Multistate systems and importance measures - part 2

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Continuous-Time Markov Chains

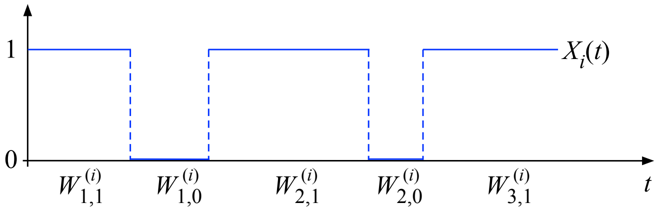
A continuous-time Markov chain with stationary transition probabilities and state space $\mathcal{X}$ is a stochastic process such that:

- The times spent in the different states are independent random variables.
- The amount of time spent in state $i \in \mathcal{X}$ is exponentially distributed with mean μ_i.
- When the process leaves state i, it enters state j with some transition probability P_{ij} where:

$$\sum_{j \in \mathcal{X}} P_{ij} = 1, \quad \text{for all } i \in \mathcal{X}$$

- The transitions follow a discrete-time Markov chain.
- If the amount of time spent in state $i \in \mathcal{X}$ has a general distribution with mean μ_i, the process is called a *semi-Markov* process.

Component processes - binary case



The state of the ith component is described by the *stochstic process* $X_i(t)$. For *repairable components* the component state jumps up and down between the *functioning state*, 1, and the *failed state* 0.

For $k = 1, 2, \ldots$ we introduce the *waiting times*:

$$W_{k,1}^{(i)} = \text{The } k\text{th lifetime of component } i$$

$$W_{k,0}^{(i)} = \text{The } k\text{th repair time of component } i$$

Component processes - binary case (cont.)

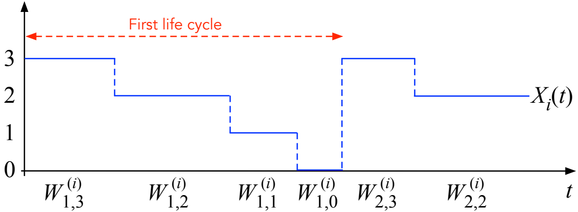
Since there are only two possible states, the transition probability matrix for the *built-in* Markov chain is given by:

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

If all the waiting times are independent and exponentially distributed, the component process $X_i(t)$ is a *Markov process*.

In the general case where the waiting times are independent but can have arbitrary distributions, the component process is a *semi-Markov* process.

Component processes - multistate case



$$T_{1,3}^{(i)} = W_{1,3}^{(i)}$$
$$T_{1,2}^{(i)} = W_{1,3}^{(i)} + W_{1,2}^{(i)}$$
$$T_{1,1}^{(i)} = W_{1,3}^{(i)} + W_{1,2}^{(i)} + W_{1,1}^{(i)}$$
$$T_{1,0}^{(i)} = W_{1,3}^{(i)} + W_{1,2}^{(i)} + W_{1,1}^{(i)} + W_{1,0}^{(i)}$$

Component processes - multistate case (cont.)

Assuming a deterministic life-cycle the transition probability matrix for the *built-in* Markov chain is given by:

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

If all the waiting times are independent and exponentially distributed, the component process $X_i(t)$ is a *Markov process*.

In the general case where the waiting times are independent but can have arbitrary distributions, the component process is a *semi-Markov* process.

Component processes with non-deterministic life-cycles

We now consider systems where the components have non-deterministic life-cycles. Thus, assuming that the state space of component $i \in C$ is $S_i = \{0, 1, \ldots, r_i\}$, the transition probability matrix for the *built-in* Markov chain of component i is:

$$\boldsymbol{P}^{(i)} = \begin{bmatrix} P_{00}^{(i)} & P_{01}^{(i)} & \cdots & P_{0,r_i}^{(i)} \\ P_{10}^{(i)} & P_{11}^{(i)} & \cdots & P_{1,r_i}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ P_{r_i,0}^{(i)} & P_{r_i,1}^{(i)} & \cdots & P_{r_i,r_i}^{(i)} \end{bmatrix}$$

Using the notation:

$$X_i^+(t) = \text{The } \textit{next state} \text{ of component } i \text{ at time } t$$

we have that:

$$P(X_i^+(t) = v | X_i(t) = u) = P_{uv}^{(i)}, \quad u, v \in S_i.$$

Component processes with non-deterministic life-cycles

Assuming that the state space of component $i \in C$ is $S_i = \{0, 1, \ldots, r_i\}$, the *backwards transition probability matrix* for the *built-in* Markov chain of component i is denoted by :

$$\boldsymbol{Q}^{(i)} = \begin{bmatrix} Q_{00}^{(i)} & Q_{01}^{(i)} & \cdots & Q_{0,r_i}^{(i)} \\ Q_{10}^{(i)} & Q_{11}^{(i)} & \cdots & Q_{1,r_i}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{r_i,0}^{(i)} & Q_{r_i,1}^{(i)} & \cdots & Q_{r_i,r_i}^{(i)} \end{bmatrix}$$

Using the notation:

$$X_i^-(t) = \text{The } \textit{previous state} \text{ of component } i \text{ at time } t$$

we have that:

$$P(X_i^-(t) = v | X_i(t) = u) = Q_{uv}^{(i)}, \quad u, v \in S_i.$$

Criticality in multistate systems

The notion of *criticality* can be generalized without any changes:

Component i is *n-critical* at time t if:

$$\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t)).$$

Component i is *p-critical* at time t if:

$$\phi(X_i^-(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t)).$$

NOTE: In the general case, given the state $X_i(t)$, the states $X_i^+(t)$ and $X_i^-(t)$ are random variables.

Multistate importance

The *importance measures* can also be generalized without any changes:

The *n-Birnbaum measure* of importance of component i at time t, denoted $I_{NB}^{(i)}(t)$, is the probability that component i is n-critical at time t:

$$I_{NB}^{(i)}(t) = P[\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t))].$$

The *p-Birnbaum measure* of importance of component i at time t, denoted $I_{PB}^{(i)}(t)$, is the probability that component i is p-critical at time t:

$$I_{PB}^{(i)}(t) = P[\phi(X_i^-(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t))].$$

Multistate importance (cont.)

The importance measures can now be calculated by conditioning on the current and next (or previous) state of component i as follows:

$$I_{NB}^{(i)}(t) = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}(t)) \neq \phi(v, \boldsymbol{X}(t))] \cdot P[X_i(t) = u] \cdot P_{uv}^{(i)} \quad (1)$$

$$I_{PB}^{(i)}(t) = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}(t)) \neq \phi(v, \boldsymbol{X}(t))] \cdot P[X_i(t) = u] \cdot Q_{uv}^{(i)} \quad (2)$$

NOTE: The new mathematical expressions must to take into account that a more general probability model is used for the component processes u and v. Thus, we must compute double sums with one term per pair of states instead of single sums.

Asymptotic importance

We henceforth focus on the asymptotic properties of the processes and ommit the time t from the notation. For component $i \in C$ we denote the stationary probabilities of the built-in Markov chain by $\pi_u^{(i)}$, $u \in S_i$. We then have the following well-known relation between the transition matrices $\boldsymbol{P}^{(i)}$ and $\boldsymbol{Q}^{(i)}$:

$$Q_{uv}^{(i)} = \frac{\pi_v^{(i)}}{\pi_u^{(i)}} P_{vu}^{(i)}, \quad u, v \in S_i. \tag{3}$$

NOTE: If the stationary distribution of the built-in Markov chain is *uniform*, i.e., if $\pi_u^{(i)} = 1/(r_i + 1)$, for all $u \in S_i$, we have:

$$\boldsymbol{Q}^{(i)} = (\boldsymbol{P}^{(i)})^T, \quad i \in C.$$

Uniform stationary distributions

An irreducible aperiodic finite Markov chain has a uniform stationary distribution if and only if $\boldsymbol{P}^{(i)}$ is a *doubly stochastic matrix*, i.e., all row sums and column sums are equal to 1.

Since the row sums are always equal to one in any transition matrix, $\boldsymbol{P}^{(i)}$ is a *doubly stochastic matrix* if and only if:

$$(1, \ldots, 1)\boldsymbol{P}^{(i)} = (1, \ldots, 1)$$

To prove the claim we start out by assuming that $\boldsymbol{P}^{(i)}$ is doubly stochastic, and let $\boldsymbol{\pi}^{(i)} = (\pi_0^{(i)}, \ldots, \pi_{r_i}^{(i)})$. Then we know that $\boldsymbol{\pi}^{(i)}$ is uniquely determined by the following equations:

$$\boldsymbol{\pi}^{(i)} P^{(i)} = \boldsymbol{\pi}^{(i)}$$

as well as $\pi_0^{(i)} + \cdots + \pi_{r_i}^{(i)} = 1$.

Uniform stationary distributions (cont.)

We then let $\tilde{\pi}_u = (1 + r_i)^{-1}$, for $u = 0, 1, \ldots, r_i$. That is, $\tilde{\boldsymbol{\pi}}$ is a *uniform* distribution on S_i, and we have:

$$\tilde{\boldsymbol{\pi}} = (1 + r_i)^{-1}(1, \ldots, 1).$$

Since $\boldsymbol{P}^{(i)}$ is doubly stochastic, it follows that:

$$\tilde{\boldsymbol{\pi}} P^{(i)} = (1 + r_i)^{-1}(1, \ldots, 1)\boldsymbol{P}^{(i)} = (1 + r_i)^{-1}(1, \ldots, 1) = \tilde{\boldsymbol{\pi}}$$

Moreover, we obviously have:

$$\tilde{\pi}_0 + \cdots + \tilde{\pi}_{r_i} = (1 + r_i)^{-1} \cdot (1 + r_i) = 1.$$

Hence, $\tilde{\boldsymbol{\pi}}$ satisfies the equations for the stationary distribution, and since these equations have a unique solution, it follows that we must have $\boldsymbol{\pi}^{(i)} = \tilde{\boldsymbol{\pi}}$. That is, $\boldsymbol{\pi}^{(i)}$ is indeed a uniform distribution.

Uniform stationary distributions (cont.)

Assume conversely that $\boldsymbol{\pi}^{(i)}$ is a uniform distribution, i.e., $\pi_u^{(i)} = (1 + r_i)^{-1}$, for $u = 0, 1, \ldots, r_i$. That is, we have:

$$\boldsymbol{\pi}^{(i)} = (1 + r_i)^{-1}(1, \ldots, 1).$$

Since $\boldsymbol{\pi}^{(i)}$ is the stationary distribution, $\boldsymbol{\pi}^{(i)}$ satisfies the following equation:

$$\boldsymbol{\pi}^{(i)} \boldsymbol{P}^{(i)} = \boldsymbol{\pi}^{(i)}$$

Inserting $\boldsymbol{\pi}^{(i)} = (1 + r_i)^{-1}(1, \ldots, 1)$ into this equation, we get:

$$(1 + r_i)^{-1}(1, \ldots, 1)\boldsymbol{P}^{(i)} = (1 + r_i)^{-1}(1, \ldots, 1)$$

Multiplying both sides of the latter equation by $(1 + r_i)$, we get that:

$$(1, \ldots, 1)\boldsymbol{P}^{(i)} = (1, \ldots, 1)$$

which proves that $\boldsymbol{P}^{(i)}$ is indeed doubly stochastic.

Asymptotic importance (cont.)

To proceed we again introduce the times spent in each state between the transitions:

$$W_{ku}^{(i)} = \text{The } k\text{th waiting time in state } u \text{ for component } i.$$

We assume that all the waiting times are independent, and that for all components $i \in C$ and states $u \in S_i$ the waiting times $W_{1u}^{(i)}, W_{2u}^{(i)}, \ldots$ are identically distributed with finite mean $\mu_u^{(i)}$.

Then it follows from standard renewal theory that the stationary distribution of X_i is given by:

$$P[X_i = u] = \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}}, \quad u \in S_i,\ i \in C. \tag{4}$$

Asymptotic importance (cont.)

Combining Eq. (4) with Eq. (1) and Eq. (2) we get the following expressions for the stationary importance measures:

$$I_{NB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot P_{uv}^{(i)} \qquad (5)$$

$$I_{PB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot Q_{uv}^{(i)} \qquad (6)$$

Comparing importance measures

Theorem (4.1)

Assume that $\mu_0^{(i)} = \cdots = \mu_{r_i}^{(i)}$. *Then* $I_{NB}^{(i)} = I_{PB}^{(i)}$.

Proof: If $\mu_0^{(i)} = \cdots = \mu_{r_i}^{(i)}$, the stationary distribution given in Eq. (4) is simplified to:

$$P[X_i = u] = \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} = \pi_u^{(i)}, \quad u \in S_i,\ i \in C.$$

Inserting this into Eq. (5) and Eq. (6), we get:

$$I_{NB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \pi_u^{(i)} \cdot P_{uv}^{(i)}$$

$$I_{PB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \pi_u^{(i)} \cdot Q_{uv}^{(i)}$$

Comparing importance measures (cont.)

We then consider the expression for $Q_{uv}^{(i)}$ given in Eq. (3):

$$Q_{uv}^{(i)} = \frac{\pi_v^{(i)}}{\pi_u^{(i)}} P_{vu}^{(i)}, \quad u, v \in S_i.$$

Inserting this into the the expression for $I_{PB}^{(i)}$ we get:

$$\begin{aligned}
I_{PB}^{(i)} &= \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \pi_u^{(i)} \cdot Q_{uv}^{(i)} \\
&= \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \pi_u^{(i)} \cdot \frac{\pi_v^{(i)}}{\pi_u^{(i)}} P_{vu}^{(i)} \\
&= \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \pi_v^{(i)} \cdot P_{vu}^{(i)} \\
&= \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \pi_u^{(i)} \cdot P_{uv}^{(i)} = I_{NB}^{(i)}
\end{aligned}$$

■

Comparing importance measures (cont.)

Theorem (4.2)

Assume that the transition matrix $\boldsymbol{P}^{(i)}$ is doubly stochastic. Then we have:

$$I_{NB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{uv}^{(i)}$$

$$I_{PB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{vu}^{(i)}$$

Comparing importance measures (cont.)

Proof: If the transition matrix $\boldsymbol{P}^{(i)}$ is doubly stochastic, we have shown that the stationary distribution of the built-in Markov chain is uniform. Hence, the stationary distribution given in Eq. (4) is simplified to:

$$P[X_i = u] = \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} = \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}}, \quad u \in S_i, \; i \in C. \tag{7}$$

Moreover, the transition matrix $\boldsymbol{Q}^{(i)}$ is equal to $(\boldsymbol{P}^{(i)})^T$. That is:

$$Q_{uv}^{(i)} = P_{vu}^{(i)}, \quad \text{for all } u, v \in S_i. \tag{8}$$

Comparing importance measures (cont.)

Hence, by inserting Eq. (7) and Eq. (8) into Eq. (5) and Eq. (6) we get:

$$
\begin{aligned}
I_{NB}^{(i)} &= \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot P_{uv}^{(i)} \\
&= \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{uv}^{(i)}
\end{aligned}
$$

$$
\begin{aligned}
I_{PB}^{(i)} &= \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot Q_{uv}^{(i)} \\
&= \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{vu}^{(i)}
\end{aligned}
$$

■

Comparing importance measures (cont.)

NOTE: Recall that:

$$I_{NB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot P[X_i = u] \cdot P_{uv}^{(i)}$$

$$I_{PB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot P[X_i = u] \cdot Q_{uv}^{(i)}$$

Hence, it follows that in general $I_{NB}^{(i)}$ and $I_{PB}^{(i)}$ depends both on the stationary distribution of X_i as well as the transition matrices $\boldsymbol{P}^{(i)}$ and $\boldsymbol{Q}^{(i)}$.

Thus, if two components have equal stationary distributions, they may still have different importance.

Similarly, if two components have equal transition probabilities, they may still have different importance.

Comparing importance measures (cont.)

EXAMPLE: We consider a multistate system (C, ϕ) where $C = \{1, 2\}$, and where $S_1 = S_2 = \{0, 1, 2\}$. We also let $f_i(u) = u$, $u = 0, 1, 2$, $i = 1, 2$.

The structure function is given by:

$$\phi(X_1(t), X_2(t)) = \min(f_1(X_1(t)), f_2(X_2(t))) = \min(X_1(t), X_2(t))$$

The transition matrices of the built-in Markov chains are:

$$\boldsymbol{P}^{(1)} = \begin{bmatrix} 0.1, & 0.3, & 0.6 \\ 0.6, & 0.1, & 0.3 \\ 0.3, & 0.6, & 0.1 \end{bmatrix}, \qquad \boldsymbol{P}^{(2)} = \begin{bmatrix} 0.7, & 0.1, & 0.2 \\ 0.2, & 0.7, & 0.1 \\ 0.1, & 0.2, & 0.7 \end{bmatrix},$$

while the mean waiting times are:

$$\mu_0^{(i)} = 2.5, \quad \mu_1^{(i)} = 3.5, \quad \mu_2^{(i)} = 4.0, \quad i = 1, 2.$$

Comparing importance measures (cont.)

Both $\boldsymbol{P}^{(1)}$ and $\boldsymbol{P}^{(2)}$ are *doubly stochastic*, implying that the stationary distributions of the built-in Markov chains are *uniform*.

Hence, we may calculate importance using Theorem 4.2. In particular, the stationary distributions can be calculated using the simplified formula given in Eq. (7), and we get for $i = 1, 2$:

$$P[X_i = 0] = \frac{\mu_0^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} = \frac{2.5}{2.5 + 3.5 + 4.0} = 0.25$$

$$P[X_i = 1] = \frac{\mu_1^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} = \frac{3.5}{2.5 + 3.5 + 4.0} = 0.35$$

$$P[X_i = 2] = \frac{\mu_2^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} = \frac{4.0}{2.5 + 3.5 + 4.0} = 0.40$$

Comparing importance measures (cont.)

To calculate $I_{NB}^{(i)}$ we need to compute the sum:[1]

$$I_{NB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot P[X_i = u] \cdot P_{uv}^{(i)}$$

Since, however, we obviously have:

$$P[\phi(u, X_2) \neq \phi(v, X_2)] = 0 \quad \text{whenever } u = v,$$

only the terms where $u \neq v$ need to be included.

Moreover, by symmetry we of course also have:

$$P[\phi(u, X_2) \neq \phi(v, X_2)] = P[\phi(v, X_2) \neq \phi(u, X_2)].$$

[1] The corresponding results for $I_{PB}^{(i)}$ are obtained in a similar fashion and are approximately the same in this case.

Comparing importance measures (cont.)

Hence, we get the following $3 \cdot 3 - 3 = 6$ non-zero probabilities for component 1:

$$P[\phi(0, X_2) \neq \phi(1, X_2)] = P[X_2 = 1] + P[X_2 = 2] = 0.35 + 0.40 = 0.75,$$
$$P[\phi(0, X_2) \neq \phi(2, X_2)] = P[X_2 = 1] + P[X_2 = 2] = 0.35 + 0.40 = 0.75,$$

$$P[\phi(1, X_2) \neq \phi(0, X_2)] = P[\phi(0, X_2) \neq \phi(1, X_2)] = 0.75,$$
$$P[\phi(1, X_2) \neq \phi(2, X_2)] = P[X_2 = 2] = 0.40,$$

$$P[\phi(2, X_2) \neq \phi(0, X_2)] = P[\phi(0, X_2) \neq \phi(2, X_2)] = 0.75,$$
$$P[\phi(2, X_2) \neq \phi(1, X_2)] = P[\phi(1, X_2) \neq \phi(2, X_2)] = 0.40.$$

Comparing importance measures (cont.)

Moreover, since these probabilities only depend on the stationary distribution of component 2, and both components have the same stationary distribution, we get exactly the same probabilities for component 2.

Thus, we have all the quantities needed in order to compute the importance measures using the formula for $I_{NB}^{(i)}$ given in Theorem 4.2, and we eventually get:

$$I_{NB}^{(1)} = 0.55425, \qquad I_{NB}^{(2)} = 0.18475$$

Comparing importance measures (cont.)

NOTE: Component 1 (0.55425) is more important than component 2 (0.18475), even though they both have the same stationary distributions.

We recall that:

$$I_{NB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot P[X_i = u] \cdot P_{uv}^{(i)}$$

and that:

$$\boldsymbol{P}^{(1)} = \begin{bmatrix} 0.1, & 0.3, & 0.6 \\ 0.6, & 0.1, & 0.3 \\ 0.3, & 0.6, & 0.1 \end{bmatrix}, \qquad \boldsymbol{P}^{(2)} = \begin{bmatrix} 0.7, & 0.1, & 0.2 \\ 0.2, & 0.7, & 0.1 \\ 0.1, & 0.2, & 0.7 \end{bmatrix},$$

For component 1 most of the weight from the transition probabilities are put on terms where $u \neq v$, and for these terms $P[\phi(u, X_2) \neq \phi(v, X_2)] > 0$.

For component 2 most of the weight from the transition probabilities are put on terms where $u = v$, and for these terms $P[\phi(X_1, u) \neq \phi(X_1, v)] = 0$.

Expected physical criticality for semi-Markov processes

The *importance measures* based on expected physical criticality can also be generalized without any changes:

The *n*-Birnbaum measure* of importance of component i at time t, denoted $I_{NB}^{*(i)}(t)$, is defined by:

$$I_{NB}^{*(i)}(t) = E|\phi(X_i(t), \boldsymbol{X}(t)) - \phi(X_i^+(t), \boldsymbol{X}(t))|.$$

The *p*-Birnbaum measure* of importance of component i at time t, denoted $I_{PB}^{*(i)}(t)$, is defined by:

$$I_{PB}^{*(i)}(t) = E|\phi(X_i^-(t), \boldsymbol{X}(t)) - \phi(X_i(t), \boldsymbol{X}(t))|.$$

Expected physical criticality for semi-Markov processes (c.)

To calculate the *n*-Birnbaum measure* and the *p*-Birnbaum measure*, we again condition on the current and next (or previous) state of component i as follows:

$$I_{NB}^{*(i)}(t) = \sum_{u,v \in S_i} E|\phi(u, \boldsymbol{X}(t)) - \phi(v, \boldsymbol{X}(t))| \cdot P[X_i(t) = u] \cdot P_{uv}^{(i)} \qquad (9)$$

$$I_{PB}^{*(i)}(t) = \sum_{u,v \in S_i} E|\phi(u, \boldsymbol{X}(t)) - \phi(v, \boldsymbol{X}(t))| \cdot P[X_i(t) = u] \cdot Q_{uv}^{(i)} \qquad (10)$$

Expected physical criticality for semi-Markov processes (c.)

Focussing on the asymptotic properties and using the same arguments as we did for $I_{NB}^{(i)}$ and $I_{PB}^{(i)}$, we get the following analogues to Eq. (5) and Eq. (6):

$$I_{NB}^{*(i)} = \sum_{u,v \in S_i} E|\phi(u, \boldsymbol{X}) - \phi(v, \boldsymbol{X})| \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot P_{uv}^{(i)} \tag{11}$$

$$I_{PB}^{*(i)} = \sum_{u,v \in S_i} E|\phi(u, \boldsymbol{X}) - \phi(v, \boldsymbol{X})| \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot Q_{uv}^{(i)} \tag{12}$$

Expected physical criticality for semi-Markov processes (c.)

The following results can easily be proved using exactly the same arguments as we used for the corresponding results for $I_{NB}^{(i)}$ and $I_{PB}^{(i)}$:

Theorem (4.4)

Assume that $\mu_0^{(i)} = \cdots = \mu_{r_i}^{(i)}$. *Then* $I_{NB}^{*(i)} = I_{PB}^{*(i)}$.

Theorem (4.5)

Assume that the transition matrix $\boldsymbol{P}^{(i)}$ *is doubly stochastic. Then we have:*

$$I_{NB}^{*(i)} = \sum_{u,v \in S_i} E|\phi(u, \boldsymbol{X}) - \phi(v, \boldsymbol{X})| \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{uv}^{(i)}$$

$$I_{PB}^{*(i)} = \sum_{u,v \in S_i} E|\phi(u, \boldsymbol{X}) - \phi(v, \boldsymbol{X})| \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{vu}^{(i)}$$

Expected physical criticality for semi-Markov processes (c.)

EXAMPLE: We consider the same multistate system (C, ϕ) where $C = \{1, 2\}$, and where $S_1 = S_2 = \{0, 1, 2\}$. We assume that $f_i(u) = u$, $u = 0, 1, 2$, $i = 1, 2$.

The structure function is given by:

$$\phi(X_1(t), X_2(t)) = \min(f_1(X_1(t)), f_2(X_2(t))) = \min(X_1(t), X_2(t))$$

The transition matrices of the built-in Markov chains are:

$$\boldsymbol{P}^{(1)} = \begin{bmatrix} 0.1, & 0.3, & 0.6 \\ 0.6, & 0.1, & 0.3 \\ 0.3, & 0.6, & 0.1 \end{bmatrix}, \qquad \boldsymbol{P}^{(2)} = \begin{bmatrix} 0.7, & 0.1, & 0.2 \\ 0.2, & 0.7, & 0.1 \\ 0.1, & 0.2, & 0.7 \end{bmatrix},$$

while the mean waiting times are:

$$\mu_0^{(i)} = 2.5, \quad \mu_1^{(i)} = 3.5, \quad \mu_2^{(i)} = 4.0, \quad i = 1, 2.$$

Expected physical criticality for semi-Markov processes (c.)

To compute $I_{NB}^{*(1)}$ we start out by determining $E|\phi(u, X_2) - \phi(v, X_2)|$ for all $u, v \in S_1$, noting that:

$$E|\phi(u, X_2) - \phi(v, X_2)| = 0 \quad \text{if } u = v$$

This implies that only the terms where $u \neq v$ need to be included. Moreover, by symmetry we have:

$$E|\phi(u, X_2) - \phi(v, X_2)| = E|\phi(v, X_2) - \phi(u, X_2)|.$$

Expected physical criticality for semi-Markov processes (c.)

Hence, we get the following $3 \cdot 3 - 3 = 6$ non-zero expectations for component 1:

$$E|\phi(0, X_2) - \phi(1, X_2)| = 1 \cdot P[X_2 = 1] + 1 \cdot P[X_2 = 2] = 0.75,$$
$$E|\phi(0, X_2) - \phi(2, X_2)| = 1 \cdot P[X_2 = 1] + 2 \cdot P[X_2 = 2] = 1.15,$$

$$E|\phi(1, X_2) - \phi(0, X_2)| = E|\phi(0, X_2) - \phi(1, X_2)| = 0.75,$$
$$E|\phi(1, X_2) - \phi(2, X_2)| = 1 \cdot P[X_2 = 2] = 0.40,$$

$$E|\phi(2, X_2) - \phi(0, X_2)| = E|\phi(0, X_2) - \phi(2, X_2)| = 1.15,$$
$$E|\phi(2, X_2) - \phi(1, X_2)| = E|\phi(1, X_2) - \phi(2, X_2)| = 0.40.$$

Expected physical criticality for semi-Markov processes (c.)

Moreover, since these expectations only depend on the stationary distribution of component 2, and both components have the same stationary distribution, we get exactly the same probabilities for component 2.

Thus, we have all the quantities needed in order to compute the importance measures using the formula for $I_{NB}^{*(i)}$ given in Theorem 4.5, and we eventually get:

$$I_{NB}^{*(1)} = 0.66225, \qquad I_{NB}^{*(2)} = 0.22075$$

For component 1 most of the weight from the transition probabilities are put on terms where $u \neq v$, and for these terms $E|\phi(u, X_2) - \phi(v, X_2)| > 0$.

For component 2 most of the weight from the transition probabilities are put on terms where $u = v$, and for these terms $E|\phi(X_1, u) - \phi(X_1, v)| = 0$.