Multistate systems and importance measures - part 3

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Failure rates

Let $\bar{F}(t) = P(T > t)$ for the lifetime T of some component, and assume that $\bar{F}(t) > 0$ for all $t \geq 0$. Then we have:

$$\bar{F}(t) = e^{-\Lambda(t)}, \quad t \geq 0,$$

where $\Lambda(t) = -\ln(\bar{F}(t))$. The function Λ is called the cumulative failure rate function of T.

The derivative of $\Lambda(t)$, denoted $\lambda(t)$, is called the failure rate function of T and can be expressed as follows:

$$\lambda(t) = \frac{\partial}{\partial t}(-\ln(\bar{F}(t))) = \frac{f(t)}{\bar{F}(t)}, \quad t \geq 0.$$

NOTE: $\lambda(t)dt$ can be interpreted as the conditional probability that the component fails in the interval $(t, t + dt]$, given that the component survived the interval $[0, t]$.

Failure rates (cont.)

NOTE: The survival function can be expressed in terms of the failure rate function as follows:

$$\bar{F}(t) = e^{-\int_0^t \lambda(u)du}, \quad t \geq 0, \tag{1}$$

Thus, a lifetime distribution is uniquely determined by its failure rate function.

If λ is increasing, we say that the lifetime has an increasing failure rate.

If λ is decreasing, we say that the lifetime has a decreasing failure rate.

- Increasing failure rates are used to describe aging. As the component gets older, it becomes more prone to failures.
- Decreasing failure rates are used to describe infant mortality. As the component gets older, early failures are eliminated or corrected.

The time transform

The cumulative failure rate function Λ can also be interpreted as a transformation of time. Based on this interpretation we introduce the time transformed random variable of a random variable T with cumulative failure rate Λ:

$$S = \Lambda(T) = \text{The cumulative failure rate at time } T$$

In order to avoid technical issues, we assume that $\lambda(t) > 0$ for all $t \geq 0$. This implies that the cumulative failure rate:

$$\Lambda(t) = \int_0^t \lambda(u)du$$

is strictly increasing. Thus, $\Lambda(T)$ has a uniquely defined inverse, Λ^{-1}.

The time transform (cont.)

We can now easily find the distribution of the random variable S:

$$\begin{aligned} P(S > s) &= P(\Lambda(T) > s) \\ &= P(T > \Lambda^{-1}(s)) \\ &= \bar{F}(\Lambda^{-1}(s)) \\ &= e^{-\Lambda(\Lambda^{-1}(s))} = e^{-s} \end{aligned}$$

Thus, we see that the random variable S is exponentially distributed with failure rate 1.

Counting processes

Definition

A stochastic process $\{N(t)\}$ *is a counting process if:*

1. $N(t) \geq 0$
2. $N(t) \in \mathbb{Z}$.
3. *If* $s < t$, *then* $N(s) \leq N(t)$
4. *If* $s < t$, *then* $N(t) - N(s)$ *represents the number of events that have occurred in* $(s, t]$.

NOTE: If $N(0) = 0$, the last property implies that:

$$N(t) = N(t) - N(0) = \text{The number of events that have occurred in } [0, t]$$

Homogenous Poisson processes

Definition

A counting process $\{N(t)\}$ *is a homogenous Poisson process with rate* λ *if:*

1. $N(0) = 0$
2. $\{N(t)\}$ *has stationary and independent increments.*
3. $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$
4. $P(N(t+h) - N(t) \geq 2) = o(h)$

NOTE: If $W_1, W_2, \ldots$ are the waiting times between events in a homogenous Poisson process with rate λ, then:

$$W_1, W_2, \ldots \text{ are independent and exponentially distributed with rate } \lambda$$

Non-homogenous Poisson processes

Definition

A counting process $\{N(t)\}$ *is a non-homogenous Poisson process with rate function* $\lambda(t)$ *if:*

1. $N(0) = 0$
2. $\{N(t)\}$ *has independent increments.*
3. $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$
4. $P(N(t+h) - N(t) \geq 2) = o(h)$

We also introduce the cumulate rate function:

$$\Lambda(t) = \int_0^t \lambda(u)du$$

Moreover, we assume that $\lambda(u) > 0$ for all $u > 0$, implying that Λ is strictly increasing, and that Λ has a uniquely defined inverse Λ^{-1}.

Non-homogenous Poisson processes (cont.)

Some properties of a non-homogenous Poisson process:

$$P(N(t) = n) = \frac{[\Lambda(t)]^n}{n!} e^{-\Lambda(t)}, \quad n = 0, 1, \ldots$$

$$P(N(t+v) - N(t) = n) = \frac{[\Lambda(t+v) - \Lambda(t)]^n}{n!} e^{-(\Lambda(t+v)-\Lambda(t))} \quad n = 0, 1, \ldots$$

$$P(N(t+v) - N(t) = 0) = e^{-(\Lambda(t+v)-\Lambda(t))}$$

Non-homogenous Poisson processes (cont.)

Let $T_1 < T_2 < \cdots$ denote the points of time when the process $\{N(t)\}$ jumps. We also let $T_0 = 0$ and introduce the waiting times between the event times:

$$W_k = T_k - T_{k-1}, \quad k = 1, 2, \ldots$$

We then have:

$$\begin{aligned} P(W_k > w_k | T_{k-1} = t_{k-1}) &= P(N(t_{k-1} + w_k) - N(t_{k-1}) = 0) \\ &= e^{-(\Lambda(t_{k-1}+w_k)-\Lambda(t_{k-1}))} \end{aligned}$$

Non-homogenous Poisson processes (cont.)

We then introduce the waiting times between the time transformed event times:

$$V_k = \Lambda(T_k) - \Lambda(T_{k-1}), \quad k = 1, 2, \ldots$$

We then have:

$$\begin{aligned}
P(V_k > v_k | T_{k-1} = t_{k-1}) &= P(\Lambda(T_k) - \Lambda(t_{k-1}) > v_k | T_{k-1} = t_{k-1}) \\
&= P(\Lambda(T_k) > \Lambda(t_{k-1}) + v_k | T_{k-1} = t_{k-1}) \\
&= P(T_k > \Lambda^{-1}(\Lambda(t_{k-1}) + v_k) | T_{k-1} = t_{k-1}) \\
&= P(W_k > \Lambda^{-1}(\Lambda(t_{k-1}) + v_k) - t_{k-1} | T_{k-1} = t_{k-1})
\end{aligned}$$

Non-homogenous Poisson processes (cont.)

We then substitute:

$$w_k = \Lambda^{-1}(\Lambda(t_{k-1}) + v_k) - t_{k-1},$$

and note that:

$$\begin{aligned}\Lambda(t_{k-1} + w_k) &= \Lambda(t_{k-1} + \Lambda^{-1}(\Lambda(t_{k-1}) + v_k) - t_{k-1}) \\ &= \Lambda(\Lambda^{-1}(\Lambda(t_{k-1}) + v_k)) \\ &= \Lambda(t_{k-1}) + v_k\end{aligned}$$

Hence, we get that:

$$\begin{aligned}P(V_k > v_k | T_{k-1} = t_{k-1}) &= P(W_k > \Lambda^{-1}(\Lambda(t_{k-1}) + v_k) - t_{k-1} | T_{k-1} = t_{k-1}) \\ &= P(W_k > w_k | T_{k-1} = t_{k-1}) \\ &= e^{-(\Lambda(t_{k-1}+w_k)-\Lambda(t_{k-1}))} = e^{-v_k}\end{aligned}$$

Non-homogenous Poisson processes (cont.)

This shows that a non-homogenous Poisson process with rate function $\lambda(t)$ can be constructed from a homogenous Poisson process as follows:

We start out by generating independent and identically distributed waiting times:

$$V_1, V_2, \ldots$$

from the exponential distribution with failure rate 1.

In order to compute the resulting event times, $T_1, T_2, \ldots$, of the non-homogenous Poisson process we note that:

$$V_k = \Lambda(T_k) - \Lambda(T_{k-1}), \quad k = 1, 2, \ldots$$

Using $T_0 = 0$ as a starting point, the event times, $T_1, T_2, \ldots$, can be calculated recursively by:

$$T_k = \Lambda^{-1}[\Lambda(T_{k-1}) + V_k], \quad k = 1, 2, \ldots$$

Renewal processes

Definition

A counting process $\{N(t)\}$ is a renewal process if the waiting times between events, $W_1, W_2, \ldots$ are independent and identically distributed with cumulative distribution function F.

NOTE: If $\{N(t)\}$ is a renewal process and the waiting times between events, $W_1, W_2, \ldots$ are exponentially distributed with rate λ, then $\{N(t)\}$ is a homogenous Poisson process.

Thus, renewal processes are generalizations of homogenous Poisson processes, similar to how semi-Markov processes are generalizations of Markov processes.

The trend-renewal model

Motivated by renewal processes and the connection between non-homogenous Poisson processes and homogenous Poisson processes, we now consider a more general type of process where we allow the time transformed waiting times:

$$V_1, V_2, \ldots$$

to be generated from an arbitrary distribution on the positive axis.

Definition (Trend-renewal model)

Let $\lambda(t)$ be a positive intensity function defined for all $t \geq 0$, satisfying $\Lambda(t) \equiv \int_0^t \lambda(u)du < \infty$ for all $t \geq 0$, and $\Lambda(\infty) = \infty$.

Furthermore, let F be a cumulative distribution function such that $F(0) = 0$.

A counting process is a trend-renewal process with respect to F and λ, and written as $TRP(F, \lambda)$, if the waiting times between the time-transformed event times $\Lambda(T_0), \Lambda(T_1), \ldots$ are independent and identically distributed with cumulative distribution function F.

The trend-renewal model (cont.)

To explain this model in more detail, we again consider the waiting times between the time-transformed event times:

$$V_k = \Lambda(T_k) - \Lambda(T_{k-1}), \quad k = 1, 2, \ldots$$

By the definition of the trend-renewal model, $V_1, V_2, \ldots$ are independent, identically distributed and:

$$P(V_k \leq v) = F(v), \quad k = 1, 2, \ldots$$

Thus, the process $TRP(F, \lambda)$ may be constructed by generating a sequence of independent random variables $V_1, V_2, \ldots$ from the distribution F.

In order to compute the resulting event times $T_1, T_2, \ldots$, we recall that:

$$V_k = \Lambda(T_k) - \Lambda(T_{k-1}), \quad k = 1, 2, \ldots$$

The trend-renewal model (cont.)

Thus, the event times, $T_1, T_2, \ldots$, can be computed recursively by:

$$T_k = \Lambda^{-1}[\Lambda(T_{k-1}) + V_k], \quad k = 1, 2, \ldots$$

Alternatively, since $T_0 = 0$ and that $\Lambda(0) = 0$, we can compute the event times, $T_1, T_2, \ldots$, using the following expansion:

$$T_1 = \Lambda^{-1}[\Lambda(T_0) + V_1] = \Lambda^{-1}(V_1)$$

$$T_2 = \Lambda^{-1}[\Lambda(T_1) + V_2] = \Lambda^{-1}(V_1 + V_2)$$

$$T_3 = \Lambda^{-1}[\Lambda(T_2) + V_3] = \Lambda^{-1}(V_1 + V_2 + V_3)$$

$$\ldots \quad \ldots$$

The trend-renewal model (cont.)

NOTE: For a given process $TRP(F, \lambda)$ and constant $c > 0$, we may define alternative functions for $t \geq 0$:

$$\tilde{\lambda}(t) = c\lambda(t)$$
$$\tilde{\Lambda}(t) = \int_0^t \tilde{\lambda}(u)du = c\int_0^t \lambda(u)du = c\Lambda(t)$$

Then it is easy to verify that:

$$\tilde{\Lambda}^{-1}(u) = \Lambda^{-1}(c^{-1} \cdot u)$$

This follows since:

$$\tilde{\Lambda}^{-1}(\tilde{\Lambda}(t)) = \Lambda^{-1}(c^{-1} \cdot \tilde{\Lambda}(t)) = \Lambda^{-1}(c^{-1} \cdot c\Lambda(t)) = \Lambda^{-1}(\Lambda(t)) = t$$

The trend-renewal model (cont.)

We then let:

$$\tilde{V}_k = \tilde{\Lambda}(T_k) - \tilde{\Lambda}(T_{k-1}), \quad k = 1, 2, \ldots$$

be the waiting times obtained using the alternative time-transform $\tilde{\Lambda}$. Then:

$$\tilde{V}_k = \tilde{\Lambda}(T_k) - \tilde{\Lambda}(T_{k-1}) = c\Lambda(T_k) - c\Lambda(T_{k-1}) = cV_k, \quad k = 1, 2, \ldots$$

Hence, we get that:

$$\begin{aligned}
\tilde{\Lambda}^{-1}(\tilde{V}_1 + \cdots + \tilde{V}_k) &= \Lambda^{-1}(c^{-1}[\tilde{V}_1 + \cdots + \tilde{V}_k]) \\
&= \Lambda^{-1}(c^{-1}[cV_1 + \cdots + cV_k]) \\
&= \Lambda^{-1}(V_1 + \cdots + V_k) = T_k, \quad k = 1, 2, \ldots
\end{aligned}$$

Thus, the alternative function $\tilde{\Lambda}$ and alternative waiting times, $\tilde{V}_1, \tilde{V}_2, \ldots$ generate the same event times, $T_1, T_2, \ldots$, as the original function, Λ, and the original waiting times, $V_1, V_2, \ldots$.

The trend-renewal model (cont.)

By definition of the trend-renewal model $TRP(F, \lambda)$, the cumulative distribution function of the original waiting times is F.

The cumulative distribution function of the alternative waiting times is given by:

$$\tilde{F}(v) = P(\tilde{V}_k < v) = P(cV_k < v) = F(v/c), \quad k = 1, 2, \ldots$$

Since the alternative function $\tilde{\Lambda}$ and alternative waiting times $\tilde{V}_1, \tilde{V}_2, \ldots$ generate the same event times, $T_1, T_2, \ldots$, as the original function, Λ, and the original waiting times $V_1, V_2, \ldots$, we conclude that:

$$TRP(F, \lambda) \quad \Leftrightarrow \quad TRP(\tilde{F}, \tilde{\lambda})$$

To avoid ambiguity, some authors only consider trend-renewal processes where the expected waiting times between the time-transformed event times are equal to 1.

In our context, this is ambiguity does not cause problems, so we ignore this issue.

A multistate trend-renewal model

NOTE: A trend-renewal model can only be used for counting processes (typically counting failure events). In the context of multistate components we need a similar framework generalizing semi-Markov processes.

For $s \in S$ and $k = 1, 2, \ldots$ we introduce:

$$T'_{ks} = \text{The } k\text{th time the component enters state } s$$
$$T_{ks} = \text{The } k\text{th time the component leaves state } s$$

For $s \in S$ and $k = 1, 2, \ldots$ we also introduce the waiting times spent in each state between the transitions:

$$W_{ks} = \text{The } k\text{th waiting time in state } s = T_{ks} - T'_{ks}$$

Finally, we let $\Lambda_s(t) = \int_0^t \lambda_s(u)du$ denote the time transform applied to event times affecting the state $s \in S$.

The Global time model

In this model we assume that the waiting times for all the states are affected by the same global clock.

Thus, e.g., if a component ages by time, this aging occurs regardless of the state the component is in.

In particular, the component ages also when it is under repair.

A global time model is also appropriate for components which are subject to seasonal effects, since such effects typically affects the component regardless of state.

The Global time model (cont.)

We let the waiting times between the time-transformed event times be defined as:

$$V_{ks} = \Lambda_s(T_{ks}) - \Lambda_s(T'_{ks})$$

Assuming that $V_{1s}, V_{2s}, \ldots$ are independent and identically distributed with cumulative distribution function F_s, these quantities can easily be generated using Monte Carlo simulation.

The event times, $T_{1s}, T_{2s}, \ldots$ can then be calculated for all $s \in S$ using the following recursive relation:

$$T_{ks} = \Lambda_s^{-1}[\Lambda_s(T'_{ks}) + V_{ks}], \quad k = 1, 2, \ldots$$

NOTE: In this case we do not necessarily have $T'_{1,s} = 0$. In fact, since we use a global clock, the times $T'_{1,s}, T'_{2,s}, \ldots$ depends on the amount of time spent in the other states.

The Local time model

In this model we assume that the waiting times for the states are affected by individual local clocks for each state.

Thus, e.g., aging in one given state is independent of the time spent in the other states.

In particular, aging in each of the functioning states is independent of repair times.

Such a model, however, is not likely to fit a case where the component is subject to global effects.

The Local time model (cont.)

We recall that the waiting times spent in each state between the transitions are:

$$W_{ks} = T_{ks} - T'_{ks}$$

We then let $U_{0s} = 0$, and introduce the following quantities for $k = 1, 2, \ldots$ and $s \in S$:

$$U_{ks} = W_{1s} + \cdots + W_{ks} = U_{k-1,s} + W_{ks}.$$

Thus, $U_{1s}, U_{2s}, \ldots$ are the points of time when the component leaves state s, given that we use a clock which is stopped when the component is not in state s. We will refer to these points of time as the *net event times* for state s.

The Local time model

We then let the waiting times between the time-transformed net event times be defined as:

$$V_{ks} = \Lambda_s(U_{ks}) - \Lambda_s(U_{k-1,s}), \quad k = 1, 2, \ldots$$

Assuming that $V_{1s}, V_{2s}, \ldots$ are independent and identically distributed with cumulative distribution function F_s, these quantities can easily be generated using Monte Carlo simulation.

The Local time model

The net event times $U_{1s}, U_{2s}, \ldots$ for state s can be calculated for all $s \in S$ using:

$$U_{ks} = \Lambda_s^{-1}[\Lambda_s(U_{k-1,s}) + V_{ks}], \quad k = 1, 2, \ldots$$

Thus, recalling that $U_{0s} = 0$ and that $\Lambda_s(0) = 0$, we have that:

$$U_{1s} = \Lambda_s^{-1}[\Lambda_s(U_{0s}) + V_{1s}] = \Lambda^{-1}(V_{1s})$$
$$U_{2s} = \Lambda_s^{-1}[\Lambda_s(U_{1s}) + V_{2s}] = \Lambda^{-1}(V_{1s} + V_{2s})$$
$$\ldots \quad \ldots$$

The Local time model

Finally, in order to find the event times $T_{1s}, T_{2s}, \ldots$, we recall that $T_{ks} = T'_{ks} + W_{ks}$, $k = 1, 2, \ldots$.

Hence, by inserting the net event times we get that:

$$T_{ks} = T'_{ks} + U_{ks} - U_{k-1,s}, \quad k = 1, 2, \ldots$$

Specific rate functions

Lakey and Rigdon (1992) introduced the power law model with two parameters $\alpha, \beta > 0$, where:

$$\lambda(t) = \alpha\beta t^{\alpha-1}, \quad t \geq 0.$$

The corresponding cumulative rate function is given by:

$$\Lambda(t) = \int_0^t \lambda(u)du = \beta t^{\alpha}.$$

NOTE: If $\alpha > 1$, the intensity is increasing, while if $\alpha < 1$, the intensity is decreasing. To run simulations we also need the inverse of the cumulative intensity function. For the power law model this is given by:

$$\Lambda^{-1}(u) = (u/\beta)^{1/\alpha}.$$

Specific rate functions (cont.)

A more general model could allow non-monotonic intensity functions as well. This can be accomplished by using an additive model:

$$\lambda(t) = \sum_{j=1}^{m} \gamma_j \lambda_j(t),$$

where $\lambda_1, \ldots, \lambda_m$ are intensity functions, and where $\gamma_1, \ldots, \gamma_m$ are non-negative constants. By applying appropriate scaling, it is easy and convenient to choose these constants so that:

$$\sum_{j=1}^{m} \gamma_j = 1.$$

If this is done, λ becomes a convex combination of the intensity functions $\lambda_1, \ldots, \lambda_m$. By integrating the intensity function we find that the cumulative intensity function is:

$$\Lambda(t) = \sum_{j=1}^{m} \gamma_j \Lambda_j(t),$$

Specific rate functions (cont.)

For additive intensity functions it may not be possible to find analytical expressions for Λ^{-1}. However, if we can find analytical expressions for $\Lambda_1^{-1}, \ldots, \Lambda_m^{-1}$, then Λ^{-1} can easily be determined numerically.

Assume more specifically, that for a given u we are able to compute $t_j = \Lambda_j^{-1}(u)$, $j = 1, \ldots, m$, and that:

$$t_{\min} = \min_{1 \leq j \leq m} t_j, \text{ and } t_{\max} = \max_{1 \leq j \leq m} t_j.$$

Then it is easy to show that:

$$t_{\min} \leq \Lambda^{-1}(u) \leq t_{\max}.$$

By using these lower and upper bounds we can easily find $\Lambda^{-1}(u)$ numerically using e.g., the bisection method.

Importance measures in the non-homogenous case

We consider a multistate system (C, ϕ) where $C = \{1, 2\}$, and where both components have only three possible states, i.e., $S_1 = S_2 = \{0, 1, 2\}$.

For simplicity we let $f_i(s) = s$, for all $s \in S_i$, $i = 1, 2$.

The structure function is given by:

$$\phi(X_1(t), X_2(t)) = \min(f_1(X_1(t)), f_2(X_2(t))).$$

The transition matrices of the built-in Markov chains are:

$$\boldsymbol{P}^{(1)} = \boldsymbol{P}^{(2)} = \begin{bmatrix} 0.0, & 0.1, & 0.9 \\ 0.9, & 0.0, & 0.1 \\ 0.1, & 0.9, & 0.0 \end{bmatrix},$$

Importance measures in the non-homogenous case (cont.)

For $s \in S_i$, $i \in C$ and $k = 1, 2, \ldots$ we introduce:

$$T_{ks}^{(i)'} = \text{The } k\text{th time Component } i \text{ enters state } s$$
$$T_{ks}^{(i)} = \text{The } k\text{th time Component } i \text{ leaves state } s$$

Moreover, we introduce the waiting times spent in each state between the transitions:

$$\begin{aligned} W_{ks}^{(i)} &= \text{Component } i\text{'s } k\text{th waiting time in state } s \\ &= T_{ks}^{(i)} - T_{ks}^{(i)'}, \end{aligned}$$

as well as the net event times:

$$U_{ks}^{(i)} = W_{1s}^{(i)} + \cdots + W_{ks}^{(i)} = U_{k-1,s}^{(i)} + W_{ks}^{(i)}.$$

Importance measures in the non-homogenous case (cont.)

The time transform applied to event times affecting the state $s \in S_i$, $i \in C$ is denoted by:

$$\Lambda_s^{(i)}(t) = \int_0^t \lambda_s^{(i)}(u)du.$$

Finally, we assume that:

$$V_{1s}^{(i)}, V_{2s}^{(i)}, \ldots, \quad s \in S_i,$$

are independent and identically distributed with cumulative distribution function $F_s^{(i)}$.

The global and local time models

For the global time model we then have:

$$T_{ks}^{(i)} = (\Lambda_s^{(i)})^{-1}[\Lambda_s(T_{ks}^{(i)'}) + V_{ks}^{(i)}],$$

while for the local time model we have:

$$U_{ks}^{(i)} = (\Lambda_s^{(i)})^{-1}[\Lambda_s(U_{ks}^{(i)}) + V_{ks}^{(i)}].$$

More specifically we assume that $V_{1s}^{(i)}, V_{2s}^{(i)}, \ldots$ are independent and exponentially distributed with expected value $\mu_s^{(i)}$, where:

$$\mu_0^{(1)} = \mu_0^{(2)} = 0.5,$$
$$\mu_1^{(1)} = \mu_1^{(2)} = 2.0,$$
$$\mu_2^{(1)} = \mu_2^{(2)} = 7.5.$$

The global and local time models (cont.)

In order to see the effect of the trend-renewal model more clearly, we use the following intensity functions:

$$\lambda_0^{(1)}(t) = \lambda_1^{(1)}(t) = 1,$$
$$\lambda_0^{(2)}(t) = \lambda_1^{(2)}(t) = \lambda_2^{(2)}(t) = 1.$$

The corresponding cumulative intensity functions then become:

$$\Lambda_0^{(1)}(t) = \Lambda_1^{(1)}(t) = t,$$
$$\Lambda_0^{(2)}(t) = \Lambda_1^{(2)}(t) = \Lambda_2^{(2)}(t) = t.$$

The global and local time models (cont.)

For state 2 of Component 1, however, a more complicated intensity function is used:

$$\lambda_2^{(1)}(t) = \sum_{j=1}^{3} \gamma_j(\alpha_j \beta_j t^{\alpha_j - 1}),$$

where the parameters of $\lambda_2^{(1)}(t)$ are:

$$\begin{aligned} \alpha_1 &= 0.75, \quad \alpha_2 = 1.0, \quad \alpha_3 = 1.6, \\ \beta_1 &= 2.0, \quad \beta_2 = 1.0, \quad \beta_3 = 0.1, \\ \gamma_1 &= 0.75, \quad \gamma_2 = 0.15, \quad \gamma_3 = 0.1. \end{aligned}$$

The corresponding cumulative intensity function then becomes:

$$\begin{aligned} \Lambda_2^{(1)}(t) &= \sum_{j=1}^{3} \gamma_j(\beta_j t^{\alpha_j}) \\ &= 0.75(2.0t^{0.75}) + 0.15t + 0.1(0.1t^{1.6}). \end{aligned}$$

The global and local time models (cont.)

Below we have plotted the intensity function $\lambda_2^{(1)}(t)$.

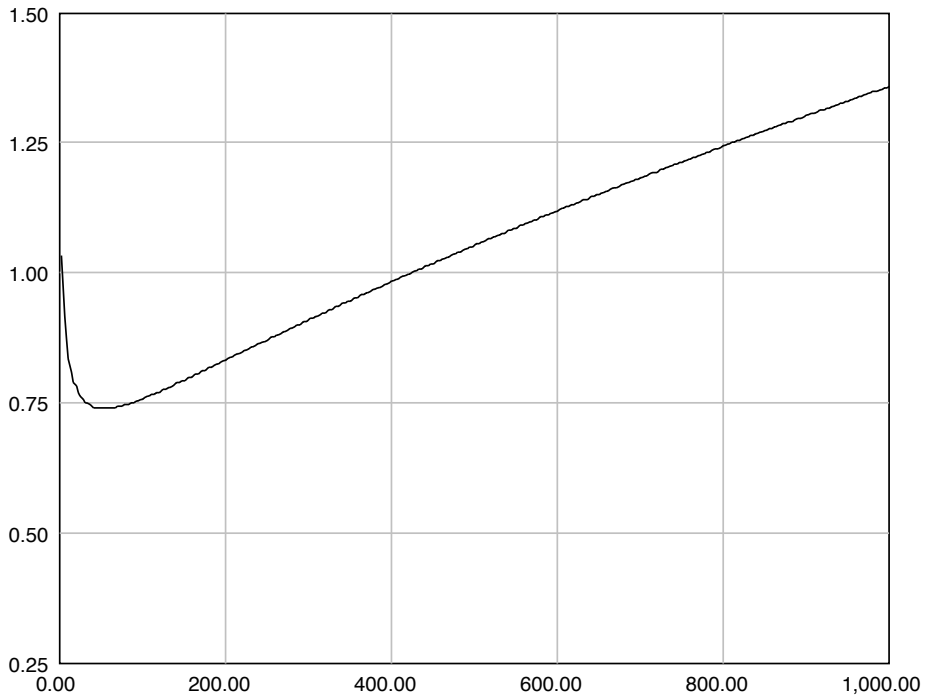

Figure: The intensity function $\lambda_2^{(1)}(t)$

We observe that $\lambda_2^{(1)}(t)$ is decreasing in the beginning, and then increasing later. This shape is often referred to as a bath tub shape.

Monte Carlo simulation using Global time model

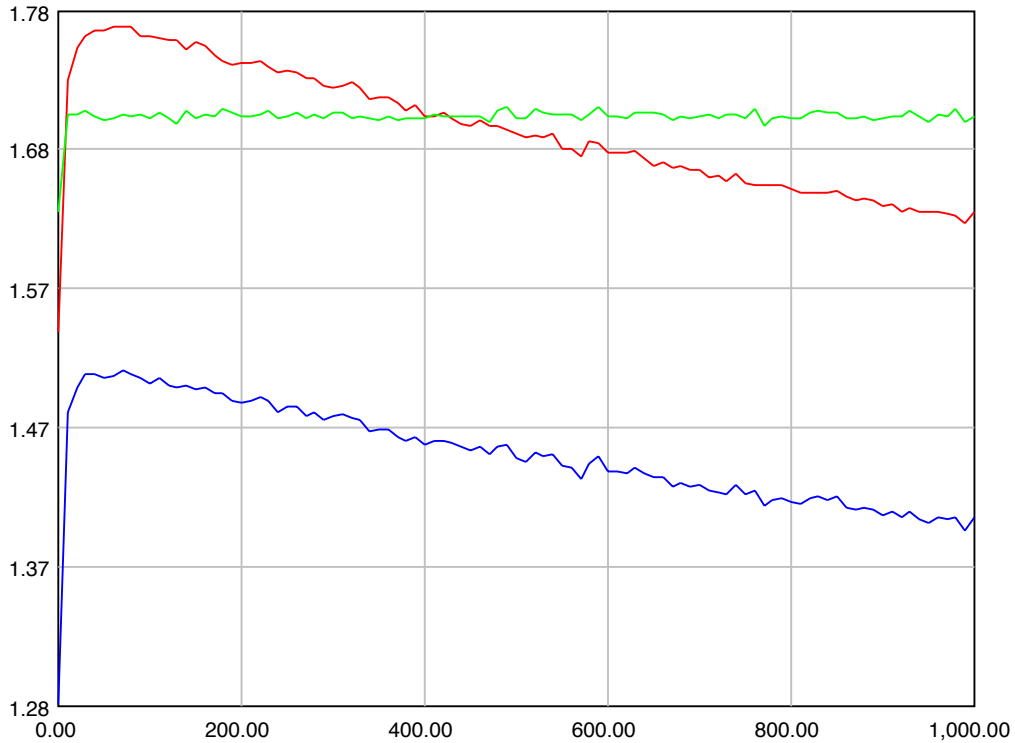

Figure: $\mathrm{E}[X_1(t)]$ (red), $\mathrm{E}[X_2(t)]$ (green) and $\mathrm{E}[\phi(t)]$ (blue) - Global time model

Monte Carlo simulation using Local time model

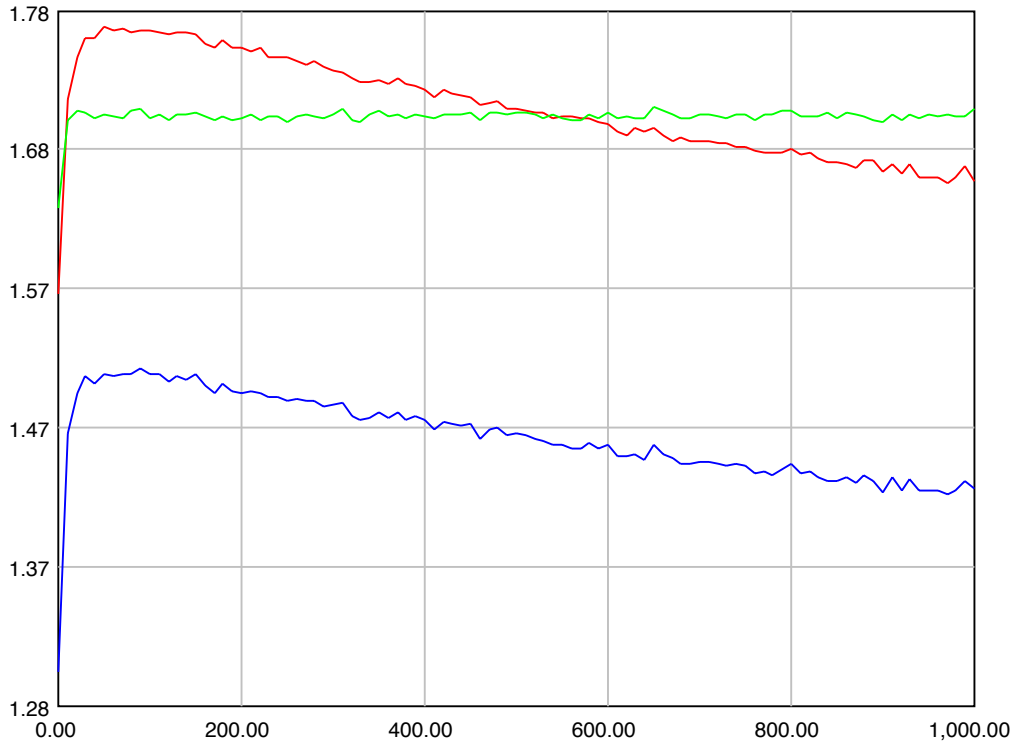

Figure: $\mathrm{E}[X_1(t)]$ (red), $\mathrm{E}[X_2(t)]$ (green) and $\mathrm{E}[\phi(t)]$ (blue) - Local time model

Monte Carlo simulation using Global time model

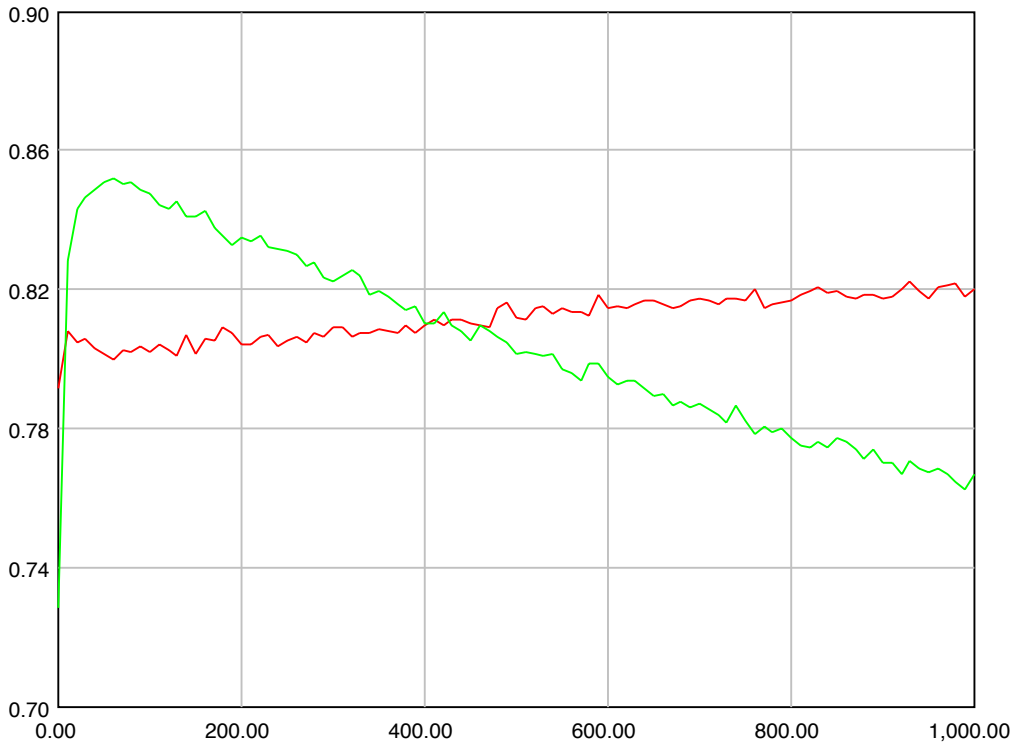

Figure: $I_{NB}^{(1)}(t)$ (red) and $I_{NB}^{(2)}(t)$ (green)

Monte Carlo simulation using Local time model

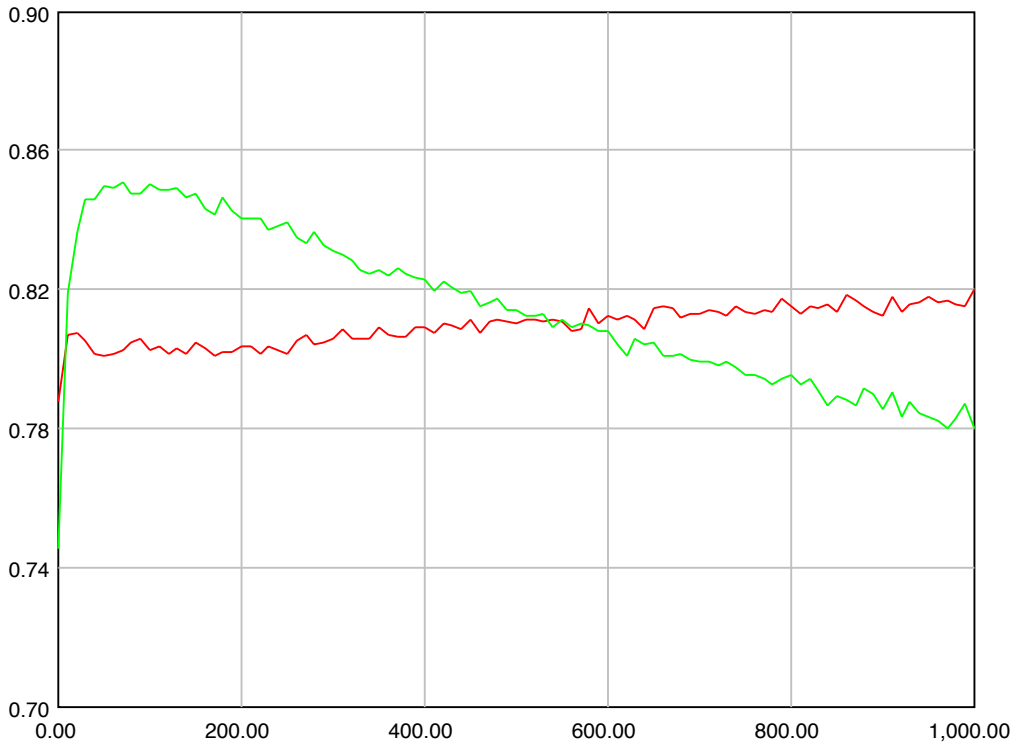

Figure: $I_{NB}^{(1)}(t)$ (red) and $I_{NB}^{(2)}(t)$ (green)

Monte Carlo simulation using Global time model

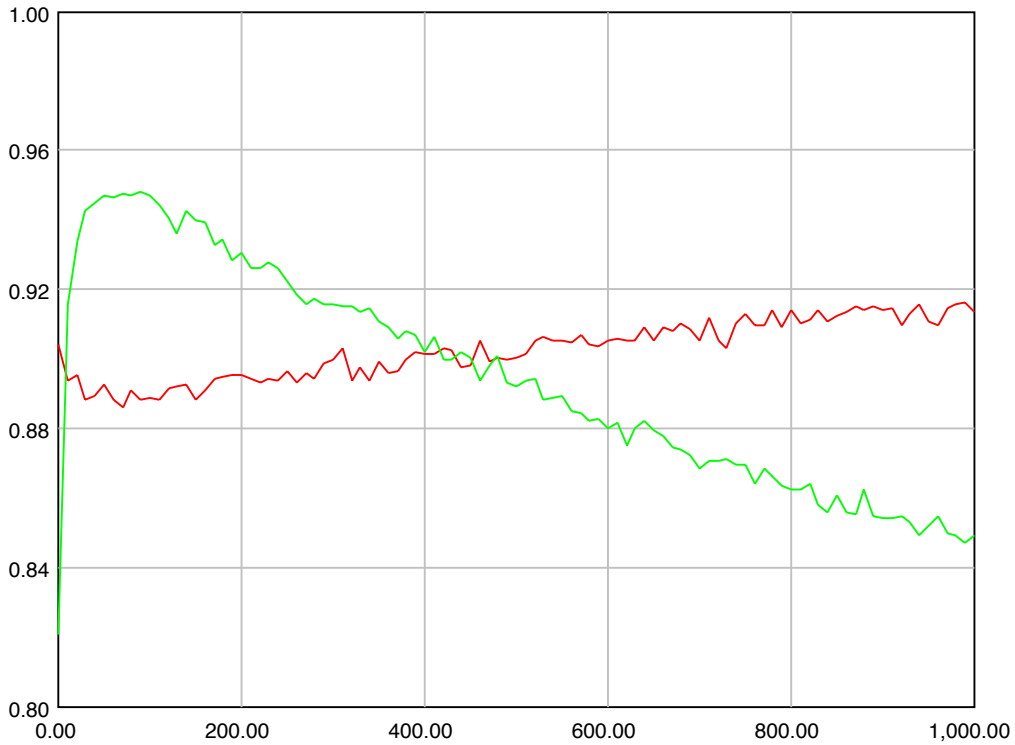

Figure: $I_{NB}^{*(1)}(t)$ (red) and $I_{NB}^{*(2)}(t)$ (green) - Global time model

Monte Carlo simulation using Local time model

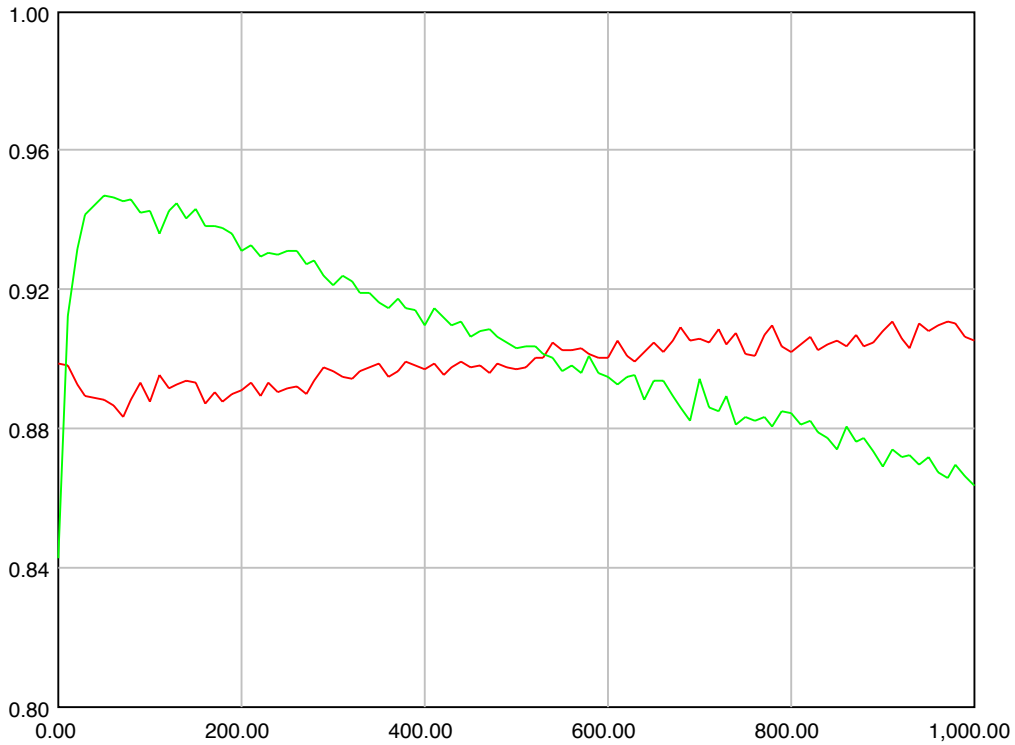

Figure: $I_{NB}^{*(1)}(t)$ (red) and $I_{NB}^{*(2)}(t)$ (green) - Local time model

Monte Carlo Result Summary

- The results show that in a non-stationary case, the importance ranking may change over time.
- In order to obtain a time-independent ranking, one must average the measures over time.
- In both plots we observe that Component 2 is more important than Component 1 in the first phase, while Component 1 is more important than Component 2 in the second phase.
- Using the Global time model, the first phase is shorter than when the Local time model is used.
- The choice between the Global and the Local model can have an impact on the results.