Environmental contours – part 1

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Environmental contours

- Let $(T, H) \in \mathbb{R}^2$ be a vector of environmental variables representing e.g., the *sea state* at some point of time t. In this presentation we let:

$$T = \text{Wave period at time } t$$

$$H = \text{Significant wave height at time } t$$

- The distribution of (T, H) is assumed to be absolutely continuous with respect to the Lebesgues measure in $\mathbb{R}^2$
- An *environmental contour* is then defined as the boundary of a set $\mathcal{B} \subseteq \mathbb{R}^2$, and denoted $\partial\mathcal{B}$.

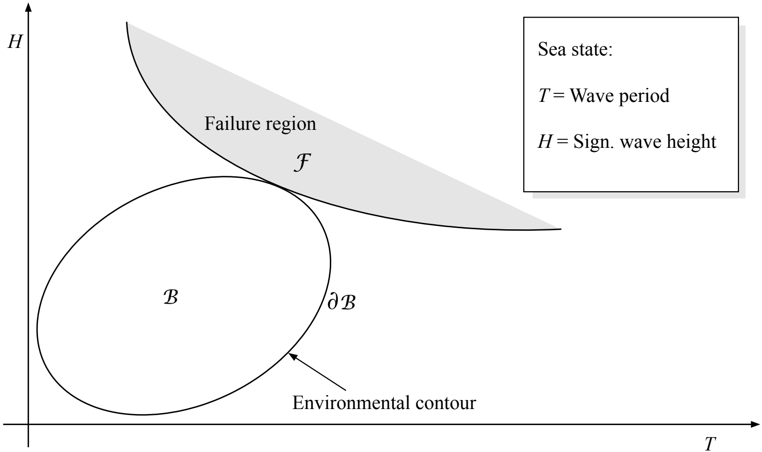
Environmental contours (cont.)

- During the design phase of some structure of interest the environmental contour can be used to identify conditions which the structure should be able to withstand
- If $(T, H) \in \mathcal{B}$, the structure should *function normally*
- The environmental contour $\partial\mathcal{B}$ represents the most severe or extreme conditions that the structure should be able to handle
- The points in $\partial\mathcal{B}$ represent possible *design requirements* for the structure.

Failure regions

- The *failure region* $\mathcal{F} \subseteq \mathbb{R}^2$ of a structure is the set of states of the environmental variables where the structure fails.
- The exact shape of the failure region of a structure is typically be unknown at this stage.
- It may still be possible to argue that the failure region belongs to a certain family denoted by $\mathcal{E}$.
- A contour $\partial\mathcal{B}$ will be evaluated with respect to the family $\mathcal{E}$.
- The family $\mathcal{E}$ depends on $\mathcal{B}$ in such a way that $\mathcal{F} \cap \mathcal{B} \subseteq \partial\mathcal{B}$ for all $\mathcal{F} \in \mathcal{E}$.
- If the size of $\mathcal{B}$ is increased (i.e., the structure is strengthened), the family of possible failure regions, $\mathcal{E}$, is reduced, and hence also the failure probability.

Environment contour and failure region



Exceedance probability

The *exceedance probability* of $\mathcal{B}$ with respect to $\mathcal{E}$ is defined as:

$$P_e(\mathcal{B}, \mathcal{E}) = \sup\{P[(T, H) \in \mathcal{F}] : \mathcal{F} \in \mathcal{E}\}.$$

NOTE: The exceedance probability is an upper bound on the failure probability of the structure assuming that the true failure region is a member of the family $\mathcal{E}$.

For a given target exceedance probability $p_e \in (0, 0.5)$ our goal is to find a minimal set $\mathcal{B}$ such that:

$$P_e(\mathcal{B}, \mathcal{E}) \leq p_e \tag{1}$$

If the set $\mathcal{B}$ satisfies (1), then $\partial\mathcal{B}$ is said to be a *valid* environmental contour.

Maximal failure regions

A failure region $\mathcal{F} \in \mathcal{E}$ is said to be *maximal* if there does not exist a region $\mathcal{F}' \in \mathcal{E}$ such that $\mathcal{F} \subset \mathcal{F}'$.

The family of maximal regions in $\mathcal{E}$ is denoted by $\mathcal{E}^*$. If $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{E}$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2$, we obviously have:

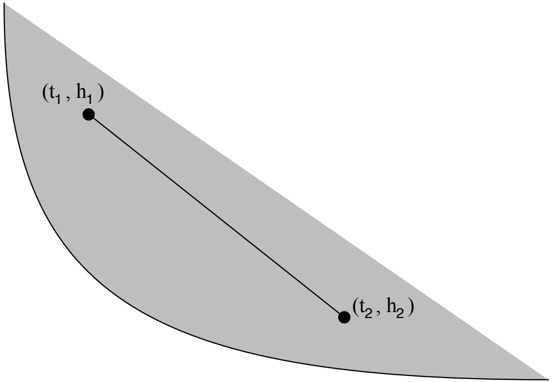
$$P[(T, H) \in \mathcal{F}_1] \leq P[(T, H) \in \mathcal{F}_2].$$

From this it follows that:

$$\begin{aligned} P_e(\mathcal{B}, \mathcal{E}) &= \sup\{P[(T, H) \in \mathcal{F}] : \mathcal{F} \in \mathcal{E}\} \\ &= \sup\{P[(T, H) \in \mathcal{F}] : \mathcal{F} \in \mathcal{E}^*\}. \end{aligned}$$

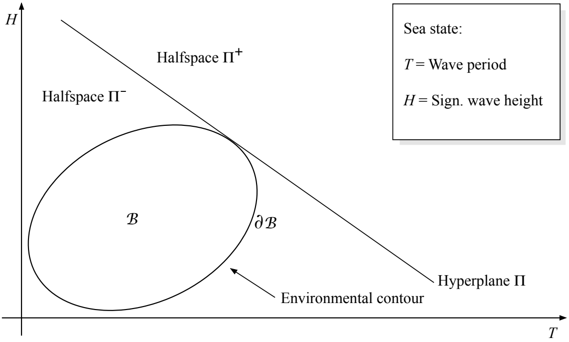
Convex failure regions

It is often natural to assume that a failure region is convex:



This means that if the structure fails at two distinct points (t_1, h_1) and (t_2, h_2), then it also fails for all states on the line segment between these points.

Supporting hyperplanes and halfspaces



- Π is a supporting hyperplane of $\mathcal{B}$
- Π^+ is a supporting halfspace of $\mathcal{B}$
- Π^- is a halfspace opposite to a supporting halfspace of $\mathcal{B}$

Convex contours

In the following we only consider contour sets $\mathcal{B}$ which are *compact* and *convex*. Furthermore, we assume that all the sets in $\mathcal{E}$ are *convex*.

For a given compact and convex set $\mathcal{B}$ we introduce the following families of sets:

$$\begin{aligned}
\mathcal{P}(\mathcal{B}) &= \text{The family of supporting hyperplanes of } \mathcal{B}, \\
\mathcal{P}^{+}(\mathcal{B}) &= \text{The family of supporting halfspaces of } \mathcal{B}, \\
\mathcal{P}^{-}(\mathcal{B}) &= \text{The family of halfspaces opposite} \\
&\quad \text{to supporting halfspaces of } \mathcal{B}
\end{aligned}$$

Convex contours (cont.)

Proposition (Halfspace failure region)

Let $\mathcal{B} \subset \mathbb{R}^2$ be a compact and convex set, and let $\mathcal{E}$ be the family of convex sets such that $\mathcal{F} \cap \mathcal{B} \subseteq \partial\mathcal{B}$ for all $\mathcal{F} \in \mathcal{E}$. Then $\mathcal{E}^* = \mathcal{P}^+(\mathcal{B})$, and hence:

$$P_e(\mathcal{B}, \mathcal{E}) = \sup_{\Pi^+ \in \mathcal{P}^+(\mathcal{B})} \{P[(T, H) \in \Pi^+]\}. \quad (2)$$

Moreover, the set $\mathcal{B}$ can be expressed as:

$$\mathcal{B} = \bigcap_{\Pi^- \in \mathcal{P}^-(\mathcal{B})} \Pi^-. \quad (3)$$

Convex contours (cont.)

The families $\mathcal{P}(\mathcal{B})$, $\mathcal{P}^+(\mathcal{B})$ and $\mathcal{P}^-(\mathcal{B})$ can be expressed in a more explicit form.

In order to explain how this can be done, we start out by letting $\theta \in [0, 2\pi)$, and define:

$$B(\mathcal{B}, \theta) = \sup_{(t,h) \in \mathcal{B}} [t \cos(\theta) + h \sin(\theta)] \tag{4}$$

We also introduce :

$$\begin{aligned}
\Pi(\mathcal{B}, \theta) &= \{(t, h) : t \cos(\theta) + h \sin(\theta) = B(\mathcal{B}, \theta)\} \\
\Pi^+(\mathcal{B}, \theta) &= \{(t, h) : t \cos(\theta) + h \sin(\theta) \geq B(\mathcal{B}, \theta)\}, \\
\Pi^-(\mathcal{B}, \theta) &= \{(t, h) : t \cos(\theta) + h \sin(\theta) \leq B(\mathcal{B}, \theta)\}.
\end{aligned}$$

Convex contours (cont.)

Since $\mathcal{B}$ is assumed to be compact, it follows that $\mathcal{B}$ is bounded, and thus, $B(\mathcal{B}, \theta)$ must be finite. Moreover, by the definition of $B(\mathcal{B}, \theta)$ it follows that:

$$t\cos(\theta) + h\sin(\theta) \leq B(\mathcal{B}, \theta), \quad \text{for all } (t, h) \in \mathcal{B}.$$

Finally, since $\mathcal{B}$ is compact, $\mathcal{B}$ is closed as well. Thus, there must exist at least one point $(t_0, h_0) \in \mathcal{B}$ such that:

$$t_0\cos(\theta) + h_0\sin(\theta) = B(\mathcal{B}, \theta)$$

From this it follows that:

$$\Pi(\mathcal{B}, \theta) \in \mathcal{P}(\mathcal{B})$$

$$\Pi^+(\mathcal{B}, \theta) \in \mathcal{P}^+(\mathcal{B})$$

$$\Pi^-(\mathcal{B}, \theta) \in \mathcal{P}^-(\mathcal{B})$$

Convex contours (cont.)

Assume conversely that $\Pi \in \mathcal{P}(\mathcal{B})$, and let Π^+ and Π^- be the corresponding supporting and opposite halfspaces separated by Π. Then Π, Π^+ and Π^- can be expressed as follows:

$$\begin{aligned} \Pi &= \{(t,h) : ta_1 + ha_2 = b\}, \\ \Pi^+ &= \{(t,h) : ta_1 + ha_2 \geq b\}, \\ \Pi^- &= \{(t,h) : ta_1 + ha_2 \leq b\} \end{aligned}$$

for suitable real numbers a_1, a_2 and b.

Without loss of generality we may assume that a_1 and a_2 are *normalized* such that $a_1^2 + a_2^2 = 1$.

Then it follows that there exists a $\theta \in [0, 2\pi)$ such that:

$$a_1 = \cos(\theta) \quad \text{and} \quad a_2 = \sin(\theta)$$

Convex contours (cont.)

Since Π^+ is a supporting halfspace of $\mathcal{B}$, we must have:

$$t\cos(\theta) + h\sin(\theta) \leq b, \quad \text{for all } (t, h) \in \mathcal{B},$$
$$t_0\cos(\theta) + h_0\sin(\theta) = b, \quad \text{for some } (t_0, h_0) \in \mathcal{B},$$

From this it follows that:

$$b = \sup_{(t,h)\in\mathcal{B}} [t\cos(\theta) + h\sin(\theta)] = B(\mathcal{B}, \theta),$$

implying that:

$$\Pi = \Pi(\mathcal{B}, \theta)$$
$$\Pi^+ = \Pi^+(\mathcal{B}, \theta)$$
$$\Pi^- = \Pi^-(\mathcal{B}, \theta)$$

Convex contours (cont.)

The following proposition summarizes these findings:

Proposition (Parametric families)

Let $\mathcal{B} \subset \mathbb{R}^2$ be a compact and convex set. Then we have:

$$\mathcal{P}(\mathcal{B}) = \{\Pi(\mathcal{B}, \theta) : \theta \in [0, 2\pi)\},$$

$$\mathcal{P}^+(\mathcal{B}) = \{\Pi^+(\mathcal{B}, \theta) : \theta \in [0, 2\pi)\},$$

$$\mathcal{P}^-(\mathcal{B}) = \{\Pi^-(\mathcal{B}, \theta) : \theta \in [0, 2\pi)\}.$$

Convex contours (cont.)

By combining the previous results we also obtain the following:

Proposition

Let $\mathcal{B} \subset \mathbb{R}^2$ be a compact and convex set, and let $\mathcal{E}$ be the family of convex sets such that $\mathcal{F} \cap \mathcal{B} \subseteq \partial\mathcal{B}$ for all $\mathcal{F} \in \mathcal{E}$. Then we have:

$$P_e(\mathcal{B}, \mathcal{E}) = \sup_{\theta \in [0, 2\pi)} \{P[(T, H) \in \Pi^+(\mathcal{B}, \theta)]\}. \tag{5}$$

Moreover, the set $\mathcal{B}$ can be expressed as:

$$\mathcal{B} = \bigcap_{\theta \in [0, 2\pi)} \Pi^-(\mathcal{B}, \theta) \tag{6}$$

Ordering of convex contours

An immediate consequence of this result is that the function B induces an *ordering* of compact and convex sets.

That is, we have the following result:

Proposition (Ordering of convex contours)

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two compact and convex sets, and assume that:

$$B(\mathcal{B}_1, \theta) \leq B(\mathcal{B}_2, \theta) \text{ for all } \theta \in [0, 2\pi).$$

Then $\mathcal{B}_1 \subseteq \mathcal{B}_2$.

Ordering of convex contours (cont.)

PROOF: If $B(\mathcal{B}_1, \theta) \leq B(\mathcal{B}_2, \theta)$ for all $\theta \in [0, 2\pi)$, this implies that:

$$\Pi^-(\mathcal{B}_1, \theta) \subseteq \Pi^-(\mathcal{B}_2, \theta) \text{ for all } \theta \in [0, 2\pi).$$

Hence, by the intersection formula we get that:

$$\mathcal{B}_1 = \bigcap_{\theta \in [0,2\pi)} \Pi^-(\mathcal{B}_1, \theta) \subseteq \bigcap_{\theta \in [0,2\pi)} \Pi^-(\mathcal{B}_2, \theta) = \mathcal{B}_2 \qquad \blacksquare$$

Parameter representations of convex contours

Another consequence of the previous results is that a compact and convex set $\mathcal{B} \subset \mathbb{R}^2$, and its boundary $\partial\mathcal{B}$ are *uniquely determined* by the function $B(\mathcal{B}, \theta)$.

In order to study the relation between $B(\mathcal{B}, \theta)$ and $\partial\mathcal{B}$ further, the following result, first proved by Minkowski in 1896, is relevant:

Proposition (Minkowski)

Let $\mathcal{B}$ be a closed convex set. Then for every point $\boldsymbol{x} \in \partial\mathcal{B}$ there exists a supporting hyperplane $\Pi \in \mathcal{P}(\mathcal{B})$ such that $\boldsymbol{x} \in \Pi$.

In our context this implies that:

Proposition (Parametric Minkowski)

Let $\mathcal{B} \subset \mathbb{R}^2$ be a compact convex set. Then for every point $(t_0, h_0) \in \partial\mathcal{B}$ there exists a $\theta \in [0, 2\pi)$ such that $(t_0, h_0) \in \Pi(\mathcal{B}, \theta)$.

Parameter representations of convex contours (c.)

The last proposition indicates that it may be possible to construct a mapping from angles $\theta \in [0, 2\pi)$ to the points in $\partial\mathcal{B}$.

In the general case, however, the relation between angles and boundary points is not straightforward.

By the definition of $B(\mathcal{B}, \theta)$ it follows that for a given $\theta \in [0, 2\pi)$ there exists at least one point $(t_0, h_0) \in \mathcal{B}$ such that:

$$t_0 \cos(\theta) + h_0 \sin(\theta) = B(\mathcal{B}, \theta), \tag{7}$$

and this point must also be on the boundary of $\mathcal{B}$.

However, (t_0, h_0) may not be the only boundary point which satisfies (7).

Parameter representations of convex contours (c.)

Example (A polygon shaped contour)

Consider a case where $\mathcal{B}$ is a convex polygon.

If, for a given θ, the vector $(\cos(\theta), \sin(\theta))$ is orthogonal to, and pointing away from one of sides of $\mathcal{B}$, then the hyperplane $\Pi(\mathcal{B}, \theta)$ intersects with all the points on this side.

On the other hand, for any $\theta' \neq \theta$, the corresponding supporting hyperplane $\Pi(\mathcal{B}, \theta')$ does not intersect with any of the points on this side (except possibly the endpoints).

Hence, it is not possible to define a mapping where each angle $\theta \in [0, 2\pi)$ is mapped to a unique point $(t_0, h_0) \in \partial\mathcal{B}$.

In order to avoid such problems we assume that $\mathcal{B}$ is *strictly convex*:

Definition

A set $\mathcal{B}$ is strictly convex if for any pair of distinct points $(t_1, h_1), (t_2, h_2) \in \mathcal{B}$, all the points on the line segment between (t_1, h_1) and (t_2, h_2) (except possibly the endpoints (t_1, h_1) and (t_2, h_2)) belong to the interior of $\mathcal{B}$.

NOTE: If $\mathcal{B}$ is a convex polygon and $(t_1, h_1), (t_2, h_2) \in \mathcal{B}$ are to adjacent corners of this polygon, then the entire line segment between (t_1, h_1) and (t_2, h_2) lies at the boundary of $\mathcal{B}$.

Thus, a convex polygon is *not* a strictly convex set.

Parameter representations of convex contours (c.)

The following proposition essentially states that for strictly convex sets there exists a well-defined mapping from angles to boundary points.

Proposition

Let $\mathcal{B} \subset \mathbb{R}^2$ *be a compact and strictly convex set. Then for every angle* $\theta \in [0, 2\pi)$ *there exists a unique point* $(t(\theta), h(\theta)) \in \partial\mathcal{B}$ *such that* $(t(\theta), h(\theta)) \in \Pi(\mathcal{B}, \theta)$.

Parameter representations of convex contours (c.)

PROOF: By the definition of $\Pi(\mathcal{B}, \theta)$ we know that there exists at least one point $(t_1, h_1) \in \partial\mathcal{B}$ such that $(t_1, h_1) \in \Pi(\mathcal{B}, \theta)$.

Assume, for a contradiction that there exists another boundary point (t_2, h_2), different from (t_1, h_1), which also belongs to the hyperplane $\Pi(\mathcal{B}, \theta)$.

Since $\Pi(\mathcal{B}, \theta)$ is convex, all the points on the line segment between (t_1, h_1) and (t_2, h_2) also belong to $\Pi(\mathcal{B}, \theta)$.

However, since $\mathcal{B}$ is assumed to be strictly convex, the points on the line segment between (t_1, h_1) and (t_2, h_2) are elements of the interior of $\mathcal{B}$, which contradicts that $\Pi(\mathcal{B}, \theta)$ is a supporting hyperplane of $\mathcal{B}$.

Hence, we conclude that $(t_1, h_1) \in \partial\mathcal{B}$ is the only boundary point which intersects with $\Pi(\mathcal{B}, \theta)$, and we define $(t(\theta), h(\theta))$ to be this point $\blacksquare$

Parameter representations of convex contours (c.)

If the function $B(\mathcal{B}, \cdot)$ is differentiable, the mapping from angles to boundary points is given by the following explicit formula:

Proposition (Parameter representation of a contour)

Let $\mathcal{B} \subset \mathbb{R}^2$ be a compact and strictly convex set, and assume that $B(\mathcal{B}, \cdot)$ defined by (4) is differentiable. Then the boundary of $\mathcal{B}$ can be expressed as:

$$\partial\mathcal{B} = \{(t(\theta), h(\theta)) : \theta \in [0, 2\pi)\}$$

where:

$$\begin{pmatrix} t(\theta) \\ h(\theta) \end{pmatrix} = \begin{bmatrix} B(\mathcal{B}, \theta) & -B'(\mathcal{B}, \theta) \\ B'(\mathcal{B}, \theta) & B(\mathcal{B}, \theta) \end{bmatrix} \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}. \quad (8)$$

Parameter representations of convex contours (c.)

PROOF: Let $\theta \in [0, 2\pi)$ and let $\delta > 0$ be small, and consider the intersection between the two supporting hyperplanes $\Pi(\mathcal{B}, \theta)$ and $\Pi(\mathcal{B}, \theta + \delta)$, denoted (t, h).

This point is found by solving the equations:

$$t \cos(\theta) + h \sin(\theta) = B(\mathcal{B}, \theta),$$

$$t \cos(\theta + \delta) + h \sin(\theta + \delta) = B(\mathcal{B}, \theta + \delta).$$

with the solution:

$$t = \frac{\sin(\theta + \delta) B(\mathcal{B}, \theta) - \sin(\theta) B(\mathcal{B}, \theta + \delta)}{\sin(\delta)}$$

$$h = \frac{-\cos(\theta + \delta) B(\mathcal{B}, \theta) + \cos(\theta) B(\mathcal{B}, \theta + \delta)}{\sin(\delta)}$$

Parameter representations of convex contours (c.)

As $\delta \to 0$ the intersection point (t, h) will converge to a point in $\Pi(\mathcal{B}, \theta)$ which we denote by $(t(\theta), h(\theta))$.

Using l'Hôpital's rule it is easy to see that $(t(\theta), h(\theta))$ is given by:

$$\begin{pmatrix} t(\theta) \\ h(\theta) \end{pmatrix} = \begin{bmatrix} B(\mathcal{B}, \theta) & -B'(\mathcal{B}, \theta) \\ B'(\mathcal{B}, \theta) & B(\mathcal{B}, \theta) \end{bmatrix} \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

where $B'(\mathcal{B}, \theta)$ denotes the derivative of $B(\mathcal{B}, \theta)$.

As θ runs through all angles in $[0, 2\pi)$, the point $(t(\theta), h(\theta))$ will move along the boundary of the set $\mathcal{B}$. Thus, the environmental contour can be expressed as:

$$\partial\mathcal{B} = \{(t(\theta), h(\theta)) : \theta \in [0, 2\pi)\}. \tag{9}$$

A necessary condition for convexity

Later on, the set $\mathcal{B}$ will be constructed by first estimating the function $B(\mathcal{B}, \cdot)$, and then reconstruct $\mathcal{B}$ using the formula for the boundary $\partial\mathcal{B}$.

IMPORTANT: The function $B(\mathcal{B}, \cdot)$ cannot be *any arbitrary function*. Since the set $\mathcal{B}$ is convex, $B(\mathcal{B}, \cdot)$ must satisfy a certain condition.

GOAL: Derive a necessary condition on $B(\mathcal{B}, \cdot)$. This condition will then be used in the estimation process.

In order to derive this condition it is convenient to extend the function $B(\mathcal{B}, \cdot)$ to a function defined for all $\theta \in \mathbb{R}$.

Since the trigonometric functions $\cos(\cdot)$ and $\sin(\cdot)$ are periodic, the extended version of $B(\mathcal{B}, \cdot)$ is periodic as well and have the property that $B(\mathcal{B}, \theta) = B(\mathcal{B}, \theta \pm 2n\pi)$ for all $n \in \mathbb{N}$.

A necessary condition for convexity (cont.)

In order to investigate this further we assume that the $B(\mathcal{B}, \cdot)$ is two times differentiable, and consider the derivative of $(t(\theta), h(\theta))$:

$$\begin{aligned} t'(\theta) &= [B(\mathcal{B},\theta)\cos(\theta) - B'(\mathcal{B},\theta)\sin(\theta)]' \\ &= B'(\mathcal{B},\theta)\cos(\theta) - B(\mathcal{B},\theta)\sin(\theta) \\ &- B''(\mathcal{B},\theta)\sin(\theta) - B'(\mathcal{B},\theta)\cos(\theta) \\ &= -[B(\mathcal{B},\theta) + B''(\mathcal{B},\theta)]\sin(\theta) \end{aligned}$$

$$\begin{aligned} h'(\theta) &= [B'(\mathcal{B},\theta)\cos(\theta) + B(\mathcal{B},\theta)\sin(\theta)]' \\ &= B''(\mathcal{B},\theta)\cos(\theta) - B'(\mathcal{B},\theta)\sin(\theta) \\ &+ B'(\mathcal{B},\theta)\sin(\theta) + B(\mathcal{B},\theta)\cos(\theta) \\ &= [B(\mathcal{B},\theta) + B''(\mathcal{B},\theta)]\cos(\theta). \end{aligned}$$

That is, we have:

$$\begin{pmatrix} t'(\theta) \\ h'(\theta) \end{pmatrix} = [B(\mathcal{B},\theta) + B''(\mathcal{B},\theta)] \cdot \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}. \tag{10}$$

A necessary condition for convexity (cont.)

Lemma (Translated contour sets)

Let $\mathcal{B} \subset \mathbb{R}^2$ be a compact and strictly convex set, and let:

$$\tilde{\mathcal{B}} = \{(\tilde{t}, \tilde{h}) = (t - t_0, h - h_0) : (t, h) \in \mathcal{B}\} \tag{11}$$

for some point $(t_0, h_0) \in \mathbb{R}^2$. Then $B(\tilde{\mathcal{B}}, \theta)$ is given by:

$$B(\tilde{\mathcal{B}}, \theta) = B(\mathcal{B}, \theta) - t_0 \cos(\theta) - h_0 \sin(\theta),$$

for all $\theta \in \mathbb{R}$. Moreover, assuming that $B(\mathcal{B}, \cdot)$ is two times differentiable, we have:

$$B'(\tilde{\mathcal{B}}, \theta) = B'(\mathcal{B}, \theta) + t_0 \sin(\theta) - h_0 \cos(\theta),$$
$$B''(\tilde{\mathcal{B}}, \theta) = B''(\mathcal{B}, \theta) + t_0 \cos(\theta) + h_0 \sin(\theta).$$

A necessary condition for convexity (cont.)

PROOF: We definition of the *B*-function we have:

$$\begin{aligned}
B(\tilde{\mathcal{B}}, \theta) &= \sup_{(\tilde{t},\tilde{h}) \in \tilde{\mathcal{B}}} \{\tilde{t}\cos(\theta) + \tilde{h}\sin(\theta)\} \\
&= \sup_{(t,h) \in \mathcal{B}} \{(t - t_0)\cos(\theta) + (h - h_0)\sin(\theta)\} \\
&= \sup_{(t,h) \in \mathcal{B}} \{t\cos(\theta) + h\sin(\theta)\} - t_0\cos(\theta) - h_0\sin(\theta) \\
&= B(\mathcal{B}, \theta) - t_0\cos(\theta) - h_0\sin(\theta)
\end{aligned}$$

Hence, we also get:

$$\begin{aligned}
B'(\tilde{\mathcal{B}}, \theta) &= B'(\mathcal{B}, \theta) + t_0\sin(\theta) - h_0\cos(\theta), \\
B''(\tilde{\mathcal{B}}, \theta) &= B''(\mathcal{B}, \theta) + t_0\cos(\theta) + h_0\sin(\theta).
\end{aligned}$$

A necessary condition for convexity (cont.)

Lemma

Let $\mathcal{B} \subset \mathbb{R}^2$ be a compact and strictly convex set, and assume that $B(\mathcal{B}, \cdot)$ is two times differentiable. Then there exists a $\theta_0 \in (0, 2\pi)$ such that:

$$B(\mathcal{B}, \theta_0) + B''(\mathcal{B}, \theta_0) > 0.$$

PROOF: Let (t_0, h_0) be an interior point of $\mathcal{B}$. Then by definition of the B-function we have:

$$t_0 \cos(\theta) + h_0 \sin(\theta) < B(\mathcal{B}, \theta) \quad \text{for all } \theta \in [0, 2\pi)$$

We then let:

$$\tilde{\mathcal{B}} = \{(\tilde{t}, \tilde{h}) = (t - t_0, h - h_0) : (t, h) \in \mathcal{B}\}$$

A necessary condition for convexity (cont.)

Hence, by the Translation contour set lemma we have:

$$B(\tilde{\mathcal{B}}, \theta) = B(\mathcal{B}, \theta) - t_0 \cos(\theta) - h_0 \sin(\theta) > 0 \quad \text{for all } \theta \in [0, 2\pi)$$

Since $B(\tilde{\mathcal{B}}, \theta)$ extended to a function defined for all $\theta \in \mathbb{R}$, is periodic, it follows that the extended version of $B'(\tilde{\mathcal{B}}, \theta)$ is periodic as well.

In particular that $B'(\tilde{\mathcal{B}}, 0) = B'(\tilde{\mathcal{B}}, 2\pi)$.

Hence, by the mean value theorem, there exists a $\theta_0 \in (0, 2\pi)$ such that $B''(\tilde{\mathcal{B}}, \theta_0) = 0$.

A necessary condition for convexity (cont.)

From this it follows that:

$$B(\tilde{\mathcal{B}}, \theta_0) + B''(\tilde{\mathcal{B}}, \theta_0) > 0$$

By the Translation contour set lemma we also that:

$$\begin{aligned} B(\tilde{\mathcal{B}}, \theta_0) + B''(\tilde{\mathcal{B}}, \theta_0) &= B(\mathcal{B}, \theta_0) - t_0 \cos(\theta) - h_0 \sin(\theta) \\ &\quad + B''(\mathcal{B}, \theta_0) + t_0 \cos(\theta) + h_0 \sin(\theta) \\ &= B(\mathcal{B}, \theta_0) + B''(\mathcal{B}, \theta_0). \end{aligned}$$

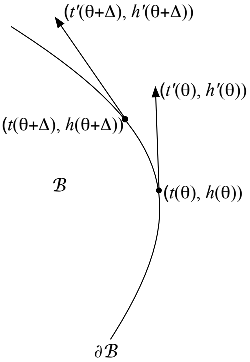
By combining the above we get that:

$$B(\mathcal{B}, \theta_0) + B''(\mathcal{B}, \theta_0) > 0$$

and thus the result is proved ■

A necessary condition for convexity (cont.)

As θ runs through $[0, 2\pi)$, the point $(t(\theta), h(\theta))$ runs counterclockwise through the boundary $\partial\mathcal{B}$.



The derivative $(t'(\theta), h'(\theta))$ is the tangent vector to $\partial\mathcal{B}$ at $(t(\theta), h(\theta))$.

A necessary condition for convexity (cont.)

The set $\mathcal{B}$ is convex if the angle between $(t'(\theta), h'(\theta))$ and $(t'(\theta+\Delta), h'(\theta+\Delta))$ is positive for any $\theta \in [0, 2\pi)$ and small $\Delta > 0$.

In order to check this, we define:

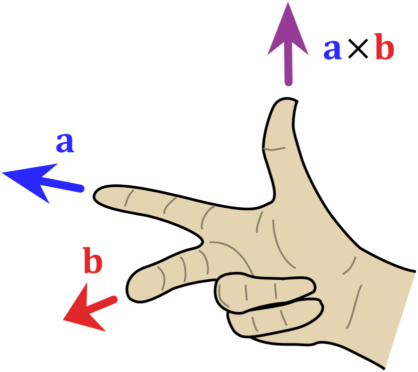
$$\boldsymbol{v}(\theta) = (t'(\theta), h'(\theta), 0), \quad \theta \in [0, 2\pi),$$

and calculate:

$$\boldsymbol{v}(\theta) \times \boldsymbol{v}(\theta+\Delta) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ t'(\theta) & h'(\theta) & 0 \\ t'(\theta+\Delta) & h'(\theta+\Delta) & 0 \end{vmatrix}$$

$$= (0,\ 0,\ t'(\theta) \cdot h'(\theta+\Delta) - h'(\theta) \cdot t'(\theta+\Delta))$$

A necessary condition for convexity (cont.)



By the *right-hand rule* of the cross-product the angle between $(t'(\theta), h'(\theta), 0)$ and $(t'(\theta + \Delta), h'(\theta + \Delta), 0)$ is positive if and only if:

$$t'(\theta) \cdot h'(\theta + \Delta) - h'(\theta) \cdot t'(\theta + \Delta) > 0.$$

A necessary condition for convexity (cont.)

We recall that:

$$\begin{pmatrix} t'(\theta) \\ h'(\theta) \end{pmatrix} = [B(\mathcal{B}, \theta) + B''(\mathcal{B}, \theta)] \cdot \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}.$$

Inserting this we get:

$$\begin{aligned} & t'(\theta) \cdot h'(\theta + \Delta) - h'(\theta) \cdot t'(\theta + \Delta) \\ &\qquad = [B(\mathcal{B}, \theta) + B''(\mathcal{B}, \theta)] \cdot [B(\mathcal{B}, \theta + \Delta) + B''(\mathcal{B}, \theta + \Delta)] \\ &\qquad\quad \cdot (-\sin(\theta)\cos(\theta + \Delta) + \cos(\theta)\sin(\theta + \Delta)) \\ &\qquad = [B(\mathcal{B}, \theta) + B''(\mathcal{B}, \theta)] \cdot [B(\mathcal{B}, \theta + \Delta) + B''(\mathcal{B}, \theta + \Delta)] \cdot \sin(\Delta). \end{aligned}$$

Since $\Delta > 0$ is small, we have $\sin(\Delta) > 0$.

A necessary condition for convexity (cont.)

Hence, the angle between $(t'(\theta), h'(\theta))$ and $(t'(\theta + \Delta), h'(\theta + \Delta))$ is positive if and only if:

$$[B(\mathcal{B}, \theta) + B''(\mathcal{B}, \theta)] \cdot [B(\mathcal{B}, \theta + \Delta) + B''(\mathcal{B}, \theta + \Delta)] > 0$$

for all $\theta \in [0, 2\pi)$ and small $\Delta > 0$.

This condition holds if and only if the sign of $B(\mathcal{B}, \theta) + B''(\mathcal{B}, \theta)$ is the same for all $\theta \in [0, 2\pi)$.

By the last Lemma there exists at least one $\theta_0 \in (0, 2\pi)$ such that:

$$B(\mathcal{B}, \theta_0) + B''(\mathcal{B}, \theta_0) > 0.$$

This implies that we must have:

$$B(\mathcal{B}, \theta) + B''(\mathcal{B}, \theta) > 0, \quad \text{for all } \theta \in [0, 2\pi)$$

A necessary condition for convexity (cont.)

Hence, we have shown the following important result:

Theorem (Convexity condition)

Let $\mathcal{B} \subset \mathbb{R}^2$ be a compact and strictly convex set, and assume that $B(\mathcal{B}, \cdot)$ is two times differentiable. Then we have:

$$B(\mathcal{B}, \theta) + B''(\mathcal{B}, \theta) > 0 \text{ for all } \theta \in [0, 2\pi). \tag{12}$$

A necessary condition for convexity (cont.)

Given a periodic function B which does not satisfy (12), it is easy to modify this function so that the condition is satisfied.

Proposition (Convexity correction)

Let $C(\cdot)$ be a periodic function with period 2π which is two times differentiable. Assuming that both C and C'' are bounded, there exists a constant C_0 such that the function $\tilde{C}(\cdot) = C_0 + C(\cdot)$ satisfies (12).

A necessary condition for convexity (cont.)

PROOF: We let:

$$c = \inf_{\theta \in [0, 2\pi)} [C(\theta) + C''(\theta)]$$

Since both C and C'' are bounded, c must be finite. If $c > 0$, $C(\cdot)$ satisfies (12). We may then let $C_0 = 0$. Hence, $\tilde{C}(\cdot) = C(\cdot)$, and thus, $\tilde{C}(\cdot)$ obviously satisfies (12) as well.

On the other hand, if $c \leq 0$, we let C_0 be some number greater than $-c$. Since $\tilde{C}'' = C''$, it follows that for all $\theta \in [0, 2\pi)$ we have:

$$\begin{aligned} \tilde{C}(\theta) + \tilde{C}''(\theta) &= C_0 + C(\theta) + C''(\theta) \\ &> -c + C(\theta) + C''(\theta) \\ &\geq -c + c = 0. \end{aligned}$$

Hence, we conclude that $\tilde{C}(\cdot)$ satisfies (12) ■