STK4400 – Week 3

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Today:

Convex and coherent risk measures.

Quantifying risk

Main goal: Present a possible way to quantify monetary risk.

In practice: Different risk measures are used to do this.

We will focus on convex risk measures, which satisfy some economically reasonable properties.

Today:

- Define convex risk measures.
- Coherent risk measures.
- Acceptance sets.
- Derive properties of convex risk measures.

Monetary measures of financial risk

We will introduce measures for the risk of a financial position, X, which takes a random value at some set terminal time.

This value depends on the current world scenario.

Examples of X: Value of portfolio of stocks. Value of personal finances (house, mortgage, savings, stocks etc.).

Intuitive measure of risk: Variance?

Problem with using variance as measure of risk: Does not separate between positive and negative deviations $\rightarrow$ not suitable for financial risk (losing money is bad, making money is good).

Economically reasonable axioms for risk measures

To fix this, Artzner et al. set up economically reasonable axioms that a measure of risk should satisfy.

Hence, they introduced coherent risk measures.

Later generalized to convex risk measures.

Measure theory

Need a few concepts from measure theory (see Shilling for more):

Assume we are given a scenario space Ω.

May be a finite set $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ or an infinite set.

On this space, we can define a σ-algebra $\mathcal{F}$, i.e. a family of subsets of Ω that contains the empty set $\emptyset$ and is closed under complements and countable unions.

The elements in the σ-algebra $\mathcal{F} \in \Omega$ are called measurable sets.

$(\Omega, \mathcal{F})$ is then called a measurable space.

Measurable functions

A measurable function is a function $f : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ (where $(\Omega', \mathcal{F}')$ is another measurable space) such that for any measurable set, $E \in \mathcal{F}'$, the inverse image (preimage), $f^{-1}(E)$, is a measurable set, i.e.,

$$f^{-1}(E) := \{\omega \in \Omega : f(\omega) \in E\} \in \mathcal{F}.$$

A random variable is a real-valued measurable function.

On a measurable space $(\Omega, \mathcal{F})$ one can define a measure, i.e. a non-negative countably additive function $\mu : \Omega \to \mathbb{R}$ such that $\mu(\emptyset) = 0$.

Then, $(\Omega, \mathcal{F}, \mu)$ is called a measure space.

Signed measures and probability measures

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A signed measure is the same as a measure, but without the non-negativity requirement.

A probability measure is a measure P such that $P(\Omega) = 1$. Let $\mathcal{P}$ denote the set of all probability measures on $(\Omega, \mathcal{F})$, and $\mathbb{V}$ the set of all measures on $(\Omega, \mathcal{F})$.

Then $\mathbb{V}$ is a vector space (also called linear space), and $\mathcal{P} \subseteq \mathbb{V}$ is a convex set (check this yourself as an exercise!).

Financial positions

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Now: Let Ω be a scenario space (possible states of the world).

No further assumptions on Ω: May be infinite.

Measure space: $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is a given σ-algebra on Ω and P is a given probability measure on $(\Omega, \mathcal{F})$.

Financial position: A mapping $X : \Omega \to \mathbb{R}$.

$X(\omega)$ is the value of the position at the terminal time if the state ω occurs.

Formally: X is a random variable.

Vector space of random variables

Let $\mathbb{X}$ be a given vector space of such random variables $X : \Omega \to \mathbb{R}$, which contains the constant functions.

For $c \in \mathbb{R}$, let $c1 \in \mathbb{X}$ denote the constant function $c1(\omega) = c$ for all $\omega \in \Omega$.

Example:

$$L^p(\Omega, \mathcal{F}, P) := \{f : f \text{ measurable and} \Big(\int_\Omega |f(\omega)|^p dP(\omega) \Big)^{1/p} < \infty\}$$

for $1 \leq p \leq \infty$.

Convex risk measures

Definition (Convex risk measure)

A convex risk measure is a function $\rho : \mathbb{X} \to \mathbb{R}$ which satisfies the following for each $X, Y \in \mathbb{X}$:

(*i*) (Convexity) $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for $0 \leq \lambda \leq 1$.

(*ii*) (Monotonicity) If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.

(*iii*) (Translation invariance) If $m \in \mathbb{R}$, then $\rho(X + m\mathbf{1}) = \rho(X) - m$.

If $\rho(X) \leq 0$, X is acceptable since it does not have a positive risk. On the other hand, if $\rho(X) > 0$, X is unacceptable.

Coherent risk measures

If a convex risk measure also satisfies positive homogeneity, that is if

$$\lambda \geq 0 \Rightarrow \rho(\lambda X) = \lambda \rho(X)$$

then ρ is called a coherent risk measure.

Subadditivity

The original definition of a coherent risk measure, did not involve convexity directly, but instead required subadditivity:

Definition (Coherent risk measure)

A coherent risk measure is a function $\pi : \mathbb{X} \to \mathbb{R}$ which satisfies the following for each $X, Y \in \mathbb{X}$:

- (i) (Positive homogeneity) $\pi(\lambda X) = \lambda \pi(X)$ for $\lambda \geq 0$.
- (ii) (Subadditivity) $\pi(X + Y) \leq \pi(X) + \pi(Y)$.
- (iii) (Monotonicity) If $X \leq Y$, then $\pi(X) \geq \pi(Y)$.
- (iv) (Translation invariance) If $m \in \mathbb{R}$, then $\pi(X + m1) = \pi(X) - m$.

Interpretations of convex risk measures

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Interpret ρ as a capital requirement, that is: $\rho(X)$ is the extra amount of money which should be added to the portfolio in a risk free way to make the position acceptable for an agent.

Interpretation of the conditions in Definition 1:

Convexity: Diversification reduces risk. The total risk of loss in two portfolios should be reduced when the two are weighed into a mixed portfolio.

Interpretations of convex risk measures, ctd.

Monotonicity: The risk of loss is reduced by choosing a portfolio that has a higher value in every possible state of the world.

Translation invariance: ρ is the amount of money one needs to add to the portfolio in order to make it acceptable for an agent. Hence, if one adds a risk free amount m to the portfolio, the capital requirement should be reduced by the same amount.

Positive homogeneity?

Artzner et al. originally required positive homogeneity. Why skip this in convex risk measures?

Positive homogeneity means that risk grows linearly with X.

May not always be the case in practice!

Now: Consider convex risk measures (the results can be proved for coherent risk measures as well).

Convex combinations and max of convex risk measures are convex risk measures

Starting with n convex risk measures, one can derive more convex risk measures:

Theorem

Let $\rho_1, \rho_2, \ldots, \rho_n$ be convex risk measures.

1. *If $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$, then $\rho = \sum_{i=1}^n \lambda_i \rho_i$ is a convex risk measure as well.*
2. *$\rho = \max\{\rho_1, \rho_2, \ldots, \rho_n\}$ is a convex risk measure.*

Proof:

1. Let's check the conditions of Definition 1. Obviously, $\rho : \mathbb{X} \to \mathbb{R}$, so we check for any $X, Y \in \mathbb{X}$, $0 \leq \lambda \leq 1$:

   (i) : This follows from that a sum of convex functions is also a convex function, and that a positive constant times a convex function is still convex.

   (ii) : If $X \leq Y$, then $\rho(X) = \sum_{i=1}^{n} \lambda_i \rho_i(X) \geq \sum_{i=1}^{n} \lambda_i \rho_i(Y) = \rho(Y)$.

Proof, ctd.

(iii) : If $m \in \mathbb{R}$,

$$\begin{aligned}\rho(X+m) &= \sum_{i=1}^{n} \lambda_i \rho_i(X+m) \\ &= \sum_{i=1}^{n} \lambda_i(\rho_i(X) - m) \\ &= \sum_{i=1}^{n} \lambda_i \rho_i(X) - m \sum_{i=1}^{n} \lambda_i \\ &= \rho(X) - m.\end{aligned}$$

2. The proof is left as an exercise.

The acceptance set

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Associated with every convex risk measure ρ, there is a natural set of all acceptable portfolios: The acceptance set, $\mathcal{A}_\rho$, of ρ.

Definition (The acceptance set of a convex risk measure, $\mathcal{A}_\rho$)

A convex risk measure ρ induces a set

$$\mathcal{A}_\rho = \{X \in \mathbb{X} : \rho(X) \leq 0\}$$

The set $\mathcal{A}_\rho$ is called the acceptance set of ρ.

Associated measure of risk

Conversely, given a class $\mathcal{A} \subseteq \mathbb{X}$, one can associate a quantitative risk measure $\rho_{\mathcal{A}}$ to it.

Definition (Associated measure of risk)

Let $\mathcal{A} \subseteq \mathbb{X}$ be a set of "acceptable" random variables. This set has an associated measure of risk $\rho_{\mathcal{A}}$ defined as follows: For $X \in \mathbb{X}$, let

$$\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}. \quad (1)$$

$\rho_{\mathcal{A}}(X)$ measures how much one must add to the portfolio X, in a risk free way, to get the portfolio into the set $\mathcal{A}$ of acceptable portfolios.

Risk measures and acceptance sets

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Note: This is the same interpretation as for a convex risk measure.

Hence:

- One can either start with a risk measure, and derive an acceptance set.
- Or one can start with a set of acceptable random variables, and derive a risk measure.

Illustration of the risk measure $\rho_{\mathcal{A}}$ associated with a set $\mathcal{A}$ of acceptable portfolios

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Let $\Omega = \{\omega_1, \omega_2\}$, and let $X : \Omega \to \mathbb{R}$ be a portfolio. Let $x = (X(\omega_1), X(\omega_2))$.

If the set of acceptable portfolios is as in Figure 24 , the risk measure $\rho_{\mathcal{A}}$ associated with the set $\mathcal{A}$ can be illustrated as below.

Illustration of the risk measure $\rho_{\mathcal{A}}$

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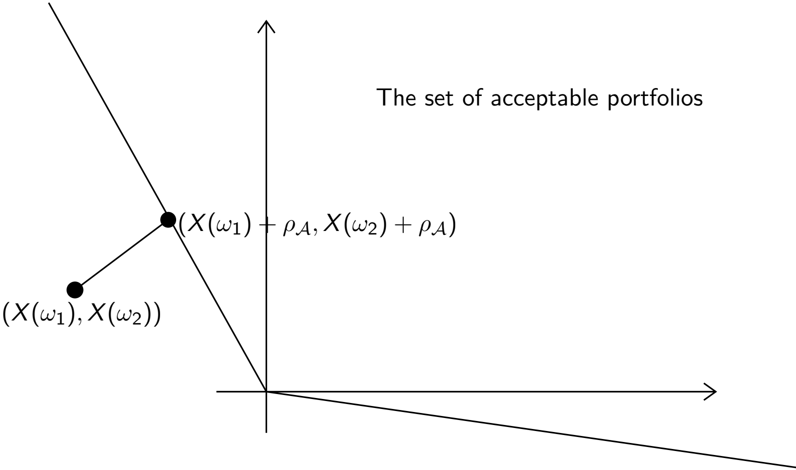

Connection between risk measure and acceptance sets

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Theorem

Let ρ be a convex risk measure with acceptance set $\mathcal{A}_\rho$. Then:

(i) $\rho_{\mathcal{A}_\rho} = \rho$

(ii) $\mathcal{A}_\rho$ *is a nonempty, convex set.*

(iii) *If* $X \in \mathcal{A}_\rho$ *and* $Y \in \mathbb{X}$ *are such that* $X \leq Y$, *then* $Y \in \mathcal{A}_\rho$

(iv) *If* ρ *is a coherent risk measure, then* $\mathcal{A}_\rho$ *is a convex cone.*

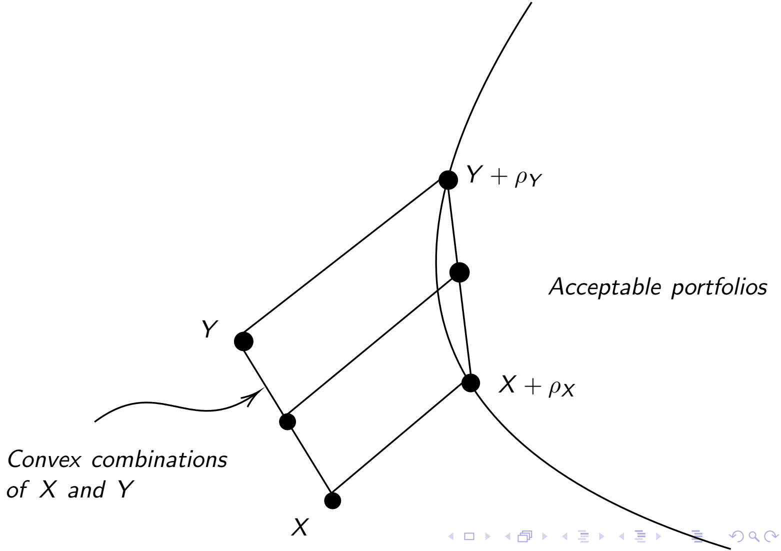
Conversely, let $\mathcal{A}$ *be a nonempty, convex subset of* $\mathbb{X}$. *Let* $\mathcal{A}$ *be such that if* $X \in \mathcal{A}$ *and* $Y \in \mathbb{X}$ *satisfy* $X \leq Y$, *then* $Y \in \mathcal{A}$. *Then, the following holds:*

(v) $\rho_{\mathcal{A}}$ *is a convex risk measure.*

(vi) *If* $\mathcal{A}$ *is a convex cone, then* $\rho_{\mathcal{A}}$ *is a coherent risk measure.*

(vii) $\mathcal{A} \subseteq \mathcal{A}_{\rho_{\mathcal{A}}}$.

Illustration of proof of Theorem 6 part (v)



Proof:

(*i*) For any $X \in \mathbb{X}$

$$\begin{aligned}
\rho_{\mathcal{A}_\rho}(X) &= \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}_\rho\} \\
&= \inf\{m \in \mathbb{R} : m + X \in \{Y \in \mathbb{X} : \rho(Y) \leq 0\}\} \\
&= \inf\{m \in \mathbb{R} : \rho(m + X) \leq 0\} \\
&= \inf\{m \in \mathbb{R} : \rho(X) - m \leq 0\} \\
&= \inf\{m \in \mathbb{R} : \rho(X) \leq m\} \\
&= \rho(X)
\end{aligned}$$

where we have used the definition of a convex risk measure (Definition 1) and an acceptance set (Definition 4).

(*ii*) $\mathcal{A}_\rho \neq \emptyset$ because $X = 0 \in \mathcal{A}_\rho$. Since ρ is a convex function, $\mathcal{A}_\rho$ is a convex set.

Proof, ctd.

(*iii*) The proof is left as an exercise.

(*iv*) The proof is left as an exercise.

Proof part (v)

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We check Definition 1: $\rho_{\mathcal{A}} : \mathbb{X} \to \mathbb{R}$.

Convexity: For $0 \leq \lambda \leq 1$, $X, Y \in \mathbb{X}$

$$
\begin{aligned}
\rho_{\mathcal{A}}(\lambda X + (1-\lambda)Y) &= \inf\{m \in \mathbb{R} : m + \lambda X + (1-\lambda)Y \in \mathcal{A}\} \\
&\leq \lambda \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\} \qquad (2) \\
&\quad + (1-\lambda)\inf\{m \in \mathbb{R} : m + Y \in \mathcal{A}\} \\
&= \lambda \rho_{\mathcal{A}}(X) + (1-\lambda)\rho_{\mathcal{A}}(Y)
\end{aligned}
$$

where the inequality follows because

$$\lambda \rho_{\mathcal{A}}(X) + (1-\lambda)\rho_{\mathcal{A}}(Y) = K + L,$$

is a real number which will make the portfolio become acceptable.

Why is this true?

Proof part (v), ctd.

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$K + L$ makes the portfolio acceptable because

$$\begin{aligned}(K + L) + (\lambda X + (1 - \lambda)Y) = (K + \lambda X) + (L + (1 - \lambda)Y)\\ = \lambda(\inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\} + X) +\\ (1 - \lambda)(\inf\{m \in \mathbb{R} : m + Y \in \mathcal{A}\} + Y) \in \mathcal{A}\end{aligned}$$

since $\mathcal{A}$ is convex (see Figure 26).

Proof part (v), ctd.

Monotonicity: In addition, if $X, Y \in \mathbb{X}, X \leq Y$

$$\begin{aligned} \rho_{\mathcal{A}}(X) &= \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\} \\ &\geq \inf\{m \in \mathbb{R} : m + Y \in \mathcal{A}\} \\ &= \rho_{\mathcal{A}}(Y) \end{aligned}$$

since $X \leq Y$.

Translation invariance: Finally, for $k \in \mathbb{R}$ and $X \in \mathbb{X}$

$$\begin{aligned} \rho_{\mathcal{A}}(X + k) &= \inf\{m \in \mathbb{R} : m + X + k \in \mathcal{A}\} \\ &= \inf\{s - k \in \mathbb{R} : s + X \in \mathcal{A}\} \\ &= \inf\{s \in \mathbb{R} : s + X \in \mathcal{A}\} - k \\ &= \rho_{\mathcal{A}}(X) - k. \end{aligned}$$

Hence, $\rho_{\mathcal{A}}$ is a convex risk measure.

Proof, part (*vi*)

From (*v*), all that remains to show is positive homogeneity.

For $\alpha > 0$

$$\begin{aligned}\rho_{\mathcal{A}}(\alpha X) &= \inf\{m \in \mathbb{R} : m + \alpha X \in \mathcal{A}\} \\ &= \inf\{m \in \mathbb{R} : \alpha(\frac{m}{\alpha} + X) \in \mathcal{A}\} \\ &= \inf\{m \in \mathbb{R} : \frac{m}{\alpha} + X \in \mathcal{A}\} \\ &= \inf\{\alpha k \in \mathbb{R} : k + X \in \mathcal{A}\} \\ &= \alpha \inf\{k \in \mathbb{R} : k + X \in \mathcal{A}\} \\ &= \alpha \rho_{\mathcal{A}}(X)\end{aligned}$$

where we have used that $\mathcal{A}$ is a convex cone in equality number three. Hence, $\rho_{\mathcal{A}}$ is a coherent risk measure.

Proof part (*vii*)

Note that $\mathcal{A}_{\rho_{\mathcal{A}}} = \{X \in \mathbb{X} : \rho_{\mathcal{A}}(X) \leq 0\} = \{X \in \mathbb{X} : \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\} \leq 0\}$.

Let $X \in \mathcal{A}$, then

$$\inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\} \leq 0,$$

since $m = 0$ will suffice (because $X \in \mathcal{A}$). Hence $X \in \mathcal{A}_{\rho_{\mathcal{A}}}$.

This concludes the proof.

Next week!

Derive an alternative, dual characterisation of convex risk measures.

To do so, we need convex duality theory (from last week!).