference on Structural Safety and Reliability, Innsbruck, Austria, 551-557. CRC Press. Taylor and Francis Group, 1993.

Picture sources:

- sciencenordic.com/windmills-sea-can-break-matches