Optimal design

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The problem

In this presentation, we will consider the problem of design optimisation. We will minimise the risk of the cost of a structural design.

The cost of the structural design is composed of two parts: A fixed failure cost K which occurs in case of system failure and a cost function $\kappa(\boldsymbol{x})$ which only depends on the chosen design $\boldsymbol{x}$.

This risk-of-cost minimisation will be done with respect to two different risk measures: Value-at-risk and conditional value-at-risk.

The problem ctd.

Conditional value-at-risk is a convex risk measure, which takes into account not just whether a system functions or fails, but to which extent it fails.

We connect the value-at-risk and conditional value-at-risk to environmental contours via functions $C(\boldsymbol{u})$ and $\bar{C}(\boldsymbol{u})$.

This connection to the functions $C(\boldsymbol{u})$ and $\bar{C}(\boldsymbol{u})$ allow us to get an alternative characterisation of the risk-minimisation problems.

Value-at-risk

Let X be some risk, and introduce $S_X(x) = P(X > x)$. The α-level value-at-risk associated with the risk X, denoted by $V_\alpha[X]$, is given by $S_X^{-1}(\alpha)$. More formally, we define:

$$V_\alpha[X] = S_X^{-1}(\alpha) = \inf\{x : P(X > x) \le \alpha\}. \quad (1)$$

In the special case where X is absolutely continuously distributed, we have:

$$V_\alpha[X] = S_X^{-1}(\alpha) = x \text{ if and only if } P(X > x) = \alpha.$$

More generally, if S_X is strictly decreasing, we have that:

$$V_\alpha[X] = x \text{ if and only if } P(X > x) \le \alpha \le P(X \ge x). \quad (2)$$

Finally, if X is a *discrete random variable*, we have that:

$$V_\alpha[X] = x \text{ if and only if } P(X > x) \le \alpha < P(X \ge x). \quad (3)$$

Value-at-risk (cont.)

Proposition (Monotone transform)

For any strictly increasing continuous function ϕ we have:

$$V_\alpha[\phi(X)] = S_{\phi(X)}^{-1}(\alpha) = \phi(S_X^{-1}(\alpha)) \tag{4}$$

PROOF: We note that since ϕ is strictly increasing, it follows by (1) that:

$$\begin{aligned} V_\alpha[\phi(X)] &= \inf\{y : P(\phi(X) > y) \leq \alpha\} \\ &= \inf\{y : P(X > \phi^{-1}(y)) \leq \alpha\}. \end{aligned}$$

We then substitute $y = \phi(x)$ and $\phi^{-1}(y) = x$, and get:

$$\begin{aligned} V_\alpha[\phi(X)] &= \inf\{\phi(x) : P(X > x) \leq \alpha\} \\ &= \phi(\inf\{x : P(X > x) \leq \alpha\}) \\ &= \phi(S_X^{-1}(\alpha)). \end{aligned}$$

Value-at-risk (cont.)

Corollary (Linearity)

For $a > 0$ and $b \in \mathbb{R}$ we have:

$$V_\alpha[aX + b] = aV_\alpha[X] + b.$$

PROOF: The result follows directly from the monotonicity property by noting that:

$$\phi(X) = aX + b$$

is a strictly increasing function for all $a > 0$ and $b \in \mathbb{R}$.

Value-at-risk and optimal design

Let $\boldsymbol{V} = (V_1, \ldots, V_n) \in \mathcal{V}$ be a vector of environmental variables and let $\alpha \in (0, 1)$ be a given probability representing an acceptable level of risk. We assume that we have determined a function $C(\boldsymbol{u})$ defined for all unit vectors $\boldsymbol{u} \in \mathbb{R}^n$ such that:

$$P[\boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})] = \alpha, \text{ for all } \boldsymbol{u} \in \mathbb{R}^n. \tag{5}$$

We also introduce the following notation:

$$\Pi(\boldsymbol{u}) = \{\boldsymbol{V} \in \mathcal{V} : \boldsymbol{u}'\boldsymbol{V} = C(\boldsymbol{u})\},$$

$$\Pi^+(\boldsymbol{u}) = \{\boldsymbol{V} \in \mathcal{V} : \boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})\},$$

$$\Pi^-(\boldsymbol{u}) = \{\boldsymbol{V} \in \mathcal{V} : \boldsymbol{u}'\boldsymbol{V} \leq C(\boldsymbol{u})\}$$

Hence, we have:

$$P[\boldsymbol{V} \in \Pi^+(\boldsymbol{u})] = P[\boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})] = \alpha, \text{ for all } \boldsymbol{u} \in \mathbb{R}^n. \tag{6}$$

Value-at-risk and optimal design (cont.)

Next we let $\boldsymbol{x} = (x_1, \ldots, x_m)$ be a vector of design variables for a given system representing various parameters such as capacity, thickness, strength etc.

Every design is referred to simply by its corresponding vector of design variables, i.e., $\boldsymbol{x}$. The set of possible designs is denoted by $\mathcal{X}$.

The *performance function* of a system is denoted by g, and is assumed to be a function of both $\boldsymbol{V}$ and $\boldsymbol{x}$:

$$g = g(\boldsymbol{V}, \boldsymbol{x}).$$

The performance function is used to identify environmental conditions where the system fails. More specifically, the system fails if and only if $g(\boldsymbol{V}, \boldsymbol{x}) > 0$.

Value-at-risk and optimal design (cont.)

The cost of a system failure is denoted by K. We also introduce a deterministic function $\kappa = \kappa(\boldsymbol{x})$ representing the cost of the design $\boldsymbol{x}$, and assume that:

$$\kappa(\boldsymbol{x}) < K \text{ for all } \boldsymbol{x} \in \mathcal{X}.$$

The total cost, denoted H, is then given by:

$$H(\boldsymbol{V}, \boldsymbol{x}) = K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] + \kappa(\boldsymbol{x}).$$

The α-level value-at-risk of a given design, denoted $V_\alpha(H)$, is given by:

$$V_\alpha(H) = S_H^{-1}(\alpha),$$

where $S_H(h) = 1 - F_H(h) = P(H > h)$. Thus, $V_\alpha(H)$ is the $(1 - \alpha)$-percentile of the distribution of H.

Our main objective is to choose a design $\boldsymbol{x}$ so that $V_\alpha(H)$ is minimised.

Value-at-risk and optimal design (cont.)

Since $\kappa(\boldsymbol{x})$ is deterministic, it follows by the linearity of V_α that:

$$V_\alpha[H] = V_\alpha[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]] + \kappa(\boldsymbol{x}).$$

We observe that $K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]$ is a discrete random variable with only two possible values, 0 and K. Its distribution is given by:

$$P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] = K] = P[g(\boldsymbol{V}, \boldsymbol{x}) > 0],$$

$$P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] = 0] = P[g(\boldsymbol{V}, \boldsymbol{x}) \leq 0].$$

By (3) we know that:

$$V_\alpha[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]] = y,$$

if and only if:

$$P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > y] \leq \alpha < P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \geq y]$$

Value-at-risk and optimal design (cont.)

In particular, we have $P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > K] = 0 < \alpha$ implying that:

$$V_\alpha[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]] = K,$$

if and only if:

$$P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \geq K] = P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > \alpha$$

Furthermore, we have $P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \geq 0] = 1 > \alpha$ implying that:

$$V_\alpha[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]] = 0,$$

if and only if:

$$P[K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > 0] = P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq \alpha$$

Value-at-risk and optimal design (cont.)

Summarising this we get:

$$V_\alpha(K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]) = \begin{cases} K & \text{if } P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > \alpha \\ 0 & \text{if } P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq \alpha \end{cases}$$

From this it follows that:

$$V_\alpha(H) = \begin{cases} K + \kappa(\boldsymbol{x}) & \text{if } P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > \alpha \\ \kappa(\boldsymbol{x}) & \text{if } P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq \alpha \end{cases}$$

Since we have assumed that $\kappa(\boldsymbol{x}) < K$ for all $\boldsymbol{x} \in \mathcal{X}$, it follows that an optimal design $\boldsymbol{x}$ must be chosen so that:

$$P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq \alpha \tag{7}$$

Value-at-risk and optimal design (cont.)

Theorem (Halfspace condition)

A sufficient condition for (7) to hold is that $g(\boldsymbol{V}, \boldsymbol{x}) \leq 0$ for all $\boldsymbol{V}$ such that $\boldsymbol{u}'\boldsymbol{V} \leq C(\boldsymbol{u})$, where $\boldsymbol{u} \in \mathbb{R}^n$ is a suitably chosen unit vector.

PROOF: The condition implies that if $g(\boldsymbol{V}, \boldsymbol{x}) > 0$, then $\boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})$.

Hence, by (5) we get that:

$$P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq P[\boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})] = \alpha.$$

Hence, we conclude that (7) is satisfied.

Value-at-risk and optimal design (cont.)

We then let $\boldsymbol{u} \in \mathbb{R}^n$ be a unit vector and consider the following subclass of designs:

$$\mathcal{X}(\boldsymbol{u}) = \{\boldsymbol{x} \in \mathcal{X} : g(\boldsymbol{V}, \boldsymbol{x}) \leq 0 \text{ for all } \boldsymbol{V} \in \Pi^-(\boldsymbol{u})\}.$$

By the halfspace condition theorem we know that the condition (7) is satisfied for all designs $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{u})$.

Hence, an optimal design within the subclass $\mathcal{X}(\boldsymbol{u})$ can be found by minimising $\kappa(\boldsymbol{x})$ with respect to $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{u})$.

Different choices of the unit vector $\boldsymbol{u}$ will generate different optimal designs. However, the choice of $\boldsymbol{u}$ may often be a result of initial concept decisions related to the system of interest. Thus, it may not be necessary to consider multiple subclasses of design.

Example: Structural reliability

We consider a system whose performance depends on the non-negative environmental variables, $\boldsymbol{V} = (V_1, \ldots, V_n) \in \mathcal{V}$. The system fails if:

$$A\boldsymbol{V} > \boldsymbol{x}$$

where $A = A^{m \times n}$ is a matrix, and the design $\boldsymbol{x} = (x_1, \ldots, x_m)$ is a vector of *strengths*.

The cost of the design $\boldsymbol{x}$ is given by:

$$\kappa(\boldsymbol{x}) = c_1 x_1 + \cdots + c_m x_m.$$

We want to minimise $\kappa(\boldsymbol{x})$ subject to $P[A\boldsymbol{V} > \boldsymbol{x}] \leq \alpha$. Since this failure probability may be difficult to compute, we instead minimise $\kappa(\boldsymbol{x})$ subject to:

$$\{\boldsymbol{V} \in \mathcal{V} : A\boldsymbol{V} > \boldsymbol{x}\} \subseteq \{\boldsymbol{V} \in \mathcal{V} : \boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})\}. \tag{8}$$

Example: Structural reliability

It follows that if the design $\boldsymbol{x}$ satisfies (8), then:

$$P[A\boldsymbol{V} > \boldsymbol{x}] \leq P[\boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})] = \alpha.$$

For a given design $\boldsymbol{x}$, we can then check if it satisfies (8) by solving the following LP-problem:

$$\text{Minimise } \boldsymbol{u}'\boldsymbol{V} \text{ subject to } A\boldsymbol{V} \geq \boldsymbol{x}. \tag{9}$$

Let $\boldsymbol{V}_0$ denote the solution to (9). Then $\boldsymbol{x}$ satisfies (8) if and only if:

$$\boldsymbol{u}'\boldsymbol{V}_0 > C(\boldsymbol{u}).$$

By using a suitable iteration method one can then find a design $\boldsymbol{x}$ which minimises $\kappa(\boldsymbol{x})$ subject to (8).

Conditional value-at-risk

Problem: VaR ignores the size of the outcomes in the tail of the distribution.

Example

$\mathrm{VaR}_{0.05}(\mathrm{X})$ is the $x-$value such that only 5% of the outcomes of X are larger (i.e., worse in our context) than this value. Hence, $\mathrm{VaR}_{0.05}(\mathrm{X})$ ignores the size, and hence the consequences, of all values above this level.

Based on the previous definition of value-at-risk, conditional value at risk (CVaR) is defined as

$$CVaR_{\alpha}(X) := \frac{1}{\alpha}\int_0^{\alpha} V_u(X)du \qquad (10)$$

That is, we compute the average of the value-at-risk in the α% worst cases.

Conditional value-at-risk ctd.

CVaR is frequently used in mathematical finance, and to some extent in insurance mathematics.

Note that $\rho := CVaR$ is a convex risk measure, i.e.

(*i*) (Convexity) $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for $0 \leq \lambda \leq 1$.

(*ii*) (Monotonicity) If $X \geq Y$, then $\rho(X) \geq \rho(Y)$.

(*iii*) (Translation invariance) If $m \in \mathbb{R}$, then $\rho(X + m\mathbf{1}) = \rho(X) - m$.

NB: The monotonicity property is the other way around from what is common in financial mathematics because we view large positive values as bad (failure of system). In finance, greatly negative values are bad (losses).

Conditional value at risk ctd.

Proposition

For any strictly increasing continuous function ϕ we have:

$$CVaR_\alpha[\phi(X)] = \frac{1}{\alpha}\int_0^\alpha \phi(S_X^{-1}(u))du. \tag{11}$$

PROOF: From the definition of CVaR (10):

$$\begin{aligned} CVaR_\alpha[\phi(X)] &= \frac{1}{\alpha}\int_0^\alpha S_{\phi(X)}^{-1}(u)du \\ &= \frac{1}{\alpha}\int_0^\alpha \phi(S_X^{-1}(u))du \end{aligned}$$

where the final equality holds because of equation (4) of the monotone transform proposition for VaR.

Jensen's inequality

Theorem (Jensen's inequality)

Let $(\Omega, \mathcal{F}, P)$ be a probability space (i.e., $P(\Omega) = 1$). Let $g : \Omega \to \mathbb{R}$ be a P-integrable function. Also, assume that $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function. Then,

$$\varphi(\int_\Omega g(\omega)dP(\omega)) = \int_\Omega \varphi(g(\omega))dP(\omega).$$

From Jensen's inequality, we find that for $f : [a, b] \to \mathbb{R}$,

$$\varphi(\frac{1}{b-a}\int_a^b f(x)dx) \leq \frac{1}{b-a}\int_a^b \varphi(f(x))dx.$$

Conditional value at risk ctd.

Proposition

For any strictly increasing continuous function and convex function ϕ we have:

$$\phi(CVaR_\alpha[X]) \leq CVaR_\alpha[\phi(X)]. \quad (12)$$

PROOF:

$$\begin{aligned}
\phi(CVaR_\alpha[X]) &= \phi(\tfrac{1}{\alpha}\int_0^\alpha S_X^{-1}(u)) \\
&\leq \tfrac{1}{\alpha}\int_0^\alpha \phi(S_X^{-1}(u))du \\
&= \tfrac{1}{\alpha}\int_0^\alpha S_{\phi(X)}^{-1}(u)du \\
&= CVaR_\alpha(\phi(X)).
\end{aligned}$$

Here, the inequality holds from Jensen's inequality. The second to last equality follows because of equation (4) of the monotone transform proposition for VaR.

Conditional value-at-risk ctd.

Corollary (Linearity of CVaR)

For $a > 0$ and $b \in \mathbb{R}$ we have:

$$CVaR_\alpha[aX + b] = aCVaR_\alpha[X] + b.$$

PROOF: By using the definition of CVaR and the linearity of VaR, we see that

$$\begin{aligned} CVaR_\alpha(aX + b) &= \frac{1}{\alpha}\int_0^\alpha V_u(aX + b)du \\ &= \frac{1}{\alpha}\int_0^\alpha \{aV_u(X) + b\}du \\ &= a(\frac{1}{\alpha}\int_0^\alpha V_u(X)du) + b \\ &= aCVaR_\alpha(X) + b. \end{aligned}$$

CVaR and optimal design

Parallel to the VaR-case, we would like to choose an optimal design $\boldsymbol{x}$ such that the conditional value at risk of the total cost is minimized:

CVaR design optimization problem:

$$\min_{\boldsymbol{x}} CVaR_{\alpha}(H)$$

where, as before, $H = K \cdot \mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0] + \kappa(\boldsymbol{x})$.

α is assumed to be given. The value of α is determined by the firm based on the required return period of the system.

From the linearity of CVaR

$$CVaR_{\alpha}(H) = K \cdot CVaR(\mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]) + \kappa(\boldsymbol{x}).$$

CVaR and optimal design ctd.

From the definition of VaR, VaR_u is decreasing in u.

Example

Check this for the previously computed VaR:

Assume $u_1 < u_2$. If $P(\mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]) > u_2$, then

$$P(\mathrm{I}[g(\boldsymbol{V}, \boldsymbol{x}) > 0]) > u_2 > u_1.$$

Hence, from the previously calculated VaR, we see that $VaR_{u_2} = K \implies VaR_{u_1} = K$, but not the other way around. Hence,

$$VaR_{u_2} \leq VaR_{u_1}.$$

CVaR and optimal design ctd.

Also, note that

$$\begin{aligned} CVaR_\alpha(\mathrm{I}[g(\boldsymbol{V},\boldsymbol{x})>0]) &= \frac{1}{\alpha}\int_0^\alpha VaR_u(\mathrm{I}[g(\boldsymbol{V},\boldsymbol{x})>0])du \\ &= \frac{1}{\alpha}\int_0^{\min\{P(\mathrm{I}[g(\boldsymbol{V},\boldsymbol{x})>0]),\alpha\}} 1du \\ &\geq VaR_\alpha(\mathrm{I}[g(\boldsymbol{V},\boldsymbol{x})>0]). \end{aligned}$$

Here, the second equality follows from the previous computation of VaR. The inequality follows since the integral is always greater than or equal 0. Also, if $P(\mathrm{I}[g(\boldsymbol{V},\boldsymbol{x})>0])>\alpha$, so

$$\min\{P(\mathrm{I}[g(\boldsymbol{V},\boldsymbol{x})>0]),\alpha\}=\alpha,$$

$CVaR = 1$ (the same as VaR).

This is also true in general, so $CVaR_\alpha$ is more conservative than VaR_α.

CVaR and optimal design ctd.

In our case, the previous calculations imply that

$$CVaR_\alpha(H) = \begin{cases} K + \kappa(\boldsymbol{x}) & \text{if } P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] > \alpha \\ K\frac{P[g(\boldsymbol{V},\boldsymbol{x})>0]}{\alpha} + \kappa(\boldsymbol{x}) & \text{if } P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq \alpha \end{cases}$$

Note that $\frac{P[g(\boldsymbol{V},\boldsymbol{x})>0]}{\alpha} \leq 1$ in the second case above, since $P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq \alpha$.

Also, note that our assumption that $\kappa(\boldsymbol{x}) < K$ for all $\boldsymbol{x} \in \mathcal{X}$, is no longer enough to guarantee that the optimal design should be chosen where $P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq \alpha$.

CVaR and optimal design ctd.

A sufficient condition to ensure that the optimal design should be chosen where $P[g(\boldsymbol{V}, \boldsymbol{x}) > 0] \leq \alpha$ is:

$$\kappa(\boldsymbol{x}_1) - \kappa(\boldsymbol{x}_2) \leq \frac{K}{\alpha}(\alpha - P[g(\boldsymbol{V}, \boldsymbol{x}_1) > 0]) \tag{13}$$

for all $\boldsymbol{x}_1$ is such that $P[g(\boldsymbol{V}, \boldsymbol{x}_1) > 0] \leq \alpha$ and $\boldsymbol{x}_2$ such that $P[g(\boldsymbol{V}, \boldsymbol{x}_2) > 0] > \alpha$.

Note that this slightly resembles a Lipschitz condition.

Halfspaces

Let $\boldsymbol{V} = (V_1, \ldots, V_n) \in \mathcal{V}$ be a vector of environmental variables and let $\alpha \in (0, 1)$ be a given probability representing an acceptable level of risk. Assume, like before, that we have determined a function $C(\boldsymbol{u})$ defined for all unit vectors $\boldsymbol{u} \in \mathbb{R}^n$ such that:

$$P[\boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})] = \alpha, \text{ for all } \boldsymbol{u} \in \mathbb{R}^n. \tag{14}$$

Now, define a function, $\bar{C}(\boldsymbol{u})$, as follows

$$\bar{C}(\boldsymbol{u}) := \mathrm{E}[\boldsymbol{u}'\boldsymbol{V} | \boldsymbol{u}'\boldsymbol{V} > C(\boldsymbol{u})]. \tag{15}$$

Halfspaces

Furthermore, introduce the following notation:

$$\bar{\Pi}(\boldsymbol{u}) = \{\boldsymbol{V} \in \mathcal{V} : \boldsymbol{u}'\boldsymbol{V} = \bar{C}(\boldsymbol{u})\},$$

$$\bar{\Pi}^{+}(\boldsymbol{u}) = \{\boldsymbol{V} \in \mathcal{V} : \boldsymbol{u}'\boldsymbol{V} > \bar{C}(\boldsymbol{u})\},$$

$$\bar{\Pi}^{-}(\boldsymbol{u}) = \{\boldsymbol{V} \in \mathcal{V} : \boldsymbol{u}'\boldsymbol{V} \leq \bar{C}(\boldsymbol{u})\}$$

Define

$$\Gamma(\boldsymbol{u}, \boldsymbol{V}) := \boldsymbol{u} \cdot \boldsymbol{V} - \bar{C}(\boldsymbol{u}). \tag{16}$$

CVaR and optimal design ctd.

For a fixed (but arbitrary) unit vector $\boldsymbol{u}$, let $\bar{\mathcal{X}}(\boldsymbol{u})$ denote the set of designs $\boldsymbol{x}$ such that $g(\cdot, \boldsymbol{x})$ dominated by $\Gamma(\boldsymbol{u}, \cdot)$.

Then, for any $\boldsymbol{x} \in \bar{\mathcal{X}}(\boldsymbol{u})$,

$$\begin{aligned}
P(g(\boldsymbol{V}, \boldsymbol{x}) > 0) &\leq P(\Gamma(\boldsymbol{u}, \boldsymbol{x}) > 0) \\
&= P(\boldsymbol{u} \cdot \boldsymbol{V} - \bar{C}(\boldsymbol{u}) > 0) \\
&= P(\boldsymbol{u} \cdot \boldsymbol{V} > \bar{C}(\boldsymbol{u})) \\
&= P(\bar{\Pi}^+(\boldsymbol{u})) \\
&\leq P(\Pi^+(\boldsymbol{u})) \\
&= \alpha.
\end{aligned}$$

where the last inequality follows because $\bar{C}(\boldsymbol{u}) > C(\boldsymbol{u})$, so by the definitions of $\bar{\Pi}^+(\boldsymbol{u})$ and $\Pi^+(\boldsymbol{u})$, we find that $\bar{\Pi}^+(\boldsymbol{u}) \subseteq \Pi^+(\boldsymbol{u})$. Hence,

$$P(\bar{\Pi}^+(\boldsymbol{u})) \leq P(\Pi^+(\boldsymbol{u})).$$

CVaR and optimal design ctd.

Therefore, we have proved that if $\boldsymbol{x} \in \bar{\mathcal{X}}(\boldsymbol{u})$, then

$$P(g(\boldsymbol{V}, \boldsymbol{x}) > 0) \leq \alpha. \tag{17}$$

We summarize this in the following theorem.

Theorem (Domination condition)

A sufficient condition for (17) *to hold is that* $g(\cdot, \boldsymbol{x})$ *is dominated by a function* $\Gamma(\boldsymbol{u}, \cdot)$ *of the form* (16), *where* $\boldsymbol{u} \in \mathbb{R}^n$ *is a suitably chosen unit vector.*

Value-at-risk and optimal design (cont.)

Assume condition (13) is satisfied. Then, we know that the optimal design should be chosen such that equation (17) holds.

Let $\boldsymbol{u} \in \mathbb{R}^n$ be a suitably chosen unit vector.

By the domination condition theorem we know that the condition (17) is satisfied for all designs $\boldsymbol{x} \in \bar{\mathcal{X}}(\boldsymbol{u})$.

Hence, an optimal design is found by minimising

$$K \frac{P[g(\boldsymbol{V}, \boldsymbol{x}) > 0]}{\alpha} + \kappa(\boldsymbol{x})$$

with respect to $\boldsymbol{x} \in \bar{\mathcal{X}}(\boldsymbol{u})$.

Choosing the unit vector $\boldsymbol{u}$

Different choices of the unit vector $\boldsymbol{u}$ will generate different optimal designs. However, the choice of $\boldsymbol{u}$ may often be a result of initial concept decisions related to the system of interest. Thus, it may not be necessary to consider multiple subclasses of design.

If a firm has N different initial concepts, $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ under consideration, the minimisation problem can be solved for each of these $\boldsymbol{u}_i$'s, $i = 1, \ldots, N$.

This results in N potentially optimal designs $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$.

To find the optimal concept, the firm can simply compare the objective function values, i.e. $V_\alpha(H(\boldsymbol{x_i}))$ or $CVaR_\alpha(H(\boldsymbol{x_i}))$, $i = 1, \ldots, N$, of these designs.

Choosing the unit vector $\boldsymbol{u}$ ctd.

Assume that for a fixed design $\boldsymbol{x}$, we know that the corresponding performance function $g(\cdot, \boldsymbol{x})$ is monotone in some V_i-component, $i = 1, \ldots, n$.

Then, roughly speaking, one should choose the unit vector $\boldsymbol{u}$ such that it follows the monotonicity.

That is, if g is non-decreasing in V_i, so $V_i \leq \bar{V}_i$ implies that $g((V_i, \boldsymbol{V}), \boldsymbol{x}) \leq g((\bar{V}_i, \boldsymbol{V}), \boldsymbol{x})$, then u should be chosen such that $u_i \in (0, 1)$.

If g is non-increasing in V_i, so $V_i \leq \bar{V}_i$ implies that $g((V_i, \boldsymbol{V}), \boldsymbol{x}) \geq g((\bar{V}_i, \boldsymbol{V}), \boldsymbol{x})$, then u should be chosen such that $u_i \in (-1, 0)$.

Choosing the unit vector $\boldsymbol{u}$ ctd.

We make the previous statement more precise: Consider the classical, $C(\boldsymbol{u})$, case. The buffered case is parallel.

Assume that there exists $\boldsymbol{V}$, $V_i \leq \bar{V}_i$ where the system fails in $(\bar{V}_i, \boldsymbol{V})$, but functions in $(V_i, \boldsymbol{V})$.

Note that this assumption is slightly stricter than g being monotone in component i. It corresponds to monotonicity as well as criticality of the i'th environmental component.

Also, assume for contradiction that we have choose $u_i \in (-1, 0)$.

By assumption, there exists $\boldsymbol{V}$, V_i and $\bar{V}_i$ such that $g(V_i, \boldsymbol{V}) \leq 0$ and $g(\bar{V}_i, \boldsymbol{V}) > 0$. That is, the system fails in $(\bar{V}_i, \boldsymbol{V})$, but functions in $(V_i, \boldsymbol{V})$.

Choosing the unit vector $\boldsymbol{u}$ ctd.

There exists a vector $\boldsymbol{u}$ such that the (by scaling) unit vector $(u_i, \boldsymbol{u})$ satisfies $(\bar{V}_i, \boldsymbol{V}) \in \Pi^-((u_i, \boldsymbol{u}))$ and $(V_i, \boldsymbol{V}) \in \Pi^+((u_i, \boldsymbol{u}))$.

From the definitions of $\Pi^+((u_i, \boldsymbol{u})), \Pi^-((u_i, \boldsymbol{u}))$, this implies that the system should function in $(\bar{V}_i, \boldsymbol{V})$ and fail in $(V_i, \boldsymbol{V})$. But this contradicts the assumption.

Hence, choosing $u_i \in (-1, 0)$ leads to a contradiction, so u_i should be chosen in the only other way possible, namely such that $u_i \in (0, 1)$.

The arguments in the case where g is non-increasing in V_i is parallel.

An alternative optimisation problem

So far, we have minimised the risk of the cost of a structural design.

An alternative optimisation problem is to minimise the expected cost under a risk constraint:

$$\min E[H(\boldsymbol{x}, \boldsymbol{V})]$$

$$\text{such that}$$

$$\text{risk}(g(\boldsymbol{x}, \boldsymbol{V})) \leq \alpha$$

Here, the risk-function, which depends on the performance function of the system, can be either value-at-risk or conditional value-at-risk.

By looking at this optimisation problem, the environmental contour is a representation of the constraint.