Multi-reservoir production optimization

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Multi-reservoir production optimization

- Part 1. Optimization in the deterministic case:

  Huseby and Haavardsson (2008) *A Framework For Multi-reservoir Production Optimization*, Sections 1-4

- Part 2. Optimization in the stochastic case:

  Huseby and Sødal (2012) *Sequential optimization of oil production from multiple reservoirs under uncertainty*

Actual production restricted by processing capacity

Consider the oil production from a field consisting of n reservoirs that share a processing facility with a constant process capacity $K > 0$.

$$\boldsymbol{Q}(t) = (Q_1(t), \ldots, Q_n(t)) = \text{Cumulative production functions}$$
$$\boldsymbol{f}(t) = (f_1(t), \ldots, f_n(t)) = \text{Pot. production rate (PPR) functions}$$

We assume that the *ultimately recoverable volumes* from the n reservoirs are respectively $V_1, \ldots, V_n$, and that:

$$0 \leq Q_i(t) \leq V_i, \quad i = 1, \ldots n.$$

Moreover, we assume that:

$$f_i(t) = f_i(Q_i(t)), \quad t \geq 0, \; i = 1, \ldots n.$$

Typically, f_i will be a decreasing function of Q_i, $i = 1, \ldots, n$.

Actual production restricted by processing capacity

If the sum of the potential production rates exceeds the capacity K of the processing facility, i.e.,

$$\sum_{i=1}^{n} f_i(t) > K$$

the production needs to be *choked*.

$$\boldsymbol{q}(t) = (q_1(t), \ldots, q_n(t)) = \text{Actual production rates after choking}$$

$$q(t) = \sum_{i=1}^{n} q_i(t) = \text{Total production rate at time } t$$

$$Q(t) = \sum_{i=1}^{n} Q_i(t) = \text{Total cumulative production at time } t$$

Production strategy

A *production strategy* is defined for all $t \geq 0$:

$$\boldsymbol{b} = \boldsymbol{b}(t) = (b_1(t), \ldots, b_n(t)),$$

where $b_i(t)$ represents the *choke factor*, i.e., the fraction of the potential production rate of the ith reservoir that is actually produced at time t, $i = 1, \ldots, n$.

The *actual production rates* from the reservoirs after the production is choked are given by:

$$q_i(t) = \frac{dQ_i(t)}{dt} = b_i(t) f_i(Q_i(t)), \quad i = 1, \ldots, n.$$

Valid production strategies

To satisfy the physical constraints of the reservoirs and the process facility, we require that:

$$0 \leq b_i(t) \leq 1, \quad i = 1, \ldots, n, \;\; t \geq 0.$$

$$\sum_{i=1}^{n} b_i(t) f_i(Q_i(t)) \leq K.$$

$\mathcal{B}$ denotes the class of production strategies that satisfy these physical constraints. We refer to production strategies $\boldsymbol{b} \in \mathcal{B}$ as *valid production strategies*.

Admissible production strategies

An *admissible production strategy* is defined as a valid production strategy $\boldsymbol{b}$ where the total production rate $q(t)$ satisfies the following constraint for all $t \geq 0$:

$$q(t) = \sum_{i=1}^{n} b_i(t) f_i(Q_i(t)) = \min\{K, \sum_{i=1}^{n} f_i(Q_i(t))\}.$$

$\mathcal{B}' \subseteq \mathcal{B}$ denotes the class of admissible strategies.

If $\boldsymbol{b} \in \mathcal{B}'$ and $T_K = \sup\{t \geq 0 : \sum_{i=1}^{n} f_i(Q_i(t)) \geq K\}$ is the plateau length, then:

$$\begin{aligned} q(t) &= K, && 0 \leq t \leq T_K. \\ q(t) &= \sum_{i=1}^{n} f_i(Q_i(t)), && t > T_K. \end{aligned}$$

Objective functions

An *objective function* is a mapping $\phi : \mathcal{B} \to \mathbb{R}$ such that if $\boldsymbol{b}^1, \boldsymbol{b}^2 \in \mathcal{B}$, we prefer $\boldsymbol{b}^2$ to $\boldsymbol{b}^1$ if $\phi(\boldsymbol{b}^2) \geq \phi(\boldsymbol{b}^1)$.

An *optimal production strategy* with respect to ϕ is a production strategy $\boldsymbol{b}^{opt} \in \mathcal{B}$ such that $\phi(\boldsymbol{b}^{opt}) \geq \phi(\boldsymbol{b})$ for all $\boldsymbol{b} \in \mathcal{B}$.

Monotone objective functions

Definition

An objective function ϕ *is said to be monotone if for any pair of production strategies* $\boldsymbol{b}^1, \boldsymbol{b}^2 \in \mathcal{B}$ *such that* $\boldsymbol{Q}(t, \boldsymbol{b}^1) \leq \boldsymbol{Q}(t, \boldsymbol{b}^2)$ *for all* $t \geq 0$ *we have* $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^2)$*.*

Proposition

Let ϕ *be a monotone objective function, and let* $\boldsymbol{b}^1, \boldsymbol{b}^2 \in \mathcal{B}$ *be such that* $\boldsymbol{b}^1(t) \leq \boldsymbol{b}^2(t)$ *for all* $t \geq 0$*. Then* $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^2)$*.*

Proposition

Let ϕ *be a monotone objective function, and let* $\boldsymbol{b} \in \mathcal{B}$*. Then there exists* $\boldsymbol{b}' \in \mathcal{B}'$ *such that* $\phi(\boldsymbol{b}') \geq \phi(\boldsymbol{b})$*.*

Symmetric objective functions

Definition

An objective function ϕ is said to be symmetric if it depends on a production strategy $\boldsymbol{b}$ only through the total production rate function $q(\cdot, \boldsymbol{b})$ (or equivalently through $Q(\cdot, \boldsymbol{b})$).

Proposition

Let ϕ be a symmetric objective function. Then ϕ is monotone if and only if for any pair of production strategies, $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ such that $Q(t, \boldsymbol{b}^1) \leq Q(t, \boldsymbol{b}^2)$ for all $t \geq 0$, we have $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^2)$.

Symmetric objective functions (cont.)

Proposition

Let ϕ be a symmetric objective function, and let $\boldsymbol{b} \in \mathcal{B}'$. Then $\phi(\boldsymbol{b})$ is uniquely determined by $\boldsymbol{Q}(T_K(\boldsymbol{b}))$. Thus, we may write $\phi(\boldsymbol{b}) = \phi(\boldsymbol{Q}(T_K(\boldsymbol{b})))$.

Since ϕ is assumed to be symmetric, it depends on $\boldsymbol{b}$ only through q. Furthermore, since $\boldsymbol{b} \in \mathcal{B}'$, we know that $q(t) = K$ whenever $0 \leq t \leq T_K(\boldsymbol{b})$. This implies that:

$$Q(T_K(\boldsymbol{b})) = \sum_{i=1}^{n} Q_i(T_K(\boldsymbol{b})) = KT_K(\boldsymbol{b}).$$

Hence, the plateau length $T_K(\boldsymbol{b})$ can be recovered from $\boldsymbol{Q}(T_K(\boldsymbol{b}))$ as:

$$T_K(\boldsymbol{b}) = K^{-1} \sum_{i=1}^{n} Q_i(T_K(\boldsymbol{b})).$$

Symmetric objective functions (cont.)

If $t > T_K(\boldsymbol{b})$, it follows since $\boldsymbol{b} \in \mathcal{B}'$ that:

$$q(t) = \sum_{i=1}^{n} q_i(t) = \sum_{i=1}^{n} f_i(Q_i(t))$$

By the Picard-Lindelöf's theorem $q_i(t)$ is uniquely determined for all $t > T_K(\boldsymbol{b})$ by its respective differential equation along with the boundary condition given by the value $Q_i(T_K(\boldsymbol{b}))$, $i = 1, \ldots, n$.

Thus, $q(t)$ is uniquely determined by $\boldsymbol{Q}(T_K(\boldsymbol{b}))$ for all $t \geq 0$, and hence so is ϕ ■

Symmetric objective functions (cont.)

As an example we consider the following objective function:

$$\phi(\boldsymbol{b}) = \int_0^\infty I\{q(u) \geq C\} q(u) e^{-Ru} du,$$
$$0 \leq C \leq K, \quad R \geq 0,$$

where R is a discount factor, and C is a threshold value reflecting the minimum acceptable production rate. We refer to this objective function as a *truncated discounted production* objective function.

This objective function is both *monotone* and *symmetric*. Hence, for any admissible production strategy, $\boldsymbol{b}$, $\phi(\boldsymbol{b})$ is uniquely determined by $\boldsymbol{Q}(T_K(\boldsymbol{b}))$.

Optimizing production strategies

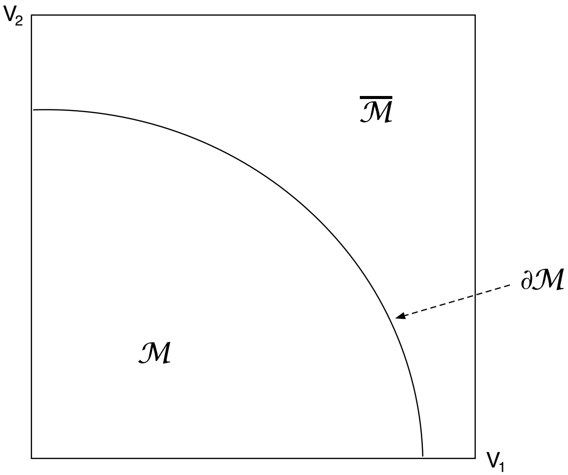
In order to study the optimization problem further we introduce the following sets:

$$\mathcal{Q} = [0, V_1] \times \cdots \times [0, V_n],$$
$$\mathcal{M} = \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) \geq K\},$$
$$\bar{\mathcal{M}} = \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) < K\}.$$

Thus, $\mathcal{Q}$ is the set of possible cumulative production vectors, $\mathcal{M}$ is the subset of $\mathcal{Q}$ where the oil can be produced at the maximum rate K, and $\bar{\mathcal{M}}$ is the subset of $\mathcal{Q}$ where the oil cannot be produced at the maximum rate K.

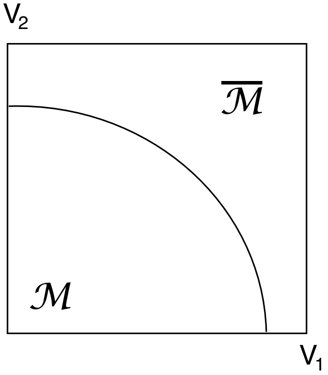
Optimizing production strategies

The sets $\mathcal{Q}$, $\mathcal{M}$ and $\bar{\mathcal{M}}$:



Optimizing production strategies

We shall see that the solution to the optimization problem depends on the shape of $\mathcal{M}$ and $\bar{\mathcal{M}}$:



Optimizing production strategies

Proposition

Consider a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$.

(i) *If $f_1, \ldots, f_n$ are convex, the set $\bar{\mathcal{M}}$ is convex.*

(ii) *If $f_1, \ldots, f_n$ are concave, the set $\mathcal{M}$ is convex.*

NOTE: If $\bar{\mathcal{M}}$ is convex, then $\bar{\mathcal{M}} \cup \partial(\mathcal{M})$ is convex as well. Similarly, if $\mathcal{M}$ is convex, then $\mathcal{M} \cup \partial(\mathcal{M})$ is convex as well.

Optimizing production strategies

Assume first that the PPR-functions are convex, and let $\boldsymbol{Q}^1 = (Q_1^1, \ldots, Q_n^1)$ and $\boldsymbol{Q}^2 = (Q_1^2, \ldots, Q_n^2)$ be two vectors in $\bar{\mathcal{M}}$. Thus, we have:

$$\sum_{i=1}^{n} f_i(Q_i^j) < K, \qquad j = 1, 2.$$

Then let $0 \leq \alpha \leq 1$, and consider the vector $\boldsymbol{Q} = (Q_1, \ldots, Q_n) = \alpha \boldsymbol{Q}^1 + (1-\alpha)\boldsymbol{Q}^2$. Since the PPR-functions are convex, we have:

$$\begin{aligned} \sum_{i=1}^{n} f_i(Q_i) &= \sum_{i=1}^{n} f_i(\alpha Q_i^1 + (1-\alpha) Q_i^2) \\ &\leq \alpha \sum_{i=1}^{n} f_i(Q_i^1) + (1-\alpha) \sum_{i=1}^{n} f_i(Q_i^2) < K \end{aligned}$$

Thus, we conclude that $\boldsymbol{Q} \in \bar{\mathcal{M}}$ as well. Hence $\bar{\mathcal{M}}$ is convex.

Optimizing production strategies

Assume then that the PPR-functions are concave, and let $\boldsymbol{Q}^1 = (Q_1^1, \ldots, Q_n^1)$ and $\boldsymbol{Q}^2 = (Q_1^2, \ldots, Q_n^2)$ be two vectors in $\mathcal{M}$. Thus, we have:

$$\sum_{i=1}^{n} f_i(Q_i^j) \geq K, \qquad j = 1, 2.$$

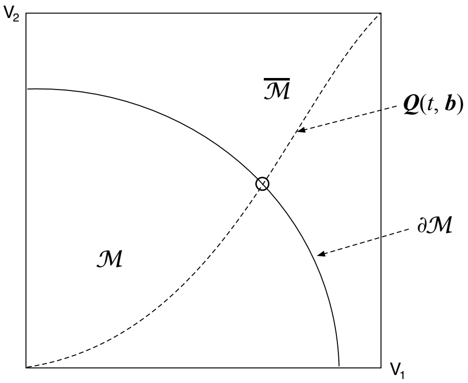
Then let $0 \leq \alpha \leq 1$, and consider the vector $\boldsymbol{Q} = (Q_1, \ldots, Q_n) = \alpha \boldsymbol{Q}^1 + (1 - \alpha)\boldsymbol{Q}^2$. Since the PPR-functions are concave, we have:

$$\begin{aligned}
\sum_{i=1}^{n} f_i(Q_i) &= \sum_{i=1}^{n} f_i(\alpha Q_i^1 + (1 - \alpha) Q_i^2) \\
&\geq \alpha \sum_{i=1}^{n} f_i(Q_i^1) + (1 - \alpha) \sum_{i=1}^{n} f_i(Q_i^2) \geq K
\end{aligned}$$

Thus, we conclude that $\boldsymbol{Q} \in \mathcal{M}$ as well. Hence $\mathcal{M}$ is convex.

Optimizing production strategies

Let $\boldsymbol{b}$ be any production strategy, and consider the points in $\mathcal{Q}$ generated by $\boldsymbol{Q}(t, \boldsymbol{b})$ as t increases.



From the boundary conditions we know that $\boldsymbol{Q}(0, \boldsymbol{b}) = \boldsymbol{0}$. Furthermore, $\boldsymbol{Q}(t, \boldsymbol{b})$ will move along some path in $\mathcal{M}$ until the boundary $\partial(\mathcal{M})$ is reached.

Optimizing production strategies

We denote the path $\{\boldsymbol{Q}(t, \boldsymbol{b}) : 0 \leq t < \infty\}$ by $\mathcal{P}(\boldsymbol{b})$.

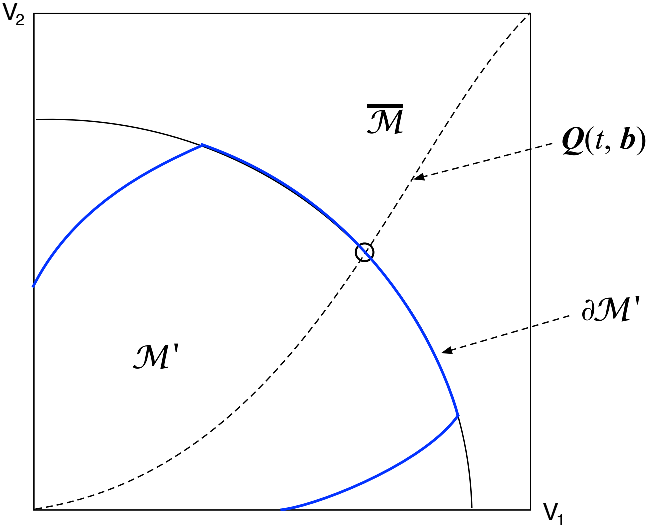
If $\boldsymbol{b} \in \mathcal{B}$, $\mathcal{P}(\boldsymbol{b})$ is said to be a *valid path*, while if $\boldsymbol{b} \in \mathcal{B}'$, $\mathcal{P}(\boldsymbol{b})$ is called an *admissible path*.

In general only a subset of $\mathcal{M}$ can be reached by admissible paths. We denote this subset by $\mathcal{M}'$.

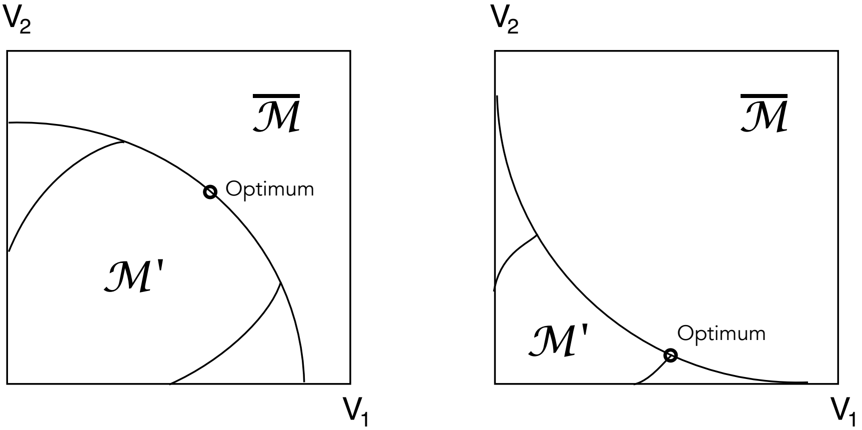
Let $\partial(\mathcal{M}') = \partial(\mathcal{M}) \cap \mathcal{M}'$.

We assume that all points in $\partial(\partial(\mathcal{M}'))$ are reachable by admissible paths.

Optimizing production strategies



Optimizing production strategies



Optimizing production strategies

Algorithm

Let ϕ be a monotone, symmetric objective function. Then a production strategy $\boldsymbol{b}$ which is optimal with respect to ϕ can be found as follows:

STEP 1. *Find $\boldsymbol{Q}^{opt} \in \partial(\mathcal{M}')$ such that $\phi(\boldsymbol{Q}^{opt}) \geq \phi(\boldsymbol{Q})$ for all $\boldsymbol{Q} \in \partial(\mathcal{M}')$.*

STEP 2. *Find a production strategy $\boldsymbol{b} \in \mathcal{B}'$ such that $\boldsymbol{Q}(T_K(\boldsymbol{b})) = \boldsymbol{Q}^{opt}$.*

Priority strategies

We now focus on the case where the PPR-functions are *convex*. Thus, the optimal strategy have an outcrossing point at the boundary of $\partial(\mathcal{M}')$. Such strategies are in a sense *extreme strategies* in the way the reservoirs are prioritized.

Definition

Consider a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$, *and let* $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ *be a permutation vector representing the prioritization order of the reservoirs.*

The priority strategy relative to $\boldsymbol{\pi}$, *denoted* $\boldsymbol{b}^{\boldsymbol{\pi}}$, *is defined by letting the production rates at time* t, $q_1(t), \ldots, q_n(t)$, *be given by:*

$$q_{\pi_i}(t) = \min[f_{\pi_i}(Q_{\pi_i}(t)), K - \sum_{j<i} q_{\pi_j}(t)], \qquad i = 1, \ldots, n.$$

Priority strategies

We observe that when assigning the production rate $q_{\pi_i}(t)$ to reservoir π_i, this is limited by $K - \sum_{j<i} q_{\pi_j}(t)$, i.e., the remaining processing capacity after assigning production rates to all the reservoirs with higher priority.

– If $f_{\pi_i}(Q_{\pi_i}(t)) \leq K - \sum_{j<i} q_{\pi_j}(t)$, reservoir π_i can be produced without any choking, and the remaining processing capacity is passed on to the reservoirs with lower priorities.

– If on the other hand $f_{\pi_i}(Q_{\pi_i}(t)) > K - \sum_{j<i} q_{\pi_j}(t)$, the production at reservoir π_i is choked so that $q_{\pi_i}(t) = K - \sum_{j<i} q_{\pi_j}(t)$. Thus, in this case *all the remaining processing capacity* is used on this reservoir, and nothing is passed on to the reservoirs with lower priorities.

Priority strategies

We introduce the following quantities ($i = 1, \ldots, n$):

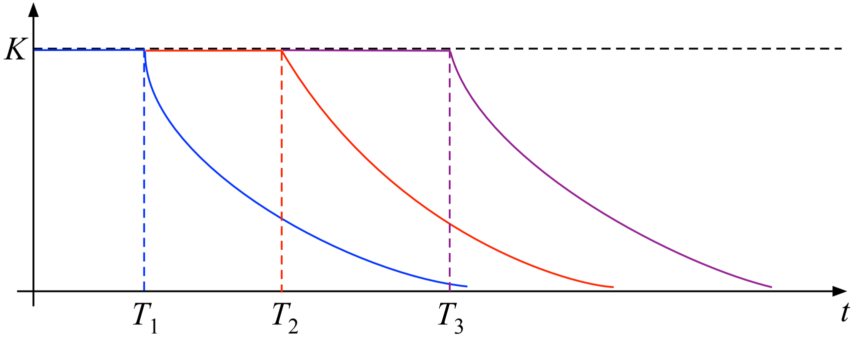
$$T_i = T_i(\boldsymbol{b}^{\boldsymbol{\pi}}) = \inf\{t \geq 0 : \sum_{j=1}^{i} f_{\pi_j}(Q_{\pi_j}(t, \boldsymbol{b}^{\boldsymbol{\pi}})) < K\}.$$

We also let $T_0 = 0$, and note that we obviously have:
$0 = T_0 \leq T_1 \leq \cdots \leq T_n = T_K(\boldsymbol{b}^{\boldsymbol{\pi}})$.

Thus, $T_1, \ldots, T_n$ defines an increasing sequence of *subplateau sets*, $[0, T_1], \ldots, [0, T_n]$, where the last one is equal to the plateau interval $[0, T_K(\boldsymbol{b}^{\boldsymbol{\pi}})]$.

$T_1, \ldots, T_n$ are called the *subplateau lengths* for the given priority strategy.

Priority strategies



Priority strategies

We now let $i \in \{1, \ldots, n\}$, and assume that $T_{i-1} < t < T_i$. Then the reservoirs $\pi_1, \ldots, \pi_{i-1}$ are produced without choking, i.e.:

$$q_{\pi_j}(t) = f_{\pi_j}(Q_{\pi_j}(t)), \qquad j = 1, \ldots, i-1.$$

Furthermore, the reservoir π_i is produced *with choking* so that:

$$q_{\pi_i}(t) = K - \sum_{j<i} q_{\pi_j}(t) = K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(t)).$$

Finally the reservoirs $\pi_{i+1}, \ldots, \pi_n$ are not produced at all.

NOTE: For $t \geq T_i$ we have:

$$f_{\pi_i}(Q_{\pi_i}(t)) \leq K - \sum_{j<i} q_{\pi_j}(t) = K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(t)).$$

Thus, from this point of time the reservoir π_i can be produced without any choking. Thus, for $t \geq T_i$ we have $q_{\pi_i}(t) = f_{\pi_i}(Q_{\pi_i}(t))$.

Priority strategies

Summarizing this we see that for $i = 1, \ldots, n$, the production rate, $q_{\pi_i}(t)$ is given by:

$$q_{\pi_i}(t) = \begin{cases} 0 & \text{if } t < T_{i-1}, \\ K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(t)) & \text{if } T_{i-1} \leq t < T_i, \\ f_{\pi_i}(Q_{\pi_i}(t)) & \text{if } t \geq T_i. \end{cases}$$

If π is a permutation vector, the corresponding priority strategy is denoted by $\boldsymbol{b}^{\pi}$.

The class of all priority strategies is denoted by $\mathcal{B}^{PR}$.

Priority strategies

Theorem

Consider a field with n reservoirs with convex PPR-functions $f_1, \ldots, f_n$. Furthermore, let ϕ be a symmetric, monotone objective function.

Then, under certain mild restrictions an optimal production strategy can be found within the class $\mathcal{B}^{PR}$.

NOTE: As there are exactly $n!$ possible permutation vectors π, this means that the class $\mathcal{B}^{PR}$ is finite. Thus, in order to find the optimal strategy, only a finite number of strategies need to be analysed.

If n is large, searching through all $n!$ strategies is very time-consuming. In such cases, however, the search can be done iteratively by starting out with a random permutation, and then at each step always choose the next permutation so that the value of objective function is increased.

Optimization with linear PPR-functions

Consider a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$, such that:

$$f_i(Q_i(t)) = D_i(V_i - Q_i(t)), \qquad i = 1, \ldots, n,$$

where $V_1, \ldots, V_n$ denotes the recoverable volumes from the n reservoirs, and where we assume that the reservoirs have been indexed so that $0 < D_1 \leq D_2 \leq \cdots \leq D_n$.

The factor D_i is referred to as the *decline factor* of the ith reservoir, $i = 1, \ldots, n$.

Optimization with linear PPR-functions

Consider the ith reservoir, and let $T \geq 0$. If this reservoir is produced without any choking, i.e., with a choking factor function $b_i(t) = 1$ for all $t \geq T$, we get:

$$q_i(t) = D_i(V_i - Q_i(T)) \exp(-D_i(t - T)), \qquad t \geq T.$$

Moreover, by integrating $q_i(t)$ from T to t we also get:

$$Q_i(t) = V_i(1 - e^{-D_i(t-T)}) + Q_i(T)e^{-D_i(t-T)}, \qquad t \geq T.$$

NOTE: $Q_i(t)$ is expressed as a convex combination of V_i and $Q_i(T)$. As t increases the weight associated with V_i increases and the weight associated with $Q_i(T)$ decreases.

A result on dominating sums

Lemma

Assume that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are such that:

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i, \qquad k = 1, \ldots, n.$$

Then for any $\mathbf{a} \in \mathbb{R}^n$ such that:

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq 0,$$

we also have:

$$\sum_{i=1}^{k} x_i a_i \geq \sum_{i=1}^{k} y_i a_i, \qquad k = 1, \ldots, n.$$

A result on dominating sums (cont.)

PROOF: Induction on k.

By the assumption it follows that $x_1 \geq y_1$, and since all the a_is are assumed to be nonnegative:

$$x_1 a_1 \geq y_1 a_1.$$

Hence, the result obviously holds for $k = 1$.

We then assume that the result is proved for $k \leq m$, and consider the case where $k = m + 1$. We then introduce $b_i = a_i - a_{m+1}$, $i = 1, \ldots, m$.
It follows that:

$$b_1 \geq b_2 \geq \ldots \geq b_m \geq 0.$$

Hence, by the induction hypothesis we have that:

$$\sum_{i=1}^{m} x_i b_i \geq \sum_{i=1}^{m} y_i b_i.$$

A result on dominating sums (cont.)

By combining this we get that:

$$\begin{aligned}
\sum_{i=1}^{m+1} x_i a_i &= \sum_{i=1}^{m} x_i(a_i - a_{m+1}) + a_{m+1} \sum_{i=1}^{m+1} x_i \\
&= \sum_{i=1}^{m} x_i b_i + a_{m+1} \sum_{i=1}^{m+1} x_i \\
&\geq \sum_{i=1}^{m} y_i b_i + a_{m+1} \sum_{i=1}^{m+1} y_i \\
&= \sum_{i=1}^{m} y_i(a_i - a_{m+1}) + a_{m+1} \sum_{i=1}^{m+1} y_i \\
&= \sum_{i=1}^{m+1} y_i a_i.
\end{aligned}$$

Thus, the result holds for $k = m + 1$ as well, and hence for $k = 1, \ldots, n$ by induction.

Optimization with linear PPR-functions

Theorem

Consider a field with n reservoirs with linear PPR-functions $f_1, \ldots, f_n$ *with decline factors* $0 < D_1 \leq D_2 \leq \cdots \leq D_n$.

Then let $\boldsymbol{b}^1$ *denote the priority strategy corresponding to the permutation* $\pi = (1, 2, \ldots, n)$*, and let* $\boldsymbol{b}^2$ *be any other valid production strategy.*

Then $Q(t, \boldsymbol{b}^1) \geq Q(t, \boldsymbol{b}^2)$ *for all* $t \geq 0$.

Thus, $\boldsymbol{b}^1$ *is optimal with respect to any monotone, symmetric objective function.*

Optimization with linear PPR-functions

PROOF: We introduce the plateau lengths $T_1, \ldots, T_n$.

If the priority strategy $\boldsymbol{b}^1$ is used, we get the following:

Reservoir 1 is produced at the rate K throughout the interval $[0, T_1]$ and will be produced without any choking for $t \geq T_1$.

Reservoirs 1 and 2 are produced at a total rate K throughout the interval $[0, T_2]$ and will be produced without any choking for $t \geq T_2$.

$\ldots$

Optimization with linear PPR-functions

We shall now prove by induction that:

$$\sum_{j=1}^{i} Q_j(t, \boldsymbol{b}^1) \geq \sum_{j=1}^{i} Q_j(t, \boldsymbol{b}^2), \qquad t \geq 0, \; i = 1, \ldots, n.$$

Thus, we start out by considering the case where $i = 1$, and assume that the priority strategy $\boldsymbol{b}^1$ is used.

If $0 \leq t \leq T_1$, then obviously:

$$Q_1(t, \boldsymbol{b}^1) = Kt.$$

If $t > T_1$, we know that reservoir 1 is produced without any choking. Thus, we have:

$$Q_1(t, \boldsymbol{b}^1) = V_1(1 - e^{-D_1(t-T_1)}) + Q_1(T_1, \boldsymbol{b}^1)e^{-D_1(t-T_1)}.$$

Optimization with linear PPR-functions

We the consider the situation where $\boldsymbol{b}^2$ is used instead.

If $0 \leq t \leq T_1$, then obviously:

$$Q_1(t, \boldsymbol{b}^2) \leq Kt = Q_1(t, \boldsymbol{b}^1).$$

If $t > T_1$, we have:

$$Q_1(t, \boldsymbol{b}^2) \leq V_1(1 - e^{-D_1(t-T_1)}) + Q_1(T_1, \boldsymbol{b}^2)e^{-D_1(t-T_1)}.$$

Thus, since $Q_1(T_1, \boldsymbol{b}^1) \geq Q_1(T_1, \boldsymbol{b}^2)$, it follows that:

$$Q_1(t, \boldsymbol{b}^1) \geq Q_1(t, \boldsymbol{b}^2) \text{ for all } t > T_1.$$

Hence, we conclude that $Q_1(t, \boldsymbol{b}^1) \geq Q_1(t, \boldsymbol{b}^2)$ for all $t \geq 0$, i.e., the induction hypothesis is proved for $i = 1$.

Optimization with linear PPR-functions

We then assume that the induction hypothesis is proved for $i = 1, \ldots, (k-1)$, and consider the case where $i = k$.

If $0 \leq t \leq T_k$, we have:

$$\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^1) = Kt \geq \sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^2).$$

In particular:

$$\sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^1) \geq \sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^2). \quad (*)$$

Optimization with linear PPR-functions

We then consider the case where $t > T_k$.

If $\boldsymbol{b}^1$ is used, the reservoirs $1, 2, \ldots, k$ are produced without any choking, thus:

$$\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^1) = \sum_{j=1}^{k} V_j(1 - e^{-D_j(t-T_k)}) + \sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^1) e^{-D_j(t-T_k)}.$$

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If, on the other hand, $\boldsymbol{b}^2$ is used, we get:

$$\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^2) \leq \sum_{j=1}^{k} V_j(1 - e^{-D_j(t-T_k)}) + \sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^2) e^{-D_j(t-T_k)}.$$

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By the induction hypothesis and (*) we have that:

$$\sum_{j=1}^{i} Q_j(T_k, \boldsymbol{b}^1) \geq \sum_{j=1}^{i} Q_j(T_k, \boldsymbol{b}^2), \qquad i = 1, \ldots, k.$$

Moreover, since $D_1 \leq D_2 \leq \cdots \leq D_k$, we have:

$$e^{-D_1(t-T_k)} \geq \cdots \geq e^{-D_k(t-T_k)}, \text{ for all } t \geq T_k.$$

Then it follows by the lemma on dominating sums that:

$$\sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^1) e^{-D_j(t-T_k)} \geq \sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^2) e^{-D_j(t-T_k)}$$

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Hence, also for $t \geq T_K$ we have:

$$
\begin{aligned}
\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^2) &\leq \sum_{j=1}^{k} V_j(1 - e^{-D_j(t-T_k)}) \\
&\quad + \sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^2) e^{-D_j(t-T_k)} \\
&\leq \sum_{j=1}^{k} V_j(1 - e^{-D_j(t-T_k)}) \\
&\quad + \sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^1) e^{-D_j(t-T_k)} \\
&= \sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^1).
\end{aligned}
$$

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By combining all this, we get for all $t \geq 0$ that:

$$\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^1) \geq \sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^2).$$

Thus, the induction hypothesis is proved for $i = k$ as well.

Hence, the result is proved by induction