Multistate systems and importance measures - part 1

Arne Bang Huseby

University of Oslo, Norway

STK 4400

Continuous-Time Markov Chains

A continuous-time Markov chain with stationary transition probabilities and state space $\mathcal{X}$ is a stochastic process such that:

- The times spent in the different states are independent random variables.
- The amount of time spent in state $i \in \mathcal{X}$ is exponentially distributed with mean μ_i.
- When the process leaves state i, it enters state j with some transition probability P_{ij} where:

$$P_{ii} = 0, \quad \text{for all } i \in \mathcal{X}$$

$$\sum_{j \in \mathcal{X}} P_{ij} = 1, \quad \text{for all } i \in \mathcal{X}$$

- The transitions follow a discrete-time Markov chain.
- If the amount of time spent in state $i \in \mathcal{X}$ has a general distribution with mean μ_i, the process is called a *semi-Markov* process.

Binary systems

A *binary system* is an ordered pair (C, ϕ) where:

$$C = \{1, \ldots, n\} \text{ is the component set.}$$

$$X_i(t) = I(\text{Component } i \text{ is functioning at time } t), \quad i \in C$$

$$\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t)) = \text{The component state vector}$$

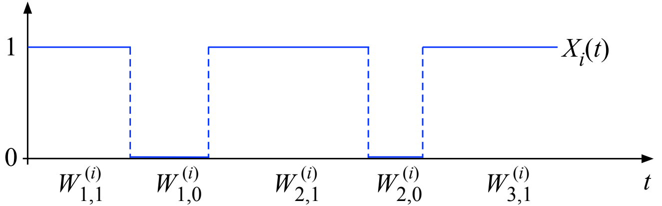
$$\phi(t) = I(\text{The system is functioning at time } t)$$

The function ϕ is called the *structure function* of the system, and we assume that:

$$\phi(t) = \phi(\boldsymbol{X}(t))$$

We also assume that $\phi(\boldsymbol{X}(t))$ is *non-decreasing* in $\boldsymbol{X}$.

Component processes



The state of the ith component is described by the *stochstic process* $X_i(t)$. For *repairable components* the component state jumps up and down between the *functioning state*, 1, and the *failed state* 0.

For $k = 1, 2, \ldots$ we introduce the *waiting times*:

$$W_{k,1}^{(i)} = \text{The } k\text{th lifetime of component } i$$

$$W_{k,0}^{(i)} = \text{The } k\text{th repair time of component } i$$

Component processes (cont.)

We assume that all the waiting times for all the components are *independent*, and that for $j = 0, 1$:

$$W_{1,j}^{(i)}, W_{2,j}^{(i)}, \ldots \text{ are identically distributed with mean } \mu_j^{(i)} < \infty.$$

We also introduce the *reliability* of the ith component at time t:

$$p_i(t) = P(X_i(t) = 1), \quad i \in C.$$

By *renewal theory* it can be shown that:

$$\lim_{t \to \infty} p_i(t) = p_i = \frac{\mu_1^{(i)}}{\mu_1^{(i)} + \mu_0^{(i)}}, \quad i \in C.$$

Component processes (cont.)

If all the waiting times are exponentially distributed, the component process $X_i(t)$ is a *Markov process*, and the transition probability matrix for the *built-in* Markov chain is given by:

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

In the general case where the waiting times can have arbitrary distributions, the component process is a *semi-Markov* process.

System reliability

The reliability of the system at time t is given by:

$$h(t) = P(\phi(t) = 1) = P(\phi(\boldsymbol{X}(t)) = 1) = h(\boldsymbol{p}(t)),$$

where $\boldsymbol{p}(t) = (p_1(t), \ldots, p_n(t))$.

By *renewal theory* it can be shown that:

$$\lim_{t \to \infty} h(t) = h(\boldsymbol{p}),$$

where $\boldsymbol{p} = \lim_{t \to \infty} \boldsymbol{p}(t) = (p_1, \ldots, p_n)$.

Finding efficient methods for calculating $h(\boldsymbol{p}(t))$ or $h(\boldsymbol{p})$ is a main topic within the field of *reliability theory*. For complex systems the reliability can be estimated using Monte Carlo simulation.

Importance Measures

Two main reasons for calculating importance of components in a system:

- Importance measures permit the analyst to determine which components merit the most additional research and development to improve overall system reliability.
- Importance measures may be used in diagnostics as a way of ranking components with respect to how likely they are to have caused problems.

Criticality

A component, $i \in C$, is said to be *critical* for the system at time t if:

$$\phi(0_i, \boldsymbol{X}(t)) \neq \phi(1_i, \boldsymbol{X}(t)).$$

Now introduce the following notation:

$$X_i^+(t) = \text{The } \textit{next state} \text{ of component } i \text{ at time } t$$

$$X_i^-(t) = \text{The } \textit{previous state} \text{ of component } i \text{ at time } t$$

Since each component only has two possible states, we have:

$$X_i^+(t) = X_i^-(t) = \begin{cases} 0 & \text{for } X_i(t) = 1 \\ 1 & \text{for } X_i(t) = 0 \end{cases}$$

NOTE: For $t = 0$, the notion of a previous state is *not defined*. However, assuming that $X(0) = 1$, we avoid this problem by simply defining $X_i^-(0) = 0$.

Criticality, cont.

It follows that component i is critical for the system at time t if:

$$\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t)). \tag{1}$$

or alternatively:

$$\phi(X_i^-(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t)). \tag{2}$$

That is, component i is critical at time t if changing the component to its next state would result in a system state change as well.

Alternatively, component i is critical at time t if changing the component to its previous state would result in a system state change as well.

The Birnbaum measure of component importance

The *Birnbaum measure* of importance of component $i \in C$ at time t, denoted $I_B^{(i)}(t)$, is the probability that the component is critical at time t.

Using our notation we have:

$$I_B^{(i)}(t) = P[\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t))]$$

$$= P[\phi(X_i^-(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t))]$$

Reliability importance examples (cont.)

Let (C, ϕ) be a 2-out-of-3 system. It is then easy to show that:

$$\phi(\boldsymbol{X}(t)) = X_1(t)X_2(t) + X_1(t)X_3(t) + X_2(t)X_3(t) - 2X_1(t)X_2(t)X_3(t)$$

In order to find $I_B^{(1)}(t)$, we note that:

$$\phi(1_1, \boldsymbol{X}(t)) = X_2(t) + X_3(t) - X_2(t)X_3(t)$$
$$\phi(0_1, \boldsymbol{X}(t)) = X_2(t)X_3(t)$$

Hence, we get that:

$$\begin{aligned} I_B^{(1)}(t) &= P[\phi(X_1(t), \boldsymbol{X}(t)) \neq \phi(X_1^+(t), \boldsymbol{X}(t))] \\ &= P[\phi(1_1, \boldsymbol{X}(t)) \neq \phi(0_1, \boldsymbol{X}(t))] \\ &= P[\phi(1_1, \boldsymbol{X}(t)) - \phi(0_1, \boldsymbol{X}(t)) = 1] \\ &= P[X_2(t) + X_3(t) - 2X_2(t)X_3(t) = 1] \\ &= p_2(t) + p_3(t) - 2p_2(t)p_3(t) \end{aligned}$$

Reliability importance examples (cont.)

By similar arguments we get that:

$$I_B^{(2)}(t) = p_1(t) + p_3(t) - 2p_1(t)p_3(t)$$
$$I_B^{(3)}(t) = p_1(t) + p_2(t) - 2p_1(t)p_2(t)$$

EXAMPLE $p_1(t) = 0.5 + z,\ p_2(t) = 0.5,\ p_3(t) = 0.5 - z$, where $0 < z < 0.5$.

$$\begin{aligned} I_B^{(1)}(t) &= p_2(t) + p_3(t) - 2p_2(t)p_3(t) \\ &= 0.5 + (0.5 - z) - 2 \cdot 0.5 \cdot (0.5 - z) = 0.5 \end{aligned}$$

$$\begin{aligned} I_B^{(2)}(t) &= p_1(t) + p_3(t) - 2p_1(t)p_3(t) \\ &= (0.5 + z) + (0.5 - z) - 2(0.5 + z)(0.5 - z) = 0.5 + 2z^2 \end{aligned}$$

$$\begin{aligned} I_B^{(3)}(t) &= p_1(t) + p_2(t) - 2p_1(t)p_2(t) \\ &= (0.5 + z) + 0.5 - 2 \cdot (0.5 + z) \cdot 0.5 = 0.5 \end{aligned}$$

Multistate systems

A *multistate system* is an ordered pair (C, ϕ) where:

$$C = \{1, \ldots, n\} \text{ is the component set.}$$
$$\phi = \phi(t) = \text{The state of the system at time } t$$

In a multistate system the components have multiple states:

$$S_i = \{0, 1, \ldots, r_i\} = \text{The state set of component } i,\ i \in C.$$

We also introduce the component state processes:

$$X_i(t) = \text{The state of component } i \text{ at time } t, \quad i \in C$$
$$\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$$

Just as in the binary case we assume that the structure function ϕ can be expressed as:

$$\phi(t) = \phi(\boldsymbol{X}(t)).$$

Component processes

At this stage we simplify the model by assuming that the components are *repairable* and have the following life cycles:

Each component starts out by being in the top state r_i:

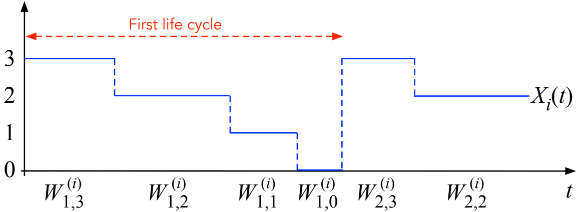
$$X_i(0) = r_i, \quad i \in C.$$

At random points of time $0 < T_{1,r_i}^{(i)} < T_{1,r_i-1}^{(i)} < \cdots < T_{1,0}^{(i)}$ the component degrades through the entire state set until it reaches state 0:

$$X_i(T_{1,r_i}^{(i)}) = r_{i-1} \quad X_i(T_{1,r_i-1}^{(i)}) = r_{i-2} \quad \cdots \quad X_i(T_{1,1}^{(i)}) = 0.$$

At time $T_{1,0}^{(i)}$ the component is repaired or replaced, and a new life cycle starts. For this life cycle the state changes occur at times: $T_{2,r_i}^{(i)} < T_{2,r_i-1}^{(i)} < \cdots < T_{2,0}^{(i)}$, etc.

Component processes (cont.)



$$
\begin{aligned}
T_{1,3}^{(i)} &= W_{1,3}^{(i)} \\
T_{1,2}^{(i)} &= W_{1,3}^{(i)} + W_{1,2}^{(i)} \\
T_{1,1}^{(i)} &= W_{1,3}^{(i)} + W_{1,2}^{(i)} + W_{1,1}^{(i)} \\
T_{1,0}^{(i)} &= W_{1,3}^{(i)} + W_{1,2}^{(i)} + W_{1,1}^{(i)} + W_{1,0}^{(i)}
\end{aligned}
$$

Component processes (cont.)

We assume that all the waiting times for all the components are *independent*, and that for $j \in S_i$:

$$W_{1,j}^{(i)}, W_{2,j}^{(i)}, \ldots \text{ are identically distributed with mean } \mu_j^{(i)} < \infty.$$

We also introduce the *probability distribution* of the ith component at time t:

$$p_{i,j}(t) = P(X_i(t) = j), \quad i \in C, \quad j \in S_i$$

By *renewal theory* it can be shown that:

$$\lim_{t \to \infty} p_{i,j}(t) = p_{i,j} = \frac{\mu_j^{(i)}}{\sum_{y \in S_i} \mu_y^{(i)}}, \quad i \in C, \quad j \in S_i$$

Component processes (cont.)

If all the waiting times are exponentially distributed, the component process $X_i(t)$ is a *Markov process*, and the transition probability matrix for the *built-in* Markov chain is given by:

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

In the general case where the waiting times can have arbitrary distributions, the component process is a *semi-Markov* process.

Physical component states

For $i \in C$ we introduce the functions:

$$f_i : S_i \rightarrow \mathbb{R} = \text{The } \textit{physical state} \text{ of component } i.$$

Thus, the physical state of component i at time t is $f_i(X_i(t))$, $i \in C$.

The functions $f_1, \ldots, f_n$ provide a convenient and intuitive way of encoding physical properties into a model.

EXAMPLE: Let component i be a pipeline. Then the physical state of the component at a given point of time may be the capacity of the pipeline at this point of time.

Physical component states, cont.

NOTE: A physical property of a component may be any real number (not just integers). In most cases, however, such properties will be *non-negative* numbers.

NOTE: The functions $f_1, \ldots, f_n$ do not need to be nondecreasing. By omitting this restriction, additional useful modeling flexibility is gained.

This allows e.g., for the inclusion of burn-in phases, maintenance as well as minimal or partial repairs of a components as part of its life cycle before it reaches its failure state.

Physical system states

It is common in multistate reliability theory to define ϕ such that it takes values in a set of *non-negative integers*.

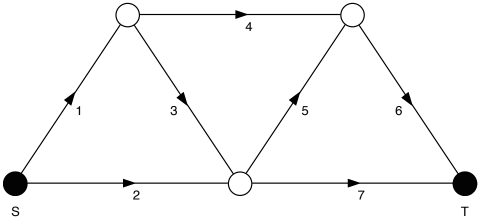
Here, however, we avoid this extra layer of abstraction, and let the structure function represent the *physical state* of the system.

Moreover, we assume that ϕ has the following form:

$$\phi(\boldsymbol{X}(t)) = \phi(f_1(X_1(t)), \ldots, f_n(X_n(t)))$$

Since the structure function represents a physical quantity, it is easier both to model and to interpret than if it had to be encoded more abstractly in terms of non-negative integers.

EXAMPLE: Flow network



$f_i(X_i(t))$ = Capacity of component i, $i = 1, \ldots, 7$.

Minimal cut sets: $K_1 = \{1, 2\}$, $K_2 = \{1, 5, 7\}$, $K_3 = \{2, 3, 4\}$, $K_4 = \{4, 5, 7\}$, $K_5 = \{2, 3, 6\}$, $K_6 = \{6, 7\}$

$$\phi(\boldsymbol{X}(t)) = \min_{1 \leq j \leq 6} \sum_{i \in K_j} f_i(X_i(t)).$$

Importance measures

For multistate systems there are many ways of defining importance measures. There is no such thing as the *best importance measure*.

Traditional uses of importance measures include:

- In design: Identifying components that should be improved
- In diagnostics: Identifying the components that are most likely to have failed

Other uses of importance measures:

- Understanding the structural and probabilistic properties of a system
- Understanding how each component affects various aspects of system performance

Criticality in multistate systems

We introduce the following notation:

$$X_i^+(t) = \text{The } \textit{next state} \text{ of component } i \text{ at time } t$$

$$X_i^-(t) = \text{The } \textit{previous state} \text{ of component } i \text{ at time } t$$

Given the life cycles of the components, these quantities are well defined, and we have:

$$X_i^+(t) = \begin{cases} X_i(t) - 1 & \text{for } X_i(t) > 0 \\ r_i & \text{for } X_i(t) = 0 \end{cases}$$

$$X_i^-(t) = \begin{cases} X_i(t) + 1 & \text{for } X_i(t) < r_i \\ 0 & \text{for } X_i(t) = r_i \end{cases}$$

NOTE: In the binary case $X_i^+(t) = X_i^-(t)$. For components with more than two possible states, however, $X_i^+(t)$ and $X_i^-(t)$ are *not* equal.

Criticality in multistate systems, cont.

We say that component i is *n-critical* at time t if:

$$\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t)).$$

Thus, component i is n-critical at time t if changing the component to its *next* state would result in a system state change as well.

We say that component i is *p-critical* at time t if:

$$\phi(X_i^-(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t)).$$

Thus, component i is p-critical at time t if changing the component to its *previous* state would result in a system state change as well.

Multistate importance

The *n-Birnbaum measure* of importance of component i at time t, denoted $I_{NB}^{(i)}(t)$, is the probability that component i is n-critical at time t:

$$I_{NB}^{(i)}(t) = P[\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t))].$$

The *p-Birnbaum measure* of importance of component i at time t, denoted $I_{PB}^{(i)}(t)$, is the probability that component i is p-critical at time t:

$$I_{PB}^{(i)}(t) = P[\phi(X_i^-(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t))].$$

NOTE: In the binary case we have $I_{NB}^{(i)}(t) = I_{PB}^{(i)}(t) = I_B^{(i)}(t)$. In the multistate case, however, we may have $I_{NB}^{(i)}(t) \neq I_{PB}^{(i)}(t)$.

Multistate importance, cont.

Assuming independent component state processes and conditioning on the state of component i at time t we get:

$$I_{NB}^{(i)}(t) = \sum_{j=1}^{r_i} P[\phi(j_i, \boldsymbol{X}(t)) \neq \phi((j-1)_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = j]$$
$$+ P[\phi(0_i, \boldsymbol{X}(t)) \neq \phi((r_i)_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = 0]$$

$$I_{PB}^{(i)}(t) = \sum_{j=0}^{r_i - 1} P[\phi((j+1)_i, \boldsymbol{X}(t)) \neq \phi(j_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = j]$$
$$+ P[\phi(0_i, \boldsymbol{X}(t)) \neq \phi((r_i)_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = r_i]$$

Changing the summation index in the last expression we get:

$$I_{PB}^{(i)}(t) = \sum_{j=1}^{r_i} P[\phi(j_i, \boldsymbol{X}(t)) \neq \phi((j-1)_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = j-1]$$
$$+ P[\phi(0_i, \boldsymbol{X}(t)) \neq \phi((r_i)_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = r_i]$$

Multistate importance, cont.

We observe that $P[X_i(t) = j]$ in the formula for $I_{NB}^{(i)}(t)$ is replaced by $P[X_i(t) = j - 1]$ in the formula for $I_{PB}^{(i)}(t)$, $j = 1, \ldots, r_i$.

Moreover, $P[X_i(t) = 0]$ in the formula for $I_{NB}^{(i)}(t)$ is replaced by $P[X_i(t) = r_i]$ in in the formula for $I_{PB}^{(i)}(t)$.

From this it follows that in the special case where the component state processes are independent and:

$$P[X_i(t) = 0] = P[X_i(t) = 1] = \cdots = P[X_i(t) = r_i],$$

we will have $I_{NB}^{(i)}(t) = I_{PB}^{(i)}(t)$.

In general, however, the two importance measures will not be equal, and may even result in different rankings.

Example 1

Consider a multistate system (C, ϕ) where $C = \{1, 2\}$.

Both components have three possible states: $S_1 = S_2 = \{0, 1, 2\}$.

In this case the component states are identical to the physical states:

$$f_i(j) = j, \quad j \in S_i \quad \text{and } i \in C.$$

The structure function is given by:

$$\phi(X_1(t), X_2(t)) = \min(f_1(X_1(t)), f_2(X_2(t))).$$

Finally, we assume that the component state variables are independent, and that for a given t we have:

$$P[X_1(t) = j] = p_j > 0, \quad j \in S_1.$$
$$P[X_2(t) = j] = q_j > 0, \quad j \in S_2.$$

Example 1, cont.

Since $f_i(j) = j$, for $j \in S_i$ and $i \in C$, the structure function can be written as:

$$\phi(X_1(t), X_2(t)) = \min(X_1(t), X_2(t)).$$

With the life cycle : $2 \to 1 \to 0 \to 2 \cdots$, we then get:

$$P[\phi(0, X_2(t)) \neq \phi(2, X_2(t))] = P[X_2(t) \geq 1] = q_1 + q_2,$$
$$P[\phi(1, X_2(t)) \neq \phi(0, X_2(t))] = P[X_2(t) \geq 1] = q_1 + q_2$$
$$P[\phi(2, X_2(t)) \neq \phi(1, X_2(t))] = P[X_2(t) = 2] = q_2,$$

$$P[\phi(X_1(t), 0) \neq \phi(X_1(t), 2)] = P[X_1(t) \geq 1] = p_1 + p_2,$$
$$P[\phi(X_1(t), 1) \neq \phi(X_1(t), 0)] = P[X_1(t) \geq 1] = p_1 + p_2,$$
$$P[\phi(X_1(t), 2) \neq \phi(X_1(t), 1)] = P[X_1(t) = 2] = p_2.$$

Example 1, cont.

Hence, since: $p_0 + p_1 + p_2 = q_0 + q_1 + q_2 = 1$, we get

$$
\begin{aligned}
I_{NB}^{(1)}(t) = p_0(q_1 + q_2) + p_1(q_1 + q_2) + p_2 q_2 &= (p_0 + p_1 + p_2)(q_1 + q_2) - p_2 q_1 \\
&= q_1 + q_2 - p_2 q_1,
\end{aligned}
$$

$$
\begin{aligned}
I_{NB}^{(2)}(t) = q_0(p_1 + p_2) + q_1(p_1 + p_2) + q_2 p_2 &= (q_0 + q_1 + q_2)(p_1 + p_2) - p_1 q_2 \\
&= p_1 + p_2 - p_1 q_2,
\end{aligned}
$$

$$
I_{PB}^{(1)}(t) = p_2(q_1 + q_2) + p_0(q_1 + q_2) + p_1 q_2 = q_1 + q_2 - p_1 q_1,
$$

$$
I_{PB}^{(2)}(t) = q_2(p_1 + p_2) + q_0(p_1 + p_2) + q_1 p_2 = p_1 + p_2 - p_1 q_1.
$$

We observe that $I_{PB}^{(1)}(t) > I_{PB}^{(2)}(t)$ if and only if $q_1 + q_2 > p_1 + p_2$.

However, if $q_1 + q_2 > p_1 + p_2$, $p_1 < p_2$ and $q_2 < q_1$, it is possible to obtain the opposite ranking with respect to the n-Birnbaum measure.

Example 1, cont.

Assume e.g., that $p_1 = 0.20$, $p_2 = 0.35$, $q_1 = 0.40$ and $q_2 = 0.20$. Then we have:

$$q_1 + q_2 = 0.60 > p_1 + p_2 = 0.55$$

$$p_2 q_1 = 0.14 > p_1 q_2 = 0.04$$

$$p_1 q_1 = 0.08$$

Hence, we get:

$$I_{NB}^{(1)}(t) = q_1 + q_2 - p_2 q_1 = 0.46,$$
$$I_{NB}^{(2)}(t) = p_1 + p_2 - p_1 q_2 = 0.51,$$

$$I_{PB}^{(1)}(t) = q_1 + q_2 - p_1 q_1 = 0.52,$$
$$I_{PB}^{(2)}(t) = p_1 + p_2 - p_1 q_1 = 0.47.$$

That is, $I_{NB}^{(1)}(t) < I_{NB}^{(2)}(t)$ while $I_{PB}^{(1)}(t) > I_{PB}^{(2)}(t)$.

Example 1, cont.

For $i \in C$, $j \in S_i$, and $k = 1, 2, \ldots$ we introduce:

$$W_{kj}^{(i)} = k\text{th waiting time in state } j \text{ for component } i,$$

We assume that all waiting times are independent and exponentially distributed with:

$$E[W_{kj}^{(1)}] = \begin{cases} 4.5 & \text{for } j = 0 \\ 2.0 & \text{for } j = 1 \\ 3.5 & \text{for } j = 2 \end{cases}$$

$$E[W_{kj}^{(2)}] = \begin{cases} 4.0 & \text{for } j = 0 \\ 4.0 & \text{for } j = 1 \\ 2.0 & \text{for } j = 2 \end{cases}$$

Asymptotically, we get that $p_1 = 0.20$, $p_2 = 0.35$, $q_1 = 0.40$ and $q_2 = 0.20$, and hence again:

$$I_{NB}^{(1)} = 0.46, \quad I_{NB}^{(2)} = 0.51, \quad I_{PB}^{(1)} = 0.52, \quad I_{PB}^{(2)} = 0.47$$

Discrete Event Simulation

Let (C, ϕ) be a binary monotone system with component state processes: $\{X_1(t)\}, \ldots, \{X_n(t)\}$.

- $E_{i1}, E_{i2}, \ldots$ are the events affecting the process $\{X_i(t)\}$
- $T_{i1}, T_{i2}, \ldots$ are the corresponding points of time for these events

Assuming that all lifetimes and repair times have *absolutely continuous distributions*, all the events happen at *distinct* points of time almost surely, i.e., all the T_{ij}s are distinct numbers.

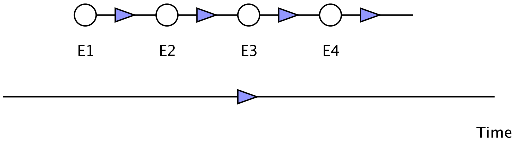
We assume that the events are sorted with respect to their respective points of time, so that $T_{i1} < T_{i2} < \cdots$.

Discrete Event Simulation (cont.)

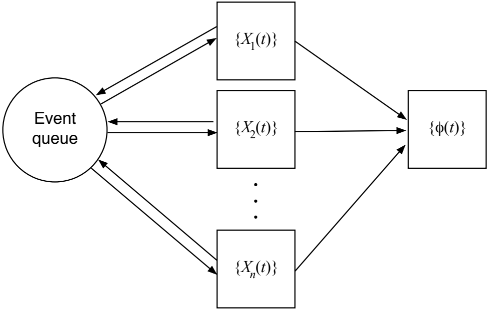
At the system level the event set is the *union* of all the component event sets.

- Let $E^{(1)}, E^{(2)}, \ldots$ denote the system events sorted with respect to their respective points of time
- Let $T^{(1)} < T^{(2)} < \cdots$ be the corresponding points of time

Each system event corresponds to a unique component event, organised in a dynamic queue sorted with respect to the points of time of the events:



Program flow



- The components post initial events to the event queue
- The event queue processes events in chronological order, and notifies the components when the events occur. As soon as an event is processed, it is removed from the queue.
- The component updates its state, posts a new event to the queue, and notifies the system about the state change

Sampling events

Although the system state and component states stay constant between events, it is of interest to sample the state values at *predefined* points of time. Thus, we introduce yet another type of event, called a *sampling events* spread out evenly on the timeline.

- Let $e_1, e_2, \ldots$ denote the sampling events
- Let $t_1 < t_2 < \cdots$ are the corresponding points of time

Typically $t_j = j \cdot \Delta$ for some suitable $\Delta > 0$, $j = 1, 2, \ldots$.

The sampling events are placed into the queue in the same way as for the ordinary events.

Example 1, cont.

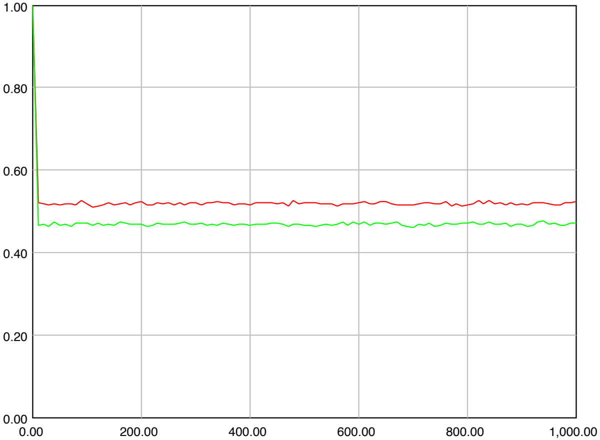


Figure: $I_{PB}^{(1)}(t)$ (red curve) $I_{PB}^{(2)}(t)$ (green curve)

Example 1, cont.

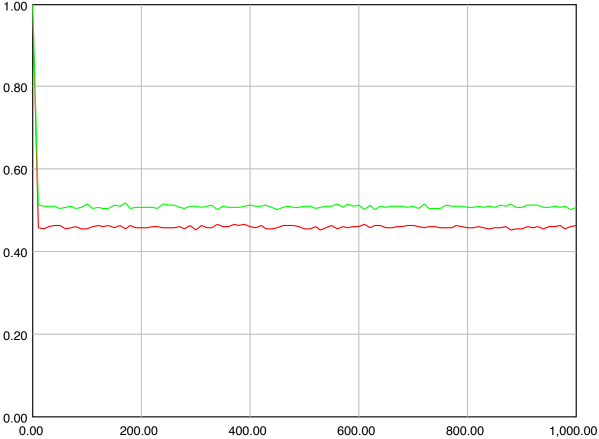


Figure: $I_{NB}^{(1)}(t)$ (red curve) $I_{NB}^{(2)}(t)$ (green curve)

Importance based on expected physical criticality

The *n*-Birnbaum measure* of importance of component i at time t, denoted $I_{NB}^{*(i)}(t)$, is the expected effect of changing i to its next state at time t:

$$I_{NB}^{*(i)}(t) = E\left|\phi(X_i(t), \boldsymbol{X}(t)) - \phi(X_i^{+}(t), \boldsymbol{X}(t))\right|.$$

The *p*-Birnbaum measure* of importance of component i at time t, denoted $I_{PB}^{*(i)}(t)$, is the expected effect of changing i to its previous state at time t:

$$I_{PB}^{*(i)}(t) = E\left|\phi(X_i^{-}(t), \boldsymbol{X}(t)) - \phi(X_i(t), \boldsymbol{X}(t))\right|.$$

NOTE: In the binary case all the different measures are the same:

$$I_{NB}^{*(i)}(t) = I_{NB}^{(i)}(t) \quad = \quad I_{PB}^{*(i)}(t) = I_{PB}^{(i)}(t) \quad = \quad I_{B}^{(i)}(t).$$

Example 2

Consider a multistate system (C, ϕ) where $C = \{1, 2\}$, and where $S_1 = \{0, 1\}$ and $S_2 = \{0, 1, 2\}$.

Moreover, we assume that:

$$f_1(j) = 2j, \quad j \in S_1,$$
$$f_2(j) = j, \quad j \in S_2.$$

As before, the structure function is given by:

$$\phi(X_1(t), X_2(t)) = \min(f_1(X_1(t)), f_2(X_2(t))).$$

Finally, we again assume that the component state processes are independent, and that for a given t we have:

$$P[X_1(t) = j] = p_j > 0, \quad j \in S_1,$$
$$P[X_2(t) = j] = q_j > 0, \quad j \in S_2.$$

Example 2, cont.

Component 1 has the life cycle: $1 \to 0 \to 1 \cdots$, implying that:

$$E|\phi(0, X_2(t)) - \phi(1, X_2(t))| = q_1 \cdot 1 + q_2 \cdot 2 = q_1 + 2q_2$$

$$E|\phi(1, X_2(t)) - \phi(0, X_2(t))| = q_1 \cdot 1 + q_2 \cdot 2 = q_1 + 2q_2$$

Component 2 has the life cycle: $2 \to 1 \to 0 \to 2 \cdots$, implying that:

$$E|\phi(X_1(t), 0) - \phi(X_1(t), 2)| = 2p_1$$

$$E|\phi(X_1(t), 1) - \phi(X_1(t), 0)| = p_1$$

$$E|\phi(X_1(t), 2) - \phi(X_1(t), 1)| = p_1$$

Hence, it follows that:

$$I_{NB}^{*(1)}(t) = p_0 \cdot (q_1 + 2q_2) + p_1 \cdot (q_1 + 2q_2) \; = \; q_1 + 2q_2$$
$$I_{NB}^{*(2)}(t) = q_0 \cdot 2p_1 + q_1 \cdot p_1 + q_2 \cdot p_1 \; = \; (1 + q_0)p_1$$

Example 2, cont.

We also have:

$$E[f_1(X_1(t))] = p_0 \cdot 0 + p_1 \cdot 2 \ = \ 2p_1$$
$$E[f_2(X_2(t))] = q_0 \cdot 0 + q_1 \cdot 1 + q_2 \cdot 2 \ = \ q_1 + 2q_2.$$

We then assume that $E[f_1(X_1(t))] = E[f_2(X_2(t))]$. This implies that:

$$I_{NB}^{*(1)}(t) = q_1 + 2q_2 = 2p_1$$
$$I_{NB}^{*(2)}(t) = (1 + q_0)p_1$$

Hence, since we must have $q_0 < 1$, we conclude that $I_{NB}^{*(1)}(t) > I_{NB}^{*(2)}(t)$.

Example 2, cont.

We recall that:

$$E|\phi(0, X_2(t)) - \phi(1, X_2(t))| = q_1 \cdot 1 + q_2 \cdot 2 = q_1 + 2q_2$$

$$E|\phi(1, X_2(t)) - \phi(0, X_2(t))| = q_1 \cdot 1 + q_2 \cdot 2 = q_1 + 2q_2$$

and:

$$E|\phi(X_1(t), 0) - \phi(X_1(t), 2)| = 2p_1$$

$$E|\phi(X_1(t), 1) - \phi(X_1(t), 0)| = p_1$$

$$E|\phi(X_1(t), 2) - \phi(X_1(t), 1)| = p_1$$

Hence, it follows that:

$$I_{PB}^{*(1)}(t) = p_1 \cdot (q_1 + 2q_2) + p_0 \cdot (q_1 + 2q_2) \;=\; q_1 + 2q_2$$

$$I_{PB}^{*(2)}(t) = q_2 \cdot 2p_1 + q_0 \cdot p_1 + q_1 \cdot p_1 \;=\; (1 + q_2)p_1$$

Example 2, cont.

We also have:

$$E[f_1(X_1(t))] = p_0 \cdot 0 + p_1 \cdot 2 \; = \; 2p_1$$
$$E[f_2(X_2(t))] = q_0 \cdot 0 + q_1 \cdot 1 + q_2 \cdot 2 \; = \; q_1 + 2q_2.$$

We then assume that $E[f_1(X_1(t))] = E[f_2(X_2(t))]$. This implies that:

$$I_{PB}^{*(1)}(t) = q_1 + 2q_2 = 2p_1$$
$$I_{PB}^{*(2)}(t) = (1 + q_2)p_1$$

Hence, since we must have $q_2 < 1$, we conclude that $I_{PB}^{*(1)}(t) > I_{PB}^{*(2)}(t)$.

Example 2, cont.

For $i \in C$, $j \in S_i$, and $k = 1, 2, \ldots$ we introduce:

$$W_{kj}^{(i)} = k\text{th waiting time in state } j \text{ for component } i,$$

We assume that all waiting times are independent and exponentially distributed with:

$$E[W_{kj}^{(1)}] = \begin{cases} 3.0 & \text{for } j = 0 \\ 3.0 & \text{for } j = 1 \end{cases}$$

$$E[W_{kj}^{(2)}] = \begin{cases} 2.0 & \text{for } j = 0 \\ 2.0 & \text{for } j = 1 \\ 2.0 & \text{for } j = 2 \end{cases}$$

Asymptotically, we get that $p_0 = p_1 = \frac{1}{2}$ while $q_0 = q_1 = q_2 = \frac{1}{3}$, and hence:

$$I_{NB}^{*(1)} = 1.0, \quad I_{NB}^{*(2)} = \frac{2}{3}, \quad I_{PB}^{*(1)} = 1.0, \quad I_{PB}^{*(2)} = \frac{2}{3}$$

Example 2, cont.

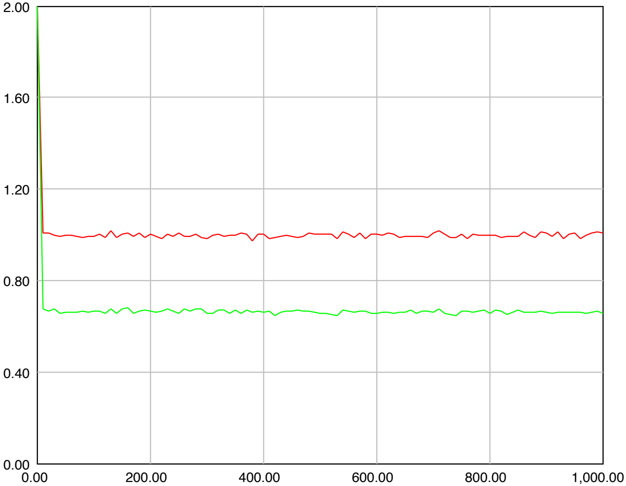


Figure: $I_{PB}^{*(1)}(t)$ (red curve) $I_{PB}^{*(2)}(t)$ (green curve)

Example 2, cont.

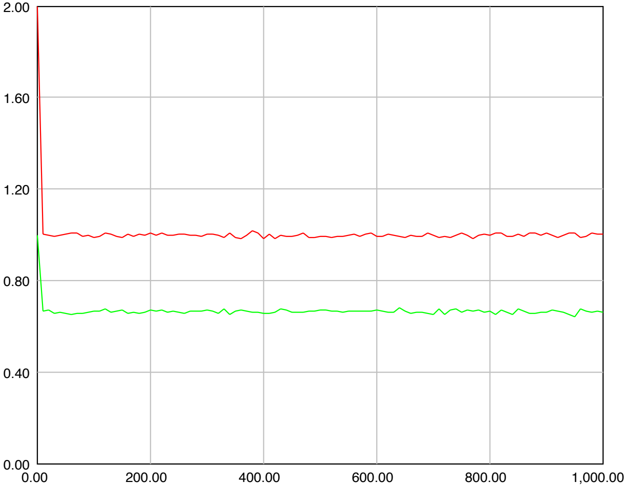


Figure: $I_{NB}^{*(1)}(t)$ (red curve) $I_{NB}^{*(2)}(t)$ (green curve)