Environmental contours – part 2

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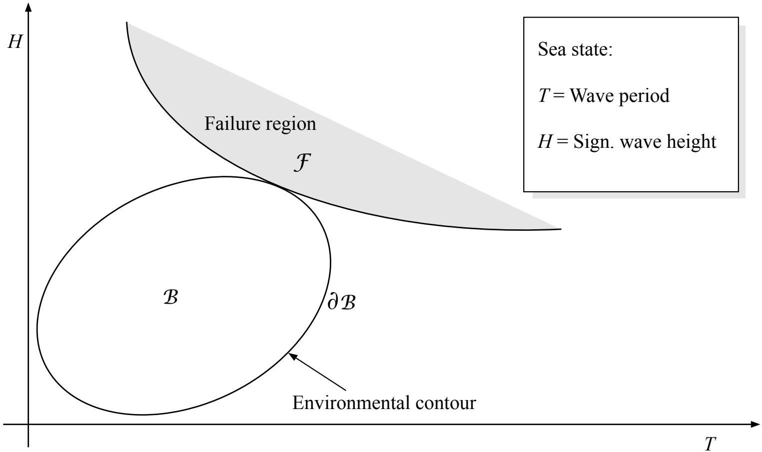
Environment contour and failure region



Sea state:

T = Wave period

H = Sign. wave height

Exceedance probability

The *exceedance probability* of $\mathcal{B}$ with respect to $\mathcal{E}$ is defined as:

$$P_e(\mathcal{B}, \mathcal{E}) = \sup\{P[(T, H) \in \mathcal{F}] : \mathcal{F} \in \mathcal{E}\}.$$

NOTE: The exceedance probability is an upper bound on the failure probability of the structure assuming that the true failure region is a member of the family $\mathcal{E}$.

For a given target exceedance probability $p_e \in (0, 0.5)$ our goal is to find a minimal set $\mathcal{B}$ such that:

$$P_e(\mathcal{B}, \mathcal{E}) \leq p_e \tag{1}$$

If the set $\mathcal{B}$ satisfies (1), then $\partial\mathcal{B}$ is said to be a *valid* environmental contour.

Convex contours

In the following we only consider contour sets $\mathcal{B}$ which are *compact* and *convex*. Furthermore, we assume that all the sets in $\mathcal{E}$ are *convex*.

For a given compact and convex set $\mathcal{B}$ we introduce the following families of sets:

$$\begin{aligned}
\mathcal{P}(\mathcal{B}) &= \text{The family of supporting hyperplanes of } \mathcal{B}, \\
\mathcal{P}^{+}(\mathcal{B}) &= \text{The family of supporting halfspaces of } \mathcal{B}, \\
\mathcal{P}^{-}(\mathcal{B}) &= \text{The family of halfspaces opposite} \\
&\quad \text{to supporting halfspaces of } \mathcal{B}
\end{aligned}$$

These families play a central part in the construction of environmental contours.

Convex contours (cont.)

Proposition (Parametric families)

Let $\mathcal{B} \subset \mathbb{R}^2$ be a compact and convex set, and define:

$$\Pi(\mathcal{B}, \theta) = \{(t, h) : t\cos(\theta) + h\sin(\theta) = B(\mathcal{B}, \theta)\}$$
$$\Pi^+(\mathcal{B}, \theta) = \{(t, h) : t\cos(\theta) + h\sin(\theta) \geq B(\mathcal{B}, \theta)\},$$
$$\Pi^-(\mathcal{B}, \theta) = \{(t, h) : t\cos(\theta) + h\sin(\theta) \leq B(\mathcal{B}, \theta)\}.$$

where $B(\mathcal{B}, \theta) = \sup_{(t,h)\in\mathcal{B}}[t\cos(\theta) + h\sin(\theta)]$*.*

We then have:

$$\mathcal{P}(\mathcal{B}) = \{\Pi(\mathcal{B}, \theta) : \theta \in [0, 2\pi)\},$$
$$\mathcal{P}^+(\mathcal{B}) = \{\Pi^+(\mathcal{B}, \theta) : \theta \in [0, 2\pi)\},$$
$$\mathcal{P}^-(\mathcal{B}) = \{\Pi^-(\mathcal{B}, \theta) : \theta \in [0, 2\pi)\}.$$

Convex contours (cont.)

Proposition (Parametric representation)

Let $\mathcal{B} \subset \mathbb{R}^2$ be a compact and convex set, and let $\mathcal{E}$ be the family of convex sets such that $\mathcal{F} \cap \mathcal{B} \subseteq \partial\mathcal{B}$ for all $\mathcal{F} \in \mathcal{E}$. Then we have:

$$P_e(\mathcal{B}, \mathcal{E}) = \sup_{\theta \in [0, 2\pi)} \{P[(T, H) \in \Pi^+(\mathcal{B}, \theta)]\}. \tag{2}$$

Moreover, the set $\mathcal{B}$ can be expressed as:

$$\mathcal{B} = \bigcap_{\theta \in [0, 2\pi)} \Pi^-(\mathcal{B}, \theta) \tag{3}$$

Parameter representations of convex contours

If the function $B(\mathcal{B}, \cdot)$ is differentiable, the mapping from angles to boundary points is given by the following explicit formula:

Proposition (Parameter representation of a contour)

Let $\mathcal{B} \subset \mathbb{R}^2$ be a compact and strictly convex set, and assume that $B(\mathcal{B}, \cdot)$ is differentiable. Then the boundary of $\mathcal{B}$ can be expressed as:

$$\partial\mathcal{B} = \{(t(\theta), h(\theta)) : \theta \in [0, 2\pi)\}$$

where:

$$\begin{pmatrix} t(\theta) \\ h(\theta) \end{pmatrix} = \begin{bmatrix} B(\mathcal{B}, \theta) & -B'(\mathcal{B}, \theta) \\ B'(\mathcal{B}, \theta) & B(\mathcal{B}, \theta) \end{bmatrix} \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}. \tag{4}$$

A necessary condition for convexity

Theorem (Convexity condition)

Let $\mathcal{B} \subset \mathbb{R}^2$ be a compact and strictly convex set, and assume that $B(\mathcal{B}, \cdot)$ is two times differentiable. Then we have:

$$B(\mathcal{B}, \theta) + B''(\mathcal{B}, \theta) > 0 \text{ for all } \theta \in [0, 2\pi). \tag{5}$$

Valid convex environmental contours

We recall that if $\mathcal{B} \subset \mathbb{R}^2$ is a set and $\mathcal{E}$ is a family of failure regions, then $\partial\mathcal{B}$ is a *valid* environmental contour if:

$$P_e(\mathcal{B}, \mathcal{E}) = \sup\{P[(T, H) \in \mathcal{F}] : \mathcal{F} \in \mathcal{E}\} \leq p_e$$

where $p_e \in (0, 0.5)$ denotes the target exceedance probability.

Assuming that $\mathcal{B} \subset \mathbb{R}^2$ be a compact and strictly convex set, and that all failure regions are convex, we may now write this as follows:

$$\begin{aligned} P_e(\mathcal{B}, \mathcal{E}) &= \sup_{\theta \in [0, 2\pi)} \{P[(T, H) \in \Pi^+(\mathcal{B}, \theta)]\} \\ &= \sup_{\theta \in [0, 2\pi)} \{P[T\cos(\theta) + H\sin(\theta) \geq B(\mathcal{B}, \theta)]\} \leq p_e, \end{aligned}$$

where as before: $B(\mathcal{B}, \theta) = \sup_{(t,h) \in \mathcal{B}}[t\cos(\theta) + h\sin(\theta)]$.

Valid convex environmental contours (cont.)

We now let $C(\theta)$ be defined for all angles $\theta \in [0, 2\pi)$ as:

$$C(\theta) = \inf\{y : P[Y(\theta) > y] \leq p_e\}, \tag{6}$$

where $Y(\theta) = T\cos(\theta) + H\sin(\theta)$. The function C is referred to as the p_e-level *percentile function* of the joint distribution of (T, H).

For $\theta \in [0, 2\pi)$ we also introduce :

$$\begin{aligned}
\Pi(\theta) &= \{(t, h) : t\cos(\theta) + h\sin(\theta) = C(\theta)\} \\
\Pi^+(\theta) &= \{(t, h) : t\cos(\theta) + h\sin(\theta) \geq C(\theta)\}, \\
\Pi^-(\theta) &= \{(t, h) : t\cos(\theta) + h\sin(\theta) \leq C(\theta)\}.
\end{aligned}$$

Valid convex environmental contours (cont.)

By the definition of $C(\theta)$ it follows that for all $\theta \in [0, 2\pi)$ we have:

$$
\begin{aligned}
&P[(T, H) \in \Pi^+(\theta)] \\
&\qquad = P[T\cos(\theta) + H\sin(\theta) \geq C(\theta)] \\
&\qquad = P[T\cos(\theta) + H\sin(\theta) > C(\theta)] = p_e
\end{aligned}
$$

If we can find a convex set $\mathcal{B}$ such that $B(\mathcal{B}, \cdot) \geq C(\cdot)$, it follows that:

$$
\begin{aligned}
P_e(\mathcal{B}, \mathcal{E}) &= \sup_{\theta \in [0, 2\pi)} \{P[(T, H) \in \Pi^+(\mathcal{B}, \theta)]\} \\
&= \sup_{\theta \in [0, 2\pi)} \{P[T\cos(\theta) + H\sin(\theta) \geq B(\mathcal{B}, \theta)]\} \\
&\leq \sup_{\theta \in [0, 2\pi)} \{P[T\cos(\theta) + H\sin(\theta) \geq C(\theta)]\} = p_e
\end{aligned}
$$

implying that $\partial\mathcal{B}$ is a valid environmental contour.

Valid convex environmental contours (cont.)

Theorem

If there exists a compact and convex set $\mathcal{B}$ such that $B(\mathcal{B}, \theta) = C(\theta)$ for all $\theta \in [0, 2\pi)$. Then $\mathcal{B}$ is the minimal compact and convex set such that $\partial\mathcal{B}$ is a valid environmental contour, and the set $\mathcal{B}$ is given by:

$$\mathcal{B} = \bigcap_{\theta \in [0,2\pi)} \Pi^{-}(\theta) \tag{7}$$

If $\mathcal{B}$ is strictly convex, and $C(\cdot)$ is differentiable, $\partial\mathcal{B}$ is given by:

$$\partial\mathcal{B} = \{(t(\theta), h(\theta)) : \theta \in [0, 2\pi)\},$$

where:

$$\begin{pmatrix} t(\theta) \\ h(\theta) \end{pmatrix} = \begin{bmatrix} C(\theta) & -C'(\theta) \\ C'(\theta) & C(\theta) \end{bmatrix} \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \tag{8}$$

Valid convex environmental contours (cont.)

Theorem

If $C(\cdot)$ is two times differentiable, a necessary condition for the existence of a strictly convex set $\mathcal{B}$ such that $B(\mathcal{B}, \theta) = C(\theta)$ for all $\theta \in [0, 2\pi)$ is that:

$$C(\theta) + C''(\theta) > 0 \text{ for all } \theta \in [0, 2\pi). \tag{9}$$

NOTE: The function $C(\cdot)$ is determined by the joint distribution of T and H. It is possible to construct distributions where $C(\cdot)$ does *not* satisfy (9).

In such cases the contour defined by the formula (8), will not be the boundary of a convex set. When this happens, $C(\cdot)$ must be adjusted.

Example: A bivariate normal distribution

Assume that T and H are independent normally distributed with $E(T) = \mu_T$, $E(H) = \mu_H$ and $SD(T) = SD(H) = \sigma$.

Thus, $\tilde{T} = (T - \mu_T)/\sigma$ and $\tilde{H} = (H - \mu_H)/\sigma$ are independent standard normally distributed.

By the rotational symmetry property of the standard bivariate normal distribution it follows that:

$$P[\tilde{T}\cos(\theta) + \tilde{H}\sin(\theta) > q_e] = p_e, \text{ for all } \theta \in [0, 2\pi),$$

where q_e denotes the $(1 - p_e)$-percentile of the standard normal distribution.

Hence, for all $\theta \in [0, 2\pi)$ we get that:

$$P[T\cos(\theta) + H\sin(\theta) > \mu_T\cos(\theta) + \mu_H\sin(\theta) + \sigma q_e] = p_e.$$

A bivariate normal distribution (cont.)

Thus, we have:

$$C(\theta) = \mu_T \cos(\theta) + \mu_H \sin(\theta) + \sigma q_e,$$
$$C'(\theta) = -\mu_T \sin(\theta) + \mu_H \cos(\theta)$$

By inserting these expressions into (8) we obtain:

$$\begin{aligned} t(\theta) &= C(\theta)\cos(\theta) - C'(\theta)\sin(\theta) \\ &= [\mu_T \cos(\theta) + \mu_H \sin(\theta) + \sigma q_e] \cos(\theta) \\ &- [-\mu_T \sin(\theta) + \mu_H \cos(\theta)] \sin(\theta) \\ &= \mu_T + \sigma q_e \cos(\theta) \end{aligned}$$

$$\begin{aligned} h(\theta) &= C'(\theta)\cos(\theta) + C(\theta)\sin(\theta) \\ &= [-\mu_T \sin(\theta) + \mu_H \cos(\theta)] \cos(\theta) \\ &+ [\mu_T \cos(\theta) + \mu_H \sin(\theta) + \sigma q_e] \sin(\theta) \\ &= \mu_H + \sigma q_e \sin(\theta) \end{aligned}$$

A bivariate normal distribution (cont.)

Putting it all together we conclude that $\partial\mathcal{B}$ can be written as:

$$\begin{pmatrix} t(\theta) \\ h(\theta) \end{pmatrix} = \begin{pmatrix} \mu_T \\ \mu_H \end{pmatrix} + \begin{bmatrix} \sigma q_e & 0 \\ 0 & \sigma q_e \end{bmatrix} \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \begin{pmatrix} \mu_T \\ \mu_H \end{pmatrix} + \sigma q_e \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

Moreover, the second derivative of C is given by:

$$C''(\theta) = [-\mu_T \sin(\theta) + \mu_H \cos(\theta)]' = -\mu_T \cos(\theta) - \mu_H \sin(\theta).$$

Hence, we get:

$$\begin{aligned} C(\theta) + C''(\theta) &= [\mu_T \cos(\theta) + \mu_H \sin(\theta) + \sigma q_e] + [-\mu_T \cos(\theta) - \mu_H \sin(\theta)] \\ &= \sigma q_e > 0, \text{ for all } \theta \in [0, 2\pi). \end{aligned}$$

In this case $\partial\mathcal{B}$ is a circle with radius σq_e centered at (μ_T, μ_H). Thus, $\partial\mathcal{B}$ is a simple closed curve, and $\mathcal{B}$ is indeed a convex set.

Estimating environmental contours

Estimating environmental contours using Monte Carlo simulations

Estimating environmental contours (cont.)

Unfortunately, finding the distribution of $T\cos(\theta) + H\sin(\theta)$ for all $\theta \in [0, 2\pi)$ is typically not easy. Thus, finding exact expressions for the C-function analytically is difficult as well.

By *difficult* we mean that it is possible at least in principle, but it involves a lot of time consuming numerical integration.

In a given practical situation the C-function can instead be estimated pointwise using Monte Carlo simulations.

In the following we shall see how this can be done.

Estimating $C(\theta)$ using Monte Carlo simulation

Assume that we have a sample from the joint distribution of (T, H) generated using Monte Carlo simulation:

$$(T_1, H_1), \dots, (T_n, H_n)$$

For a given angle $\theta \in [0, 2\pi)$ we calculate the projections of these points onto the unit vector $(\cos(\theta), \sin(\theta))$, i.e.:

$$Y_i(\theta) = T_i \cos(\theta) + H_i \sin(\theta), \quad i = 1, \dots, n$$

These projections are then sorted in ascending order:

$$Y_{(1)}(\theta) \leq Y_{(2)}(\theta) \leq \cdots \leq Y_{(n)}(\theta).$$

Estimating $C(\theta)$ using crude Monte Carlo (cont.)

Assuming that $k \leq n$ is an integer such that:

$$\frac{k}{n} \approx 1 - p_e.$$

Then $C(\theta)$ can be estimated by:

$$\hat{C}(\theta) = Y_{(k)}(\theta)$$

Challenges using crude Monte Carlo simulation

- In many typical applications p_e can be very small, i.e., less than 0.1%. In such cases a large number of simulations are needed in order to obtain stable estimates.
- Processing the results in order to obtain the contours can be very time consuming.
- Storing a large number of simulation results in the computer memory can represent a challenge.
- Most of the simulations yield results close to the central area of the joint distribution, and thus very few results provide information about the contour area.

An improved simulation method will be given later.

Polygon approximation of $\mathcal{B}$

At this stage we assume that we have estimated the C-function for a suitable set of angles $\theta_1, \dots, \theta_n \in [0, 2\pi)$, and we let $\hat{C}(\theta_1), \dots, \hat{C}(\theta_n)$ denote the corresponding estimates. Using these estimates, we define:

$$\hat{\Pi}^+(\theta_i) = \{(t,h) : t\cos(\theta) + h\sin(\theta) \geq \hat{C}(\theta_i)\}$$
$$\hat{\Pi}(\theta_i) = \{(t,h) : t\cos(\theta) + h\sin(\theta) = \hat{C}(\theta_i)\}$$
$$\hat{\Pi}^-(\theta_i) = \{(t,h) : t\cos(\theta) + h\sin(\theta) \leq \hat{C}(\theta_i)\}$$

$\mathcal{B}$ can then be approximated by a polygon of the following form:

$$\hat{\mathcal{B}} = \bigcap_{i=1}^{n} \hat{\Pi}^-(\theta_i).$$

By considering the intersections between consecutive hyperplanes $\Pi(\theta_i)$ and $\Pi(\theta_{i+1})$, we find the corners of the polygon, and hence also the polygon itself.

Polygon approximation of $\mathcal{B}$ (cont.)

Estimating the polygon $\hat{\mathcal{B}}$ using only a few simulations and hyperplanes:

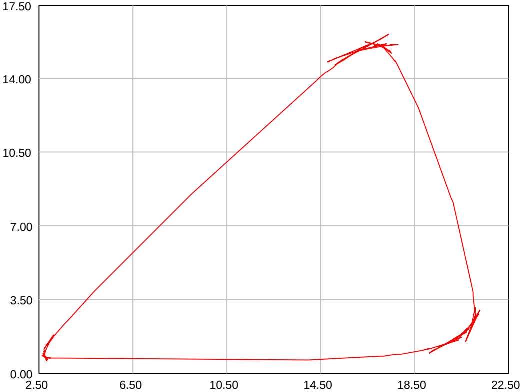


Figure: 1000 simulations, $n = 90$.

Polygon approximation of $\mathcal{B}$ (cont.)

By increasing the number of simulations and hyperplanes, a smoother contour is obtained:

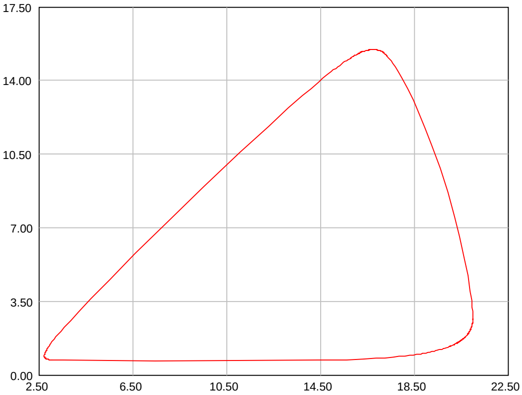


Figure: 1000000 simulations, $n = 360$.

Polygon approximation of $\mathcal{B}$ (cont.)

If we zoom in on the border of $\hat{\mathcal{B}}$, we still find substantial "irregularities":

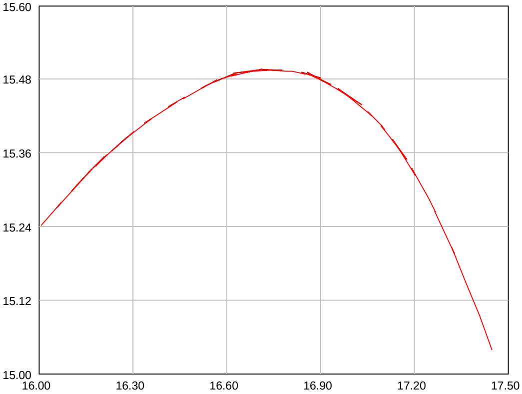


Figure: 1000000 simulations, $n = 360$.

Supporting hyperplanes and contours

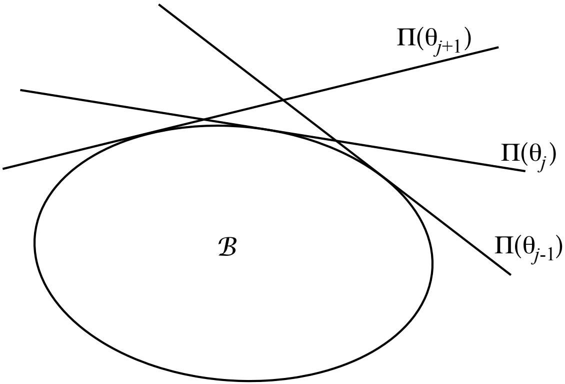


Figure: Ideal case: All hyperplanes support $\mathcal{B}$

Supporting hyperplanes and contours (cont.)

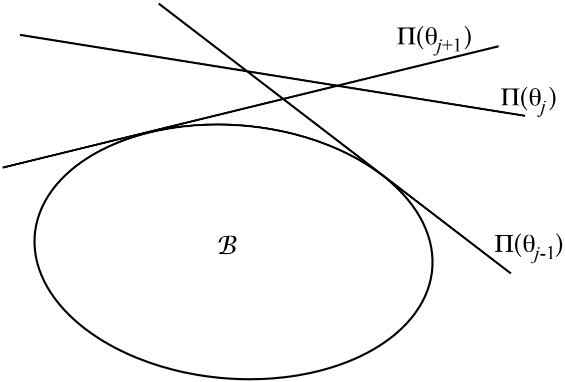


Figure: Irregular case: The hyperplane $\Pi(\theta_j)$ does *not* support $\mathcal{B}$

Numerical example based on MC simulation

EXAMPLE: An environmental contour estimated by using Monte Carlo simulation.

We let $p_e = 1.37 \cdot 10^{-5}$, which corresponds to a return period of 25 years.

The joint long-term models for *significant wave height*, denoted by H, and *wave period* denoted by T is given by:

$$f_{T,H}(t,h) = f_H(h) f_{T|H}(t|h)$$

where a three-parameter Weibull distribution is used for the significant wave height, H, and a lognormal conditional distribution is used for the wave period, T.

Numerical example (cont.)

The Weibull distribution is parameterized by a location parameter, γ, a scale parameter α, and a shape parameter β:

$$f_H(h) = \frac{\beta}{\alpha}\left(\frac{h-\gamma}{\alpha}\right)^{\beta-1} e^{-[(h-\gamma)/\alpha]^{\beta}}, \quad h \geq \gamma.$$

The lognormal distribution has two parameters, the log-mean μ and the log-standard deviation σ and is expressed as:

$$f_{T|H}(t|h) = \frac{1}{t\sqrt{2\pi}} e^{-[(\ln(t)-\mu)^2/(2\sigma^2)]}, \quad t \geq 0,$$

Numerical example (cont.)

The dependence between H and T is modelled by letting the parameters μ and σ be expressed in terms of H as follows:

$$\mu = E[\ln(T)|H = h] = a_1 + a_2 h^{a_3},$$

$$\sigma = SD[\ln(T)|H = h] = b_1 + b_2 e^{b_3 h}.$$

The parameters $a_1, a_2, a_3, b_1, b_2, b_3$ are estimated using available data from the relevant geographical location.

Numerical example based on MC simulation (cont.)

TOTAL SEA – WEST OF SHETLAND

Table: Fitted parameter for the three-parameter Weibull distribution for significant wave heights

α	β	γ
2.259	1.285	0.701

Table: Fitted parameter for the conditional log-normal distribution for wave periods

	$i = 1$	$i = 2$	$i = 3$
a_i	1.069	0.898	0.243
b_i	0.025	0.263	-0.148

Numerical example based on MC simulation (cont.)

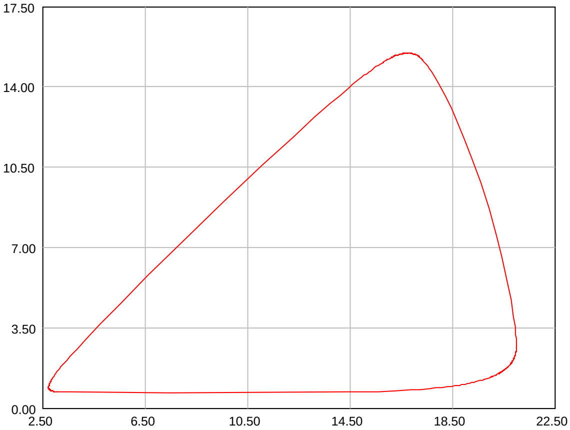
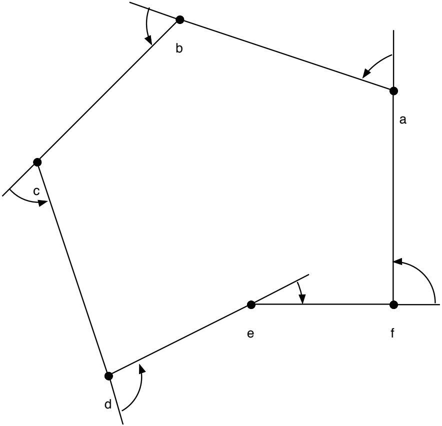


Figure: 1000000 simulations, $n = 360$.

Measuring angles along an environmental contour



Estimated angles along $\partial\mathcal{B}$.

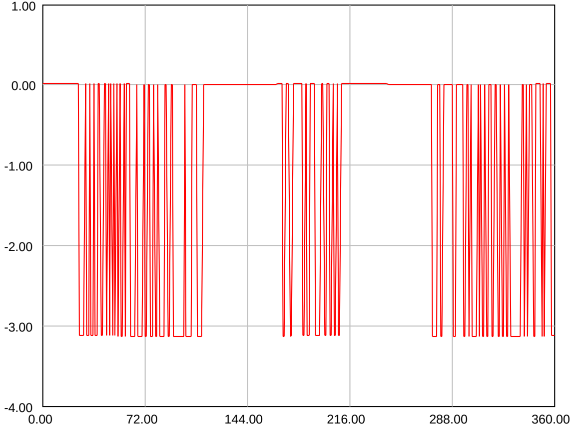


Figure: 1000000 simulations, $n = 360$.

Estimated values for $C(\theta) + C''(\theta)$

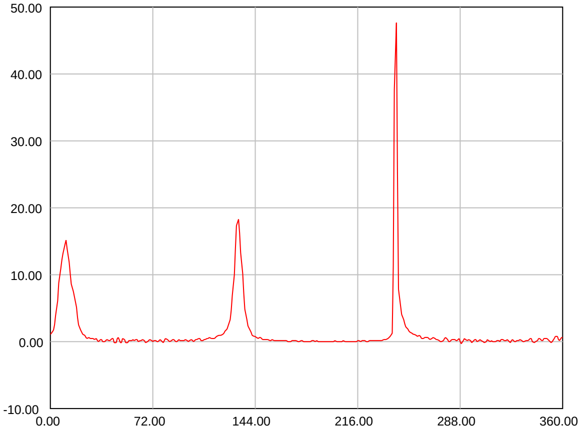


Figure: 1000000 simulations, $n = 360$.

Smoothing the estimated C-curve

To get rid of the loops along the contour, we may use a smoothed version of the estimated C-curve.

$$\tilde{C}(\theta_j) = \frac{\sum_{i=-k}^{+k} \omega_i \hat{C}(\theta_{j+i})}{\sum_{i=-k}^{+k} \omega_i}, \quad j = 1, \ldots, n,$$

for suitable integer $k \geq 0$ and weights $\omega_{-k}, \ldots, \omega_{+k}$.

NOTE: In the above formula the indices are *"looped"*, so that $\theta_{n+i} = \theta_i$, $i = 1, 2, \ldots, k$, while $\theta_{1-i} = \theta_{n+1-i}$, $i = 1, 2, \ldots, k$.

In the following example we have used $k = 5$ and:

$$\omega_{-i} = \omega_{+i} = (6 - i), \quad i = 0, 1, \ldots, 5.$$

Original (red) and smoothed (green) C-curves

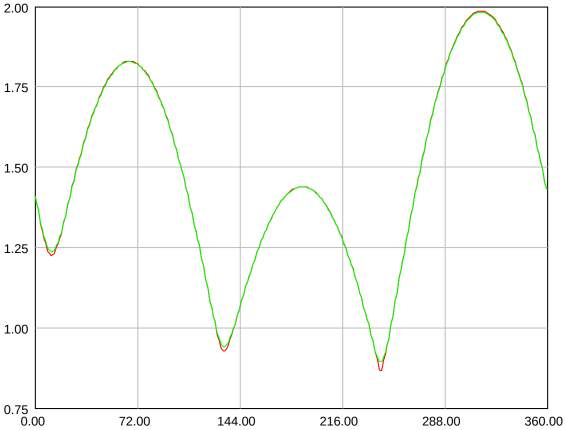


Figure: 1000000 simulations, $n = 360$.

Estimated angles along $\partial\mathcal{B}$ smoothed

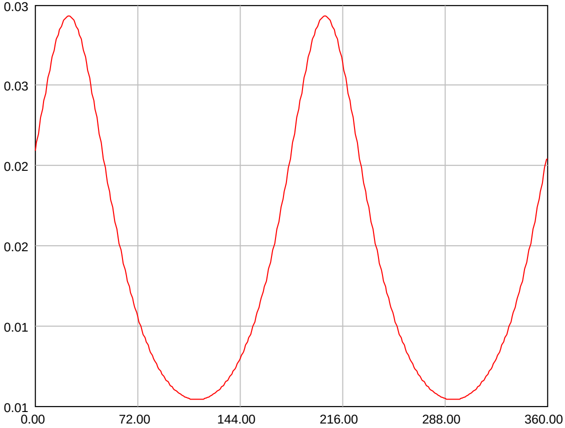


Figure: 1000000 simulations, $n = 360$.

Estimated values for $C(\theta) + C''(\theta)$ smoothed

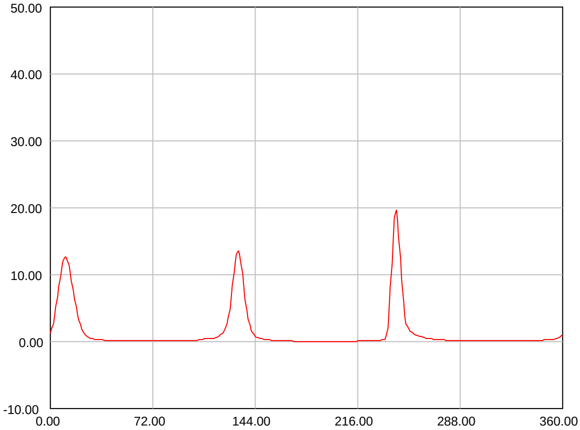


Figure: 1000000 simulations, $n = 360$.

Original and smoothed contours

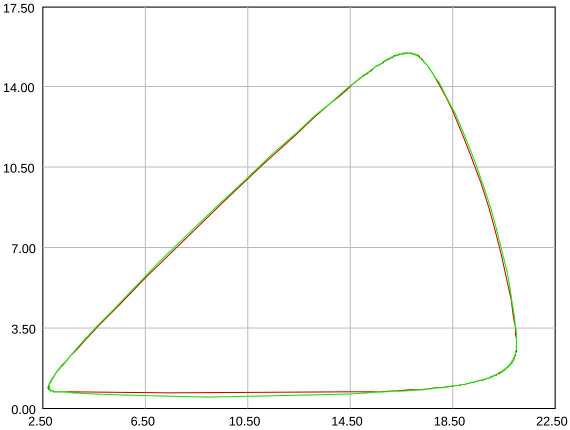


Figure: 1000000 simulations, $n = 360$.