1 Problem solving II

1.1 Introduction

The purpose of this second round of problem solving is to demonstrate how the tools of the three preceding chapters are put to work to solve practical problems in life insurance. Both liabilities and assets are considered and also (in section 13.4) their *coordination*, so-called *asset-liability management* (or ALM for short). The cases under discussion will simpler than in real life. This is at it should be. Basic training is not to be burdened with the myriad of details often encountered in practice.

Yet we do approach the demands of practical problem solving. There are in all examples aspects of mathematical modeling, uncertainty in input to those models and computational issues as well. The mathematical part has been introduced in the preceding chapters and also statistical time series models. The latter are needed to portray investment risk. We shall adhere to the earlier position (chapter 11) of projecting experience of the twentieth century. Warning against over-confidence in that approach was issued earlier (section 11.5), but it is difficult to find anything convincingly better. Interest in long, historical time series for macroeconomic variables seems to be on its way up, see, for example, ?? and ??. We shall here rely on the Wilkie system of models due to the British actuary David Wilkie; see, e.g., Wilkie (1995). In actuarial science these appear to be the most established and most widely used ones. The Wilkie models will be introduced along the way, as they are needed.

All examples considered involve stochastic simulation at some stage. How we plan and carry out these kind of experiments is thus a recurrent theme. What can be said about such matters at a general level?

First of all, simulations are laboratory experiments, run under idealised laboratory conditions. They produce uncertainty assessments. It is probably wise to regard those as *underestimates*. If things appear risky or uncertain in the laboratory, they are likely to be even worse in real life. That has actually nothing to do with simulations as such and applies to stochastic modelling generally, irrespectively of how computations are carried out; see, e.g., Kendal (1985), p.??

Then there is the question of *design*. All simulation experiments are (by definition) carried out under fully specified conditions, called $\hat{\mathcal{M}}$ in figure 1.1. We might refer to $\hat{\mathcal{M}}$ as the **ground truth**. Computer experimentation is, above all, about how we set up $\hat{\mathcal{M}}$ to reflect what we want to find out, what we know and what we do *not* know. There may be many members in the latter group. Suppose we have *not* been able to select some factor right. The *level* of risk will then be wrong, but not necessarily *comparisions* as some other factor is varied. We shall in the next section be dealing with errors in the mortality rates and how they affect projections of liability. Some specific portfolio will be used. Surely what we find out will have much relevance also for *other* portfolios, even though they differ in composition.

One way to make this idea more concrete is to assume that we are considering a risk variable Y of the form

$$Y = H(\mathbf{X}, \theta, \eta), \tag{1.1}$$

where $\mathbf{X}$ is some random vector and (θ, η) parameters. If $\mathbf{X}^*$ is a simulation of $\mathbf{X}$, then

$$Y_1^* = H(\mathbf{X}^*, \theta_1, \eta), \qquad Y_2^* = H(\mathbf{X}^*, \theta_2, \eta)$$

are Monte Carlo realisations of Y under θ_1 and θ_2 respectively. Suppose, that approximately

$$H(\mathbf{X}, \theta, \eta) \doteq H_1(\mathbf{X}, \theta) H_2(\mathbf{X}, \eta). \tag{1.2}$$

Then,

$$\frac{Y_2^*}{Y_1^*} \doteq \frac{H_2(\mathbf{X}^*, \theta_2)}{H_1(\mathbf{X}^*, \theta_1)},$$

η dropping out. Thus we would, approximately, see the effect of varying θ even if η was *not* correctly selected. That was comparision of *ratios*, for a similar argument in terms of differences; see exercise ??.

The reasoning is simplistic. In practice, we would not know such things, although we might be able to pass some judgement. However, the general idea is a sound one. It is frequently easier to evaluate the effect of change in a parameter than to estimate the actual level of a variable Y.

Planning simulation experiments can not be done according to a given recipy in a cookbook. There are an unlimited number of variations and design; interpretation must be learned by example. A number of different problems encountered in life insurance are discussed below. We start with liabilities.

1.2 Mortality risk revisited

Posing the problem

Chapter 11 dealt principally with *liabilites*. There *were* stochastic models involved, but by concentrating on expected values all uncertainty was removed from the evaluations. It was argued in chapter 3 that this was sensible since random variation is negligible compared to the total amount of money. Mortality risk in this sense is thus zero, or close to zero.

However, there is another side to this. The liability projections depend on unknown quantities. For example, where are the survival probabilites in section 10.2 to come from? Or the transition probabilites in section 10.4? From historical data, surely, and this begs the question of whether the experience we are able to draw on is extensive enough or even relevant. Above all, we must ask ourselves how far back in time we should go. Often questions like these lack clear-cut answers and are settled through a dose of pragmatism.

For example, there may, after all, be plenty of survival data. Mightn't that indicate that we should, perhaps, *not* look so far back, since patterns emerging from the more recent years are not unduly disturbed by random error in any case? The advantage of such tactics is that trends in mortality are automatically accomodated if we are able to concentrate on the more recent past. Of course, that changes if we are trying to gauge the trends themselves, rely on them to continue and build them into the projections. For a simple example, see exercise ??. To estimate such trends we would need to take a longer look back. Whether it is sound to link projections to trends we see today, is debatable. It is not an issue that can be settled by actuarial methodology alone.

What this tells us is that future development in mortality is uncertain. Does that mean that we should describe these concepts through stochastic processes like we did with interest rates in chaper 11? There *has* been some interest in such an approach lately, and the idea might even cite experience in certain countries in Eastern Europe where life expectancy actually went down throughout the last decades of the twentieth century.

Actually this holds much less appeal than for interest rates. First of all, there is little experience to hang stochastic fluctuations in mortality rates on. Over centuries man has consistently tended to live longer, which indicates that time effects in mortality, if included, should be of a *deterministic.* type, resembling more the models in chapters ?? than the stochastic processes we dealt so much with in the preceding chapters. Mortality as stochastic processes only serves to create more complicated, less transparent analyses without giving much in return. That applies to disability, as well although the issues are not quite the same. There may now be less historical data to build on, and for government funded schemes the very definition of disability may be obscure, often influenced by political currents that do not repeat themselves in the future. That obstacle applies to many

Quantity	Symbol	Assumption
Portfolio size	J	100000
Mortality	q_l	Gomperz-Makeham[1]
Time step		Annual
Interest rate	r	4% annually
Benefit	ζ	40000 Euro annually
Retirement	l_p	65 years
Maximum age	l_e	Unlimited
Payments		All in advance
Premium	π^{eq}	Equivalence

[1]Parameters in (11.24).

Table 1.1: *Portfolio description for the expirements reported in figure 13.1 and 13.2.*

macroeconomic variables as well, as we shall see in the next section.

It would be too ambitious to tackle all these issues exhaustively in an introductory book. Our more modest aim is to take a first step through the study of an ordinary pension portfolio. How do we go about to examine the impact of errors in the survival probabilites? How to determine the required size of the historical data base? If we worry about the economic consequences of the future in terms of mortality differing from the past, how do we throw light on this? As elsewhere, the simplest answer is through computer experiments. We start by looking at a specific portfolio and then take you through all the stages of such a process.

Portfolio description

A simplified portfolio of pension plans will be used to study the effect of errors in the mortality rates. The main assumptions are shown in table 13.1. There are J policies. They pay premium and receive benefit (40000 euro) only once a year, i.e. at the start of it. Each member of the pension scheme entered at the age of $l_0 = 30$ years. As usual l_ω will be used to denote the maximum age (120 years).

The age distribution was made to follow a Gomperz-Makeham law in the following sense. Suppose the recrutement (always at 30) on average exactly balances the departure due to death[1]. It can then be proved that the age distribution in the long run must become

$$\gamma_l = \rho \; {}_kp_{l_0}, \qquad k = l - l_0,$$

where ${}_kp_{l_0}$ are the survival probabilities defined in ?? and ρ is a constant of proportionality, determined so that

$$\gamma_{l_0} + \ldots + \gamma_{l_\omega} = 1.$$

If J is the total number of polcies, then

$$J_l = \gamma_l J$$

is the number of them at age l. Don't worry with the argument behind this peculiar way of laying out the portfolio, that is only to persuade you that the age distribution is a reasonable one that *could* arise in practice; see exercise ?? for details.

We are thus entring at an ordinary stage of the life of the portfolio. Some of the policy holders are contributing premium, others have reached the benefit stage. Earlier (so-called *retrospective* premium) have been transferred to the people in charge of the assets of the company. Our concern is the liabilities, defined as

$$\mathcal{R}^{\text{po}} = \sum_{l=l_{l_0}}^{l_\omega} J_l \mathcal{R}_l.$$

To compute the total the reserve we simply add those of the individual policies; of course we must multiply with the number of clients in the various age groups. All this is simplistic compared to the real world. It is, above all, assumed that all policies are identical. It would not have been difficult to change that into a scenario closer to the reality, but that would not have contributed much to the problem posed above.

The equivalence premium can be calculated as

$$\pi^{\text{eq}} = 5.458 \qquad \text{(unit: thousand euro)}$$

[1]We do not take into consideration that people in practice might leave for a number of other reasons, for example that they change job.

Link from true model

$$\begin{tikzcd} P_l \arrow[r] & {}_kp_{l_0} \arrow[r] \arrow[d] & \pi^{\mathrm{eq}} \arrow[dl] \\ & \mathcal{R}_{\mathrm{Po}} & \end{tikzcd}$$

Link from fitted model

$$\begin{tikzcd} \hat{P}_l \arrow[r] & {}_k\hat{p}_{l_0} \arrow[r] \arrow[d] & \hat{\pi}^{\mathrm{eq}} \arrow[dl] \\ & \hat{\mathcal{R}}_{\mathrm{Po}} & \end{tikzcd}$$

Figure 1.1: Relationships beween survival model and equivalence premium and reserve.

and the total reserve is

$$\mathcal{R}^{\mathrm{po}} = 17097 \quad \text{(unit: million euro)}$$

Note that these figures are computed under the true survival probabilities. In practice we only have *estimated* ones which leads to *estimated* premium $\hat{\pi}^{\mathrm{eq}}$ and *estimated* $\hat{\mathcal{R}}^{\mathrm{po}}$ reserve as well. It is to examine the *errors* $\hat{\pi}^{\mathrm{eq}} - \pi^{\mathrm{eq}}$ and $\hat{\mathcal{R}}^{\mathrm{po}} - \mathcal{R}^{\mathrm{po}}$ we now turn the attention.

Errors: A first look

The scenarios we are dealing with is shown in Figure 11.1 One-step survival probabilities P_l lead to multi-step ones ${}_kP_l$ which in turn produces the equivalence premium π^{eq} and the reserve $\mathcal{R}^{\mathrm{po}}$. The precise mathematical relationships are those in chapter 10. Our target is the similar flow form *estimates* $\hat{P}_l$ through *estimates* ${}_k\hat{P}_l$ to final estimates $\hat{\pi}^{\mathrm{eq}}$ and $\hat{\mathcal{R}}^{\mathrm{po}}$. We are forced to supply $\hat{P}_l$ instead of ${}_kP_l$, and the consequence is errors $\hat{\pi}^{\mathrm{eq}} - \pi^{\mathrm{eq}}$ and $\hat{\mathcal{R}}^{\mathrm{po}} - \mathcal{R}^{\mathrm{po}}$. What can be said about them?

The distribution will typically resemble the *normal* one if there is enough historical data to hang the estimates $\hat{P}_l$ on. Why was explained in chapter 5. With the estimation we shall use below it is actually possible derive the standard deviation of the error distribution through exact mathematical arguments. However, the resulting expressions are messy, and we end up with doing *more*, not *less* work than with simulation.

Simulation experiment: Design

Arguably portfolio created is also a part of the experimental design, but the main thing is how the historical data are generated. In real life we might have been using officuial government records. That holds the weakness that the mortalities of the population as a whole are not quite the same as in our portfolio.

In practice historical data are likely to come from the companies themselves. Perhaps the process is like the following: Imagine that at some point in the past there were J' policies, the age of the policy holders being $l'_1, \dots, l'_{J'}$, stored on file. The fatal events of this population are then followed K years ahead (up to the present time). In practice it would have been recrutement of new customers, and some other would have left for oterh causes than deaths, but we can trust this to have minor impact on the conclusions, and we ignore that issue.

Implementation

How such a scheme is simulated is elaborated in Algorithm 13.1. Details must depend on how the probabilities are estimated. In the present case the simple non-parameric estimate (??) has been used. During the period the population is monitored we then have to count

- the total number of deaths D_l,

occurring at age l, and

- the total number N_l of times

a policy holder of age l has been at risk. The probability of surviving the year from l to $l+1$ is then

$$\hat{P}_l = 1 - \frac{D_l}{N_l}.$$

The algorithm to generate Monte Carlo versions of this estimates runs as folloows:

Algorithm 13.1. MC experiment of mortality

Input: Survival probababilties P_l,

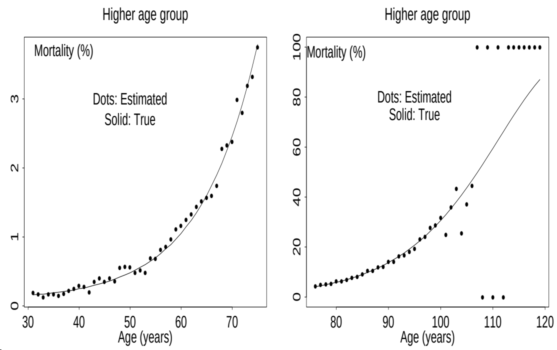

Figure 1.2: *Simulated log-returnes from models fitted historical data.*

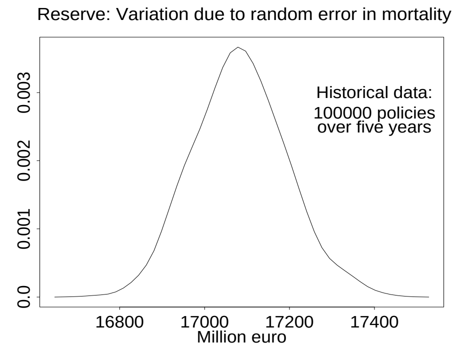

Figure 1.3: *Simulated log-returnes from models fitted historical data.*

age distribution l'_j (**historical**)

0 All l: $D^*_l \leftarrow 0$, $N^*_l \leftarrow 0$
1 For $j = 1, \dots, J'$ do
2 For $k = 0, \dots, K$ do *%Simulations*
3 $l \leftarrow l'_j + k$ *% of the fate*
4 $N^*_l \leftarrow N^*_l + 1$ *% of client j*
5 Draw $U^* \sim$ uniform
6 If $U^* > P_l$ then *%Life ended*
$D^*_l \leftarrow D^*_l + 1$, leave k-loop
7 End k-loop
8 End j-loop
9 Return all l: $P^*_l \leftarrow 1 - D^*_l / N^*_l$

The algorithm tracks the policy holders individually (lines $2-6$). Note how D^*_l and N^*_l are updated on lines 6 and 4. Each time a fatal event at age l occurs, D^*_l is augmented; each time a policy holder has reached a age l, the value of N^*_l goes up. On output the algorithm has counted the number of deaths D^*_l and the number of individuals who have at some point been at that age. The ratio D^*_l/N^*_l is then an assessment of the mortality at age l, and the survival probability is the expression on line 9.

The output is then entered the second flow chart in Figure 10.1 and equivalence premium and reserve calculated. We may repeat the procedure as many times as we please and deduce the impact of errors in mortality.

Results (and what we make of them)

The main example is 100000 policies followed over five years. Estimated one-year mortality probabilities from a single simulation have been plotted in Figure 10.1 (as the dots) and compared to the true Gomperz-Makeham model (solid lines). They certainly follow each other, though not at very high age (perhaps above 95 years) where the estimated ones (right hand plot) are meaningless. Even at younger age the estimated probabilities wriggles a little up and down; they are not completly monotone. That does not look very elegant. The question is whether it matters.

	Premium[1]		Reserve[2]	
History: 5 years	Mean	SD	Mean	SD
100000 policies	5.455	0.028	17084	108
10000 policies	5.458	0.093	17136	356

[1]In thousand euro [2]In million euro

Table 1.2: *Equivalence premium and reserve obtained from simulations from true mortality model*

If you examine the plot carefully you will detect the estimates being above the true ones for a while, then they fall under a period before climbing up gain for another while. Such a performance is consistent with the estimatation errors being *positively* correlated at neighbours. Such effects may have have important contribuions to make to errors in the reserve. Simulation has the considerable advantage that it automatically caters for it.

When we apply the estimated model to the computation of the equaivalence premium and the reserve (table 13.2 and figure 13.2) it seems that it is good enough. The standard deviation when we have drawn on 100000 policies over five years are no more than 0.6% of the total reserve. That occurs despite the inelegant behaviour when models are inspected visually (figure 10.2). And 100000 policies re not much! Even so small a number as 10000 policies dos not lead to dramattic errors, though they are now much bigger. Both experiments were run with 1000 replications.

The final experiment is an attempt to imitate what happens when we use out-dated information. Reserves calculated from the original Gomperz-Makeham model were compared what we get from changing one of the parameters (θ_2); see, e.g., (11.24) in section 10.2 about that. Underestimating survival leads to underestimation of reserves of up to 15%. Clearly that is something we might like to avoid. The practical eprecussions are dealt with next.

What we have learned

In terms of method: How such experiments are conceived and carried out.

In terms of computational speed: That experiments this size encounter no computational obstacles when implemented properly

In terms of relevance for insurance:

- Simple mortality estimates:
  Often good enough
- Random error: Didn't seem important
- Obsolete data. Dangerous!

In terms of model error: Sometimes we can live with them!

In addition there are the following comments: The fact that bias due to wrong population was so much more important than random error definitely argues in favour of using company records instead of offical statistics. It seems a good idea to update estimates once in a while and not let them be untouched for decades. Is the non-parametric estimates too primitive?

$\theta_2 \times 10^{-2}$	9.067	8.817	8.567	8.317
Mean life[1]	75.1	76.8	78.7	80.7
Reserve[2]	17.1	18.7	20.4	22.1

[1] In years for new-born [2]In billion euro

Table 1.3: *Reserve obatined under variation of the Gomperz-Makeham survival model; parameters θ_1 and θ_2 as (11.24).*

Our experience rather suggests otherwise. True, at very high age they were completely wrong, but then that didn't matter too much anyway. Two points can be cited on their behalf. The first is that they are very simple to use, the other is that they produce estimates that are *unbiased.* Once you impose a parametric model like the Gomperz-Makeham you do obtaim smooth, more elegantly looking survival curves, but you are also at the mercy of the parametric form resembling the true one. You have no gurantee of that; if the discepancy is too large ther would again be bias effects. Anyway, we have not gone into how such models are fitted in practice; for that you must dip into a book like Haberman and Petacci (1998).

1.3 Liabilities and inflation

Posing the problem

Reserves in life insurance, though large, are not that difficult to evaluate. The discount factor is, of course problematic, but that issue belongs to the realm of finacial risk and is covered in the next section. The greatest threat to the validity of the liability projections so far is underestimation of the survival probabilities. We saw above that the repercussions of an obsolete model could be severe. The same goes for future trends in mortality that have not been picked up.

Another feature of importance is *inflation.* Benefits are often linked to the development of a wage index or the price level in general. How that is introduced into the calculations and what the consequences are is the topic of this section.

Dealing with inflation: Fixed rate

Price movements of a country can be captured by a time series $\{Q_k\}$ of indexes, Suppose, for example, that Q_k is the retail price index at time t_k. In real terms a purchase for which it is paid Y_k is then worth Y_k/Q_k. It is convenient to introduce the convention that $Q_0 = 1$ which means that Y_k/Q_k is the value of Y_k in t_0-prices.

In practice the series $\{Q_k\}$ is often (but not always) increasing in k, and to preserve purchasing power a pension must somehow be linked to it. Suppose the contract specifies that the cash-flow is to be in real terms. This means that a payment $\zeta(i)$ into or out of the account at time t_k must be of size $\zeta(i)Q_k$, and the expression ?? for the reserve must be changed to

$$\mathcal{R}_l = \sum_{k=0}^{l_e-l} {}_k\bar{\zeta}_{l_0} d_k, \tag{1.3}$$

where

$$d_k = \frac{Q_k}{(1+r)^k}. \tag{1.4}$$

The presence of Q_k means that the discounting is changed from

$$1/(1+r)^k \quad \text{to} \quad Q_k/(1+r)^k.$$

We may feed on the previous inflation-free calculations if it is assumed that

$$Q_k = (1+i)Q_{k-1}, \tag{1.5}$$

where now i is a fixed growth in the retail price. Since $Q_0 = 1$, this implies that $Q_k = (1+i)^k$ and

$$\frac{Q_k}{(1+r)^k} = \left(\frac{1+i}{1+r}\right)^k = (1+r^i)^k$$

where (exercise ?)

$$r^i = \frac{r-i}{1+i}. \tag{1.6}$$

Thus, (1.3) can be rewritten

$$\mathcal{R}_l = \sum_{k=0}^{l_e-l} \frac{{}_k\bar{\zeta}_{l_0}}{(1+r^i)^k}, \tag{1.7}$$

Inflation 2%		Inflation 3%	
Premium[1]	Reserve[2]	Premium[1]	Reserve[2]
9.947	22059	13.306	25220
5.458[3]	26260[3]	5.458[3]	33329[3]

[1]In thousand euro [2]In million euro
[3] Premium charged with no inflation

Table 1.4: *Equivalence premium and reserve under regimes with inflation for same portfolio as in Table 10.2.*

and all the methods from chapter 10 can be applied with the inflation-adjusted discount factor $1/(1+r^i)$.

It is obvious that inflation-linked payments must increase funding requirements. This emerges clearly from the mathematics, as indeed it had to. From (1.6) r^i decreases with i so that $r^i < r^0 = r$ if $i > 0$, and inflation amounts to a *lower* discount factor and hence *higher* reserves. We see that clearly in Table 11.4 where regimes of 2% and 3% annual inflation are compared. The portfolio is the same as in Table 10.2.

One feature worth noting is the effect of including inflation in the calculations of premium which more than doubles from row two (inflation ignored) to to row one (inflation included) when the inflation rate is 3%. The reserve is also influenced greatly when you compare with the corresponding figure (about 17 billion) in table 13.2. It is now 30% larger when the annual the inflation rate is 2% and 50% larger when it is 3%. Undercoverage may be dramatic if inflation is ignored alltogether (last row in table 13.4.) Now the money needed to cover future benfit will be almost doubled when inflation is 3%.

Stochastic extension

The preceding argument lead to a conclusion of importance, but it also contained a major weakness. When liabilites are considered over decades, inflation is far from constant. If this issue is to be taken into account, (1.5) must be extended to

$$Q_k = (1+i_k)Q_{k-1}, \tag{1.8}$$

where i_k now is time-varying. The discount factor d_k in (1.4) now changes into

$$d_k = \frac{Q_k}{(1+r)^k} = \frac{1+i_k}{(1+r)} \frac{Q_{k-1}}{(1+r)^{k-1}} = \frac{1+i_k}{1+r} d_{k-1},$$

or

$$d_k = \frac{d_{k-1}}{1+r_k^i}, \qquad r_k^i = \frac{r-i_k}{1+i_k}. \tag{1.9}$$

The expression (1.3) for the reserve still stands, but now the discount factor is computed recursively through (1.9).

The problem is that we do not know what i_k is going to be. One thing is the fixed version $i_k \equiv i$ where it is at least possible to 'take a position' as to what level is likely in the future, quite another thing is the time-varying version. It is hard to see that we can escape a stochastic model and use of historical data. We deal with that next.

Long-term modelling

Models for price movements are part of the Wilkie system; see Wilkie (1995). The inflation model writes i_k as

$$i_k = (1+\xi^i)\exp(Z_k^i) - 1 \tag{1.10}$$

where

$$Z_k^i = a^i Z_{k-1}^i + \sigma^i \varepsilon_k^i, \tag{1.11}$$

is an autoregressive process of morder one. As usual, the series $\{\varepsilon_k^i\}$ consists of independently and standard normal variables. Inserting into (1.8) produces a stochastic version of the recursion for the price index; i.e.

$$Q_k = Q_{k-1}(1+\xi^i)\exp(Z_k^i). \tag{1.12}$$

The parameters suggested by Wilkie for the period 1923 – 94 is

$$\hat{\xi}^i = 4.8\%\ (1.2\%) \qquad \hat{a}^i = 0.58\ (0.08)$$
$$\hat{\sigma}^i = 0.040\ (0.004).$$

The numbers in parenthesis are the estimated standard errors. A feature we have seen before

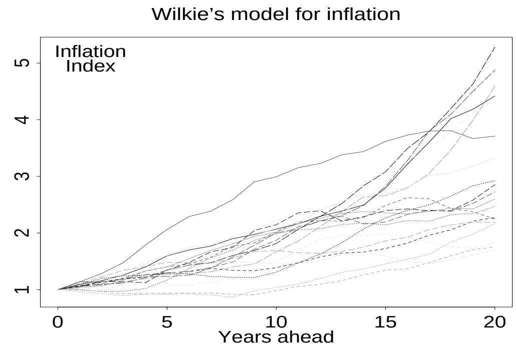

Figure 1.4: *Simulated log-returnes from models fitted historical data.*

(and which we will meet again) is the huge uncertainty in the estimate of the expectation ξ. Thus, even if the economic forces behind inflation can be trusted *not* to cause systematically different patterns in the present, twentyfirst century, the sheer statistical error in the most important parameter is large. The economic consequences are considerable; we saw that in Table 13.4.

The fitted Wilkie inflation model is simulated in Figure 13.4, starting at $i_0 = 4.8\%$; see below for how it is done. One noteworthy feature is how often *negative* inflation occurs; i.e. when the retail index simulated goes down from the previous year. Why that is so, is analysed in exercise ? through a little dose of mathematics. What should our attitude be towards a model with this property? It isn't that negative inflation can't occur, but the frequency may appear excessive. The issue is again one lacking a clear answer. One study that appears worth doing is to investigate the impact on the reserve. We deal with that next.

Simulating inflation risk

What we have learned

In terms of method: How inflation can be described mathematically and how both determinstic and stochastic versions can be integrated into present value computations of portfolio liabilites.

	Discount rate 8.7%			Discount rate 6.7%		
σ	5%	50%	95%	5%	50%	95%
0	17.7[1]	17.7	17.7	22.5	22.5	22.5
0.02	13.8	17.8	23.3	16.3	22.7	31.7
0.04	11.0	18.2	31.0	11.9	23.2	45.6

[1] All figures in **billion** euro. 100000 simulations

Table 1.5: *Present value (percentiles) of the portfolio in Table 11.2 under stochastic inflation.*

In terms of financial significance: The following main points are founded on the prceding discussion:

- Inflation is a severe threat.
- Stochastic effects are important.
- The model matters,
- and the selection is not clear-cut.

In terms of strategy: It follows from the preceding points that it is advantageous to immunize financial assets against inflation. Although easier said than one, it should be one of the considerations when designing investment portfolios; some assets follow inflation better than others; see later sections in this chapter.

1.4 20$^{\text{th}}$ century investment risk

Posing the problem

How relevant are the lessons of the twentieth century for projections ahead? The issue has been raised before (section 13.1), and it is one on which people are likely to differ in their opinions. But if the experience of the past century (or half-century) is not to be used at all, then what else? There isn't really reliable theory to fall back on, and the *implied* information in current market performance (see, e.g., section 11.?) does not suffice.

The twentieth century performance of financial assets classes is by no means to be swallowed raw and used uncritically as description of the financial risk ahead, but it does represent experience to draw on and provides useful models for laboratory experiments. We are in this section going to present the *Wilkie* system of models based on the very long article Wilkie (1995). The price index process used in the preceding section is an introductory example. Other macroeconomic quantities of central importance in insurance, are yield and dividend for equity and spot and long rate of interest. We discuss them below. Additional variables are wage and property indexes. Although highly relevant you will have to consult Wilkie's original work for those.

One thing we would like to learn is the kind of financial risks it is sound to take if future does not deviate systematically from the processes that went on in the twentieth century. Should we then go for equity rather than bonds and has that someting to do with the time horizon of our projections? What about inflation? Is it true, as frequently maintainded, that equity investments protect us better that interest rate instruments ? How the liabilities distribute over time may also be relevant, but that is deferred until sections 13.6 and 7.

All these questions (except the last one) is answered through simulation in later sections of this chapter. But first we have to present the models themselves. They are based almost solely on empirical studies of the past. The arguments behind their construction are beyond the scope of this book, and the presentation concentrates on their mathematical description. It has been an aim to provide an easy link to the basic toolbox for time series models in chapter 11, and the detailed structure and notation is completely rewritten compared to what you will find in Wilkie's original article. This apart, there is nothing new; the models are as put forward in Wilkie (1995).

The Wilkie models: Stationary part

The concept of stationarity was introduced in chapter 3, and we saw in chapter 11 how such processes through simple manipulations gave birth to non-stationary ones. Inflation is an example. It was asssumed in the preceding section that the rate of inflation $\{I_k\}$ was stationary which meant in practical terms that

	Symbol	innovation
Stationary building blocks		
Ordinary inflation	I_k	ε_k^i
Share yield	y_k	ε_k^i
Share dividend inflation	I_k^d	ε_k^d
Long interest rate	$r_k(T)$	ε_k^r
Spot interest rate	r_k	
Interest rate ratio	F_k	ε_k^f
Derived variables		
Price index	Q_k	
Dividend	D_k	
Share price	S_k	
Return on equity	R_k^e	

Table 1.6: *Macroeconomic variables of the Wilkie model*

it fluctuated around some long term average. The price process $\{Q_k\}$ that derives from it is qualitatively different in exhibiting persistent growth (interupted by the occasional decline).

Other macroeconomic quantites that behave similarly to inflation are yields on shares (denoted y_k), the inflation on share dividend (I_k^d) and the long and the short rate of interest ($r_k(T)$ and r_k respectively). The model for $\{r_k(T)\}$ must depend on the length T of the period. The one identified by Wilkie is for so-called *consols*; i.e. for loans with infinite time to expiry. We shall in this book use this model as an approximation for interest over long periods T. The Wilkie models for interest are developed in terms of the ratio $F_k = r_k/R_k(T)$.

In a well-functioning society all these quantities oscillate within certain, not clearly defined limits, hence the restriction to stationarity. The variables and their mathematical notation are summarised in the upper part of Table 13.6. Also given (third column) are the the random processes ε_k^i, ε_k^y and so on driving their fluctuations. As elsewhere those are independent in time and gaussian with mean zero and standard deviation one. Different processes ε_k^i, ε_k^y, ... are are also stochastically independent of each other. This condition is the natural way to construct models for many dynamic variables jointly, but it does *not* imply that the variables themselves must be independent, as will emerge below.

The link from the innovation processes in table 13.6 to the macroeconomic quantities is a two-step one with linear processes of the type introduced in section 11.5 as intermediates. The empirical studies conducted by Wilkie suggested the seven linear processes in (1.13-1.19), divided here into three groups; i.e.

Inflation

$$Z_k^i = a^i Z_{k-1}^i + \sigma^i \varepsilon_k^i, \tag{1.13}$$

Equity

$$Z_k^y = a^y Z_{k-1}^y + \sigma^y \varepsilon_k^y, \tag{1.14}$$

$$Z_k^{d|y} = \sigma^d(\varepsilon_k^d + b_1^d \varepsilon_{k-1}^d) + \theta^{d|y}\varepsilon_{k-1}^y, \tag{1.15}$$

$$Z_k^{d|i} = a^{d|i} Z_{k-1}^{d|i} + \underbrace{(b_0^{d|i} Z_k^i + b_1^{d|i} Z_{k-1}^i)}_{\text{Due to inflation}}, \tag{1.16}$$

Interest rates

$$Z_k^{r|y} = a^r Z_{k-1}^{r|y} + \sigma^r \varepsilon_k^r + \theta^{r|y}\varepsilon_k^y, \tag{1.17}$$

$$Z_k^{r|i} = a^{r|i} Z_{k-1}^{r|i} + \underbrace{(1 - a^{r|i}) Z_k^i}_{\text{Due to inflation}}, \tag{1.18}$$

$$Z_k^f = a^f Z_{k-1}^f + \sigma^f \varepsilon_k^f. \tag{1.19}$$

The superscripts define the variables to which they belong. Here $\{Z_k^i\}$ (for inflation), $\{Z_k^y\}$ (for equity) and, $\{Z_k^f\}$ (for the interest rate ratio) are all ordinary autoregressive processes of order 1, all moving independently of each other. How they relate to inflation, yield and the interest ratio is explained below.

Of the remaining four processes $\{Z_k^{d|y}\}$, $\{Z_k^{d|i}\}$ define share dividend *inflation* and $\{Z_k^{r|y}\}$ and $\{Z_k^{r|i}\}$) the long rate of interest. These processes have a more complicated structure than the others and contain *cross* terms; i.e. they are linked to the the movements equity yield and inflation. This dependency is reflected in their superscripts. Consider, for example, the fifth process $\{Z_k^{d|i}\}$. There is an ordinary autoregressive relationship one step back, but last term is *not* an independent ε-series. Instead the cycles are now being driven by inflation, hence the designation $Z^{d|i}$.

The macroeconomic variables in Table 13.6 is then, in turn, connected to the processes (1.13-1.19) through

Inflation

$$I_k = (1+\xi^i)\exp(Z_k^i) - 1, \tag{1.20}$$

Equity

$$y_k = \xi^y \exp(Z_k^y + \theta^{y|i} Z_k^i), \tag{1.21}$$

$$I_k^d = (1+\xi^d)\exp(Z_k^{d|y} + Z_k^{d|i}) - 1, \tag{1.22}$$

Interest rates

$$r_k(T) = \xi^r \exp(Z_k^{r|y}) + \xi^i + Z_k^{r|i}, \tag{1.23}$$

$$F_k = \xi^f \exp(Z_k^f), \tag{1.24}$$

$$r_k = F_k r_k(T). \tag{1.25}$$

There are alltogether 22 parameters in the relationships (1.13-1.25).

Estimates and interpretation

The model shown above has been identified from the period 1923 to 1994. Its parameters are shown in table 13.7 with their standard errors in parenthesis. Since some of the parameters used here are different from those in Wilkie (1995), it was necessary to recompute them by a rough approximation technique[2]. Those are the *-marked ones in Table 13.7. A common feature is the high uncertainty in almost all estimates of the ξ-parameters[3]. This is unfortunate since they are the parameters having the highest impact on the assessments of financial risk, as is explained below. However, we saw the same thing with a simpler example in section 2.?, and it is a fact of life with repercussions on how the model can be used for projecting future risk; more on that in section 13.5.

[2]It was impossible to do that accurately from the information in Wilkie's paper, since correlations between estimates are not given there. The values obtained are computed by the so-called delta-method, introduced in exercise ?, ignoring correlations between the original Wilkie estimates.

[3]The exception is $\hat{\xi}^y$ for which the standard error recorded in Table 13.7 is strikingly low.

	Inflation (I_k)	
$\hat{\xi}^i = 4.80\%$ (1.2%)	$\hat{a}^i = 0.58$ (0.08)	$\hat{\sigma}^i = 0.040$ (0.004)
	Yield(y_k)	
$\hat{\xi}^y = 4.10\%$ (0.3%*)	$\hat{a}^y = 0.55$ (0.10)	$\hat{\sigma}^y = 0.16$ (0.01)
$\hat{\theta}^{y\|i} = 1.79$ (0.59)		
	Dividend inflation (I_k^d)	
$\hat{\xi}^d = 6.50\%$ (1.8%*)	$\hat{b}_1^d = 0.57$ (0.13)	$\hat{\theta}^{d\|y} = -0.027$ (0.007*)
$\hat{\sigma}^d = 0.067$ (0.006)	$\hat{a}^{d\|i} = 0.87$ (0.08)	$\hat{b}_0^{d\|i} = 0.50$ (0.19*)
$\hat{b}_1^{d\|i} = -0.36$ (0.19*)		
	Long interest rate ($r_k(T)$)	
$\hat{\xi}^r = 3.05\%$ (0.65%*)	$\hat{a}^r = 0.90$ (0.04)	$\hat{\theta}^{r\|y} = 0.052$ (0.02*)
$\hat{\sigma}^r = 0.19$ (0.02)	$\hat{a}^{r\|i} = 0.955$ (NA)	
	Interest ratio (F_k)	
$\hat{\xi}^f = 0.80$ (0.06*)	$\hat{a}^f = 0.74$ (0.08)	$\hat{\sigma}^f = 0.18$ (0.02*)

**-marked standard errors means that a rough conversion from standard errors in Wilkie (1995) has been carried out.*

Table 1.7: *Estimated parameters for the Wilkie long-term asset models with standard errors in paranthesis.*

There is a common structure in the model worth pointing out. The random variation is represented by the seven Z-processes (1.13-1.19). All of them have *mean* zero. Suppose, for a moment, we removed all randomness and forced them to be *identically* zero. It then follows immediately from (1.20-1.25 that

$$I_k \equiv \xi^i, \quad y_k \equiv \xi^y, \quad I_k^d \equiv \xi^d \tag{1.26}$$

$$r_k(T) \equiv \xi^r + \xi^i, \quad r_k \equiv \xi^f(\xi^r + \xi^i). \tag{1.27}$$

This gives a useful interpretation of the ξ-parameters. In practice the Wilkie model allows the macro-economic variables to fluctuate around these values. Not that the expectations of I_k, y_k and so forth are *larger* than the values in (1.26) and (1.27); see exercise ?.

The non-stationary part

In applications we are often more interested in the price level process $\{Q_k\}$ than the rate of inflation, and of course we need the dividend process $\{D_k\}$ and stock prices $\{S_k\}$ as well as returns R_k^e of investing in equity; see, e.g., the second part of table 13.6. They are are defined in terms of the the stationary building blocks introduced above through the simple recursions

$$Q_k = (1 + I_k)Q_{k-1} \tag{1.28}$$

$$D_k = (1 + I_k^d)D_{k-1} \tag{1.29}$$

$$S_k = \frac{D_k}{y_k} \tag{1.30}$$

$$R_k^e = \frac{S_k + D_k}{S_{k-1}} - 1. \tag{1.31}$$

There are no new parameters in this part of the model.

The return on equity needs eleaboration. Suppose equity priced as S_{k-1} is bought at time t_{k-1}. At the end of the period the value of the stock is changed to S_k and we have also collected dividend D_k. That leads to the expression (1.31) for the return R_k^e on the investment S_{k-1}. A useful alternative form is obtained if we use (1.30) to replace S_k and S_{k-1}. Then

$$R_k^e = \frac{D_k/y_k + D_k}{D_{k-1}/y_{k-1}} - 1,$$

which becomes

$$R_k^e = \frac{D_k}{D_{k-1}}(1 + y_k)\frac{y_{k-1}}{y_k} - 1,$$

or, on inserting (1.29)

$$R_k^e = (1 + I_k^d)(1 + y_k)\frac{y_{k-1}}{y_k} - 1. \tag{1.32}$$

Since $\{I_k^d\}$ and $\{y_k\}$ are described by stationary processes, the return on equity $\{R_k^e\}$ becomes stationary too. The non-stationary effects in the dividends $\{D_k\}$ and the prices $\{S_k\}$ on equity cancel in the returns which fluctuate under the Wilkie model without persistent trends in any direction.

Unit:%, 100000 simulations.

	I_k	y_k	I_k^d	$r_k(T)$	r_k
Fixed[1]	4.8	4.1	6.5	7.8	6.3
Mean	4.9	4.2	6.9	8.2	6.8
SD.	5.2	1.0	9.4	2.0	2.5

[1] *Without andomness, i.e (13.26) and (13.27).*

Table 1.8: *Mean and standard deviation of the five stationary consituents of the Wilkie model*

One-year behaviour

Although the main purpose of the Wilkie model is to describe investment risk in the long run, it is a useful introductory exercise to examine model projections for a single year. We have in tables 13.8 and 13.9 allowed the macreoconomic variables to develope over 20 years and recorded means and standard deviations from the *last* of these *simulated* annual movements. This way of running the experiment ensures that the statistical parameters come from a typical state of the economy; see, e.g, section 3.?. How the simulations were carried out in detail is discussed below.

The mean and standard deviation of the rate of inflation, equity yield, equity dividend inflation and the long and short rate of interest and the spot rate are shown in table 13.8. It emerges that the the expressions (1.26) and (1.27) are only slightly less that the true expectations. The long rate with higher mean return and smaller standard deviation is, on the surface, better for long term investments than the spot rate.

A better picture of the potential of the various asset classes are obtained if we compare the interest rates with the return on equity. That is done in Table 13.9, both for the ordinary return (left hand side) and the inflation-adjusted one (right hand side). If R_k is the nominal

	Current			*Inflation adjusted*		
	Mean and standard deviation (unit: %)					
	R^e_k	$r_k(T)$	r_k	R^e_k	$r_k(T)$	r_k
Mean	13.3	8.2	6.8	7.9	3.3	2.0
Sd	23.0	2.0	2.5	20.9	4.9	5.1
	Correlation matrix					
	R^e_k	$r_k(T)$	r_k	R^e_k	$r_k(T)$	r_k
R^e_k	1.0	.07	.04	1.0	.-.06	-.06
$r_k(T)$	.07	1.0	.66	-.06	1.0	.93
r_k	.04	.66	1.0	-.06	.93	1.0

Table 1.9: *Mean, standard deviation and correlation matrix of annual return on equity, long and short bank deposits under the Wilkie model.*

return on an investment then

$$\frac{R_k - I_k}{1 + I_k}$$

is the returns in real terms; see ?.

There are several striking features. From the experience of the 20th century equities have much higher *expected* return than interest rate instruments. Note that this also holds when we correct for inflation. The other side of the coin is risk. Table 13.9 suggests that equities is much more risky, the standard deviation being ten times higher when we do not correct for inflation. In real terms that gap is diminished to a factor around four due to standard deviation of interest rates more than doubling. Folklore cited earlier that equities are robust towards inflation seems at least partially substantiated, but we shall study this more thoroughly in section 13.5.

It is also worth commenting on the correlation between asset returns. The two interest rates are strongly correlated, particularly when inflation adjusted. By contrast their correlation with equity is virtually zero. Such results are simplistic in the sense that the dynamic development of the assets is a result of much more complex features than merely correlations between movements at the same period. The simplest way to proceed is to examine returns over longer time horizons through simulations. That is dealt with in section 13.5.

Implementing the Wilkie models

A good summary of the Wilkie models is provided by a sketch of how they are implemented in the computer. Actually this is not much more more than a repetition of all the recursive equations above. We start with inflation.

Algorithm 13.3 Simulating inflation

Input: ξ^i, a^i, σ^i
0 $Q^*_= \leftarrow 1$, $Z^{i*}_0 \leftarrow z^i_0$ % z_0 *user selected*
1 For $k = 1, \ldots, K$ do
2 Draw $\varepsilon^{i*}_k \sim$ normal(0,1)
3 $Z^{i*}_k \leftarrow a^i Z^{i*}_{k-1} + \sigma^i \varepsilon^{i*}_k$
4 $Q^*_k \leftarrow Q^*_{k-1}(1+\xi^i)\exp(Z^{i*}_k)$
5 Endfor
6 Return Q^*_k, Z^{i*}_k, $k = 1, \ldots, K$.

On line 4 (1.20) and (1.28) have been merged into one equation. Note that the simulated quantites have been *-marked according to our usual convention. A question is how the recursive scheme should be initialised (line 0). The price index itself is easy, simply make Q^*_0 equal to one. Then all subsequent prices can be referred back to the level at the start of the scheme by dividing on Q^*_k.

For z^i_0 there are several possibilties. We may take $z^i_0 = 0$. The simulations are then run from a typical state of the economy. Often we would want to start from the situation we actually have. Let $I_0 = I^o_0$ be the observed rate of inflation. Insert $k = 0$ in (1.20) and solve it for $Z^i_0 = z^i_0$. We then obtain

$$z^i_0 = \log\left(\frac{1 + I^o_0}{1 + \xi^i}\right)$$

Note that the sequence $\{Z^{i*}_k\}$ in algorithm 13.3 may have to be stored to be used with other simulation algoritms. For equity the scheme runs as follows:

Algorithm 13.4 Simulating equity

Input: ξ^y, a^y, σ^y, $\theta^{y|i}$,
ξ^d, b^d_1, $\theta^{d|y}$, σ^d, $a^{d|i}$, $b^{d|i}_0$, $b^{d|i}_1$.

0 Select D_0^*, Z_0^{y*}, ε_0^{y*} ε_0^{d*}, $Z_0^{d|i*}$, % *See below*
1 For $k = 1, \ldots, K$ do
2 Draw ε_k^{y*}, $\varepsilon_k^{d*} \sim$ normal (0,1)
3 $Z_k^{y*} \leftarrow a^y Z_{k-1}^{y*} + \sigma^y \varepsilon_k^{y*}$
4 $y_k^* \leftarrow \xi^y \exp(Z_k^{y*} + \theta^{y|i} Z_k^{i*})$
5 $Z_k^{d|y*} \leftarrow \sigma^d(\varepsilon_k^{d*} + b_1^d \varepsilon_{k-1}^{d*}) + \theta^{d|y} \varepsilon_{k-1}^{y*}$
6 $Z_k^{d|i*} \leftarrow a^{d|i} Z_{k-1}^{d|i*} + (b_0^{d|i} Z_k^{i*} + b_1^{d|i} Z_{k-1}^{i*})$
7 $D_k^* \leftarrow D_{k-1}^*(1 + \xi^d) \exp(Z_k^{d|y^*} + Z_k^{d|i*})$
8 $S_k^* = D_k^* / y_k^*$
9 $R_k^{e*} \leftarrow (D_k^* + S_k^*)/S_{k-1}^* - 1$
10 Endfor
11 Return $R_k^{e*}, \varepsilon_k^{y*}$, $k = 1, \ldots K$.

Again this is no more than a condensation of equations given earlier. Note that we have to store the sequence ε_k^{y*}, which are needed with interest rates in a moment. How we start the scheme is the same issue as for inflation. If you go carefully through the recursion, you will discover that we need precisely the five quantities on line 0. The dividend D_0 can be selected as the current one, but for the returns, it does not matter what we choose see exercise ?.

For the other four $(Z_0^{y*}, \varepsilon_0^{y*}, \varepsilon_0^{d*}, Z_0^{d|i*})$ there are the same two possibilities as for inflation. We may either select all as zero, starting at typical state of the economy or adapt it to the present situation observed. However, the latter is now technically more complicated and we deal with it in exercises ?? and ??.

Finally there is the algorithm for the two interest rates:

Algorithm 13.5 Interest rates
Input: ξ^f, a^f, σ^f,ξ^r,
ξ^r, a^r, σ^r, $\theta^{r|y}$, ξ^i, $\theta^{r|i}$,
0 $Z_0^{f*} \leftarrow z_0^f$, $Z_0^{r*} \leftarrow z_0^r$, % z_0^f, z_0^r *user selected*
1 For $k = 1, \ldots, K$ do
2 Draw ε_k^{f*}, $\varepsilon_k^{r*} \sim$ normal (0,1)
3 $Z_k^{f^*} \leftarrow a^f Z_{k-1}^{f^*} + \sigma^f \varepsilon_k^{f*}$
4 $F_k^* \leftarrow \xi^f \exp(Z_k^{f*})$
5 $Z_k^{r*} \leftarrow a^r Z_{k-1}^{r*} + \sigma^r \varepsilon_k^{r*} + \theta^{r|y} \varepsilon_k^{y*}$
6 $r_k(T)^* \leftarrow \xi^r \exp(Z_k^{r*}) + \xi^i + \theta^{r|i} Z_k^{i*}$
7 $r_k^* \leftarrow F_k^* r_k(T)^*$
8 Endfor
9 Return $r_k(T), r_k^*$, $k = 1, \ldots K$.

For the initial values there are the same options as before. The details of adapting to the preseent state of the economy are worked out in exercise ?.

1.5 Investment risk over decades

Posing the problem
Insurance companies and public pension schemes with obligations extending over decades have to take the long view in their investments. But can financial risk over such a long time span really be evaluated? And if so, how should we go about it and what would be the uncertainty?

There are several sides to this. The simplest is how returns on single assets develope under stochastic regimes. Then there is the *portfolio* viewpoint. Normally investments are spread over different asset classes, for example equities and bonds of varying maturity. Clearly we must learn how to conduct appraisals of our *combined* holdings far ahead, uncertainty included. The problem is aggragavated by portfolios being managed according to different strategies. How the latter are exercised often depends on what happens in the future. This is something we would like to take into account at an early stage when the long-term investment program is planned. It will be demonstarted below how you do that.

Another issue is the use of risk-reducing instruments like options. Is that advantageous, or rather, under what circumstances should they be employed? A first step is to evaluate how risk is altered when the portfolio is protected through financial derivatives.

These assessments will be run under the long-term model discussed in the preceding section. That raises still another question: What are the consequences of the model being wrong? How much is the evaluations changed by that? If the 20^{th} century calibration is not to be relied on, then how should it be modified? It is not within the scope of the present book to answer that, but what it can do is to propose a framework for examining such issues. We deal with it at the end of the section.

Return on single assets

One-year returns on equity and interest rate were recorded in in table 13.9. The target now is longer time horizons. It is simplest to assume that a single euro or US$ is invested at time $t_0 = 0$. The value of the assets then evolves according to the recursion ($k = 0, \ldots, K-1$)

$$\mathcal{V}_{k+1} = (1 + R_k)\mathcal{V}_k, \qquad \mathcal{V}_0 = 1, \tag{1.33}$$

which yields the nominal return

$$R_0(K) = \mathcal{V}_K - 1 \tag{1.34}$$

over a time span of length $T = Kh$, h being one year. The corresponding *real* return is

$$R_0^{\mathrm{re}}(K) = \frac{\mathcal{V}_K}{Q_K} - 1. \tag{1.35}$$

The return R_k in (1.50) may come from a complex portfolio of assets (see below), but we shall first examine single asset classes. Three possibilites covered by the model in section 13.4 are then

$$\begin{aligned} R_k &= r_k && \text{(spot rate)} \\ &= r_k(L) && \text{(long rate)} \\ &= R_k^e && \text{(equity)} \end{aligned}$$

Note that with this set-up all funds received are placed in the same asset. For example, with equities dividend could be re-invested in several ways. When $R_k = R_k^e$ is the return throughout, this means that dividend money is always used to buy new stock. An alternative investment plan is examined later in this section.

The computer implementation of the recursion (1.50) is summarized by the following algorithm:

Algorithm 13.6 Asset risk over time

0 $\mathcal{V}_0^* \leftarrow v_0$ %*Initial investment, below* $v_0 = 1$
1 For $k = 0, \ldots, K-1$ do
2   Generate R_k^*, Q_k^* %*Wilkie, selected parts*
3    $\mathcal{V}_{k+1}^* \leftarrow \mathcal{V}_k^*(1 + R_k^*)\mathcal{V}_{k-1}^*$,
4 Endfor
5 $R_0(K)^* \leftarrow \mathcal{V}_K^* - 1$, $R_0^e(K)^* \leftarrow (\mathcal{V}_K^*/Q_K^*) - 1$
6 Return $R_0(K)^*$ and $R_0^e(K)^*$

Note that on line 2 the return R_k^* and the price index Q_k^* are generated by inserting the relevant parts of algorithms 13.3-13.5 above, depending on the asset in question. On output the algorithm produces the nominal and real return. It is possible to move computation of the returns *inside* the loop on lines 1 to 4 so that they are obtained at any point in time t_k. We utilize that below.

Suppose the scheme is run m times, producing m realisations of the returns $R_0(k)$ and $R_0^{\mathrm{re}}(k)$ at each t_k. Each set may be ranked in ascending order; the ϵm'th in size is then an approximation of the ϵ percentile of the return; see, e.g., chapter 4. The results below are based on $m = 100000$ simulations, which means that Monte Carlo error virtually vanishes in the ϵ range considered. All simulations were run from the same initial scenario typifying an 'average' state of the economy of the twentieth century. In precise terms all Z-processes in (1.13-1.19) were zero at $t_0 = 0$.

To see how the scenarios develop you may consult figure 13.6, but our initial discussion is in terms of figure 13.5 displaying *percentile curves*. If read vertically the figure relays the variability of the returns at future points in time. For example, the nominal return from the spot rate after twenty years (upper, left corner of figure 13.5) are almost 500% if we have been in real luck and no more than about 120% in the opposite case. These numbers are the 95% and the 5% percentiles respectively. With this understanding what does figure 13.5 tell us about investment risk under the risk model of the twentieth century?

Note the differing scales on the vertical

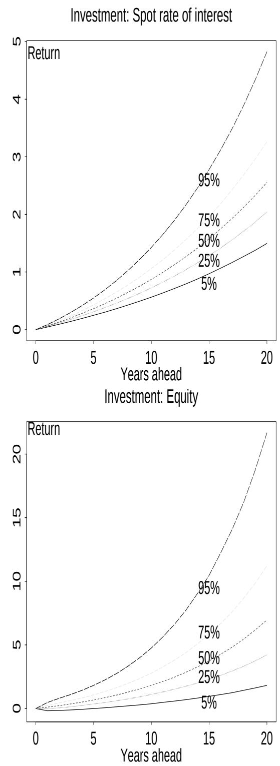

Figure 1.5: **Percentiles** *of the returns under the Wilkie model,* **current** *value to the left,* **real** *to the right.* **Varying vertical scale**

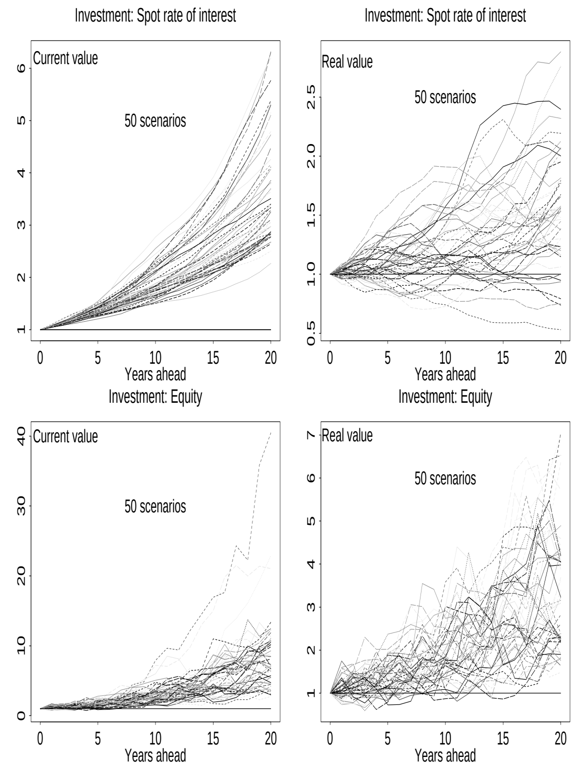

Figure 1.6: *Simulated development of investment of one euro according to Wilkies model,* **current** *value to the left,* **real** *to the right.* **Varying vertical scale**

axes. The top panel is of the spot rate of interest and lower one applies to equity; nominal returns are to the left and real ones to the right. The lesson learned seems to be this: Equity has *in nominal terms* both a larger downside and a (much) larger upside than the spot rate. In real terms its downside over time is *smaller*, not larger. In other words, if inflation-linked liabilities are to be protected, the spot rate of interest is risky! It is also worth noticing that returns, whether nominal or real, are skewedly distributed. The 50% curves in figure 13.5 are not in the middle, but drawn downwards.

Single assets: A second look
How do we proceed when equity dividend is not used to buy new shares, as in the prceding experiments, but instead put in the bank, earning the spot rate of interest? The portfolio then consists of the original shares bought plus a growing bank account. In mathematical terms the scheme develops according to $(k = 0, \ldots, K-1)$

$$\begin{aligned} \mathcal{V}_{k+1} &= \mathcal{V}^r_{k+1} + S_{k+1}, & \mathcal{V}_0 &= S_0 \\ \mathcal{V}^r_{k+1} &= (1 + r_k)\mathcal{V}_k + D_k, & \mathcal{V}^r_0 &= 0. \end{aligned} \quad (1.36)$$

Here $\mathcal{V}_k$, the portfolio value at time t_k, is the sum of the value of the stock held and the bank account $\mathcal{V}^r_k$. The original investment in equity is S_0, which is also the value of the portfolio at that time. When the system start to run, the value S_k of the stock will fluctuate, dividends D_k are added the bank account which also earns interest.

How is the nominal and real value of the portfolio simulated? We need the joint movements of (Q_k, D_k, S_k, r_k), delivered by the implementation of the Wilkie model, which are then plugged into the recursion (1.53). The real value is $\mathcal{V}_k/Q_k$. The details (the same as above) are omitted.

The 50 simulated scenarios in each of the joint plots figure 13.6 merely confirm the conclusions drawn from Figure 13.5. The spot rate investments (top panel) are run under exactly the same conditions as in figure 13.5 whereas the lower panel now is the recursion (1.53). Note that the original investment is $S_0 = 1$ and you must subtract this number to get returns.

A bond portfolio

The next example is a portfolio of bonds. As elsewhere in this chapter we shall be interested in the long view, decades ahead. The term structure of interest rates observed in the market will put a value on the portfolio today and through forward trade even in the future; see, e.g., chapter 11. Our examples will be based on the twentieth century interest rate model of section 13.4, and here there is an obstacle. The model only produced the spot rate and the very long rate whereas we need the intermediate periods as well. Problems of this type arise in the more complicated situations encountered in practice too. The solution often employed is numerical interpolation, a simple, useful technique outlined below.

Basic training deals with simplified problems, and our bond portfolio will consist of zero-coupon bonds. In real life coupon payments are nearly always present when the time to maturity is long, but that is left out to avoid technical complexity. Suppose we at the start of the scheme (at time $t_0 = 0$) have bought, either directly from the issuer, or in the second-hand market bonds that pays one money unit (say euro) at the time of maturity (and nothing before that). All bonds expire an interval $L = Jh$ after having been issued. The portfolio thus consists of bonds identical type, but their time to maturity varies.

Let $r_0(j)$ is the forward rate of interest for the period from t_0 to t_j $j = 1, \ldots, J$. The initial value of the portfolio is then

$$\mathcal{V}_0 = \sum_{j=1}^{J} \frac{B_j}{\{1 + r_0(j)\}^j}, \tag{1.37}$$

where B_j are the number bonds expiring at $t_j = jh$. Since we are dealing with one-euro bonds, this is also the capital released at that time, so that (1.54) is indeed the value of the portfolio. As with equity earlier a study of the financial risk of bonds requires a strategy of what is going to be done with cash obtained at expiry. It will here be assumed that all of it is used to buy new bonds at the price at that time.

Risk is driven by an interest rate model that portrays how the the future term structure $r_k(j)$ is going to develope[4]. The Wilkie model only provides the spot rate r_k and the long rate $r_k(L)$, where L is assumed to be the time to maturity at bond issuing. For iterest rates over intermediate periods we may invoke **numerical interpolation**. If r_k^* and $r_k(L)^*$) are simulated values of the short and long rate, then

$$r_k(j)^* = \frac{(J-j)r_k^* + (j-1)r_k(L)^*}{J-1}, \tag{1.38}$$

is a simulation of $r_k(j)$, coinciding with r_k^* at $j = 1$, with $r_k(L)^*$ at $j = J$ and moving smoothly between these two extremes as j is varied.

The simulated term structure (1.55) leads to simulated, future portfolio values $\mathcal{V}_k^*$ similar to (1.54), but there is one catch. The number of bonds B_j maturing j time units ahead does not stay constant, but fluctuate as time passes. In fact, these quantities become random variables, since the number of bonds bought when renewed depend on what has happened in the money market. Details are conveyed by the following simulation algorithm:

Algorithm 13.7 The rolling bond portfolio

0 $B_j^* \leftarrow B_j$, $j = 1, \ldots, J$

1 For $k = 1, \ldots, K$ do

   %Updating the portfolio

2   For $j = 0, \ldots, J-1$ do

$$B_j^* \leftarrow B_{j+1}^*$$

3   $B_J^* \leftarrow B_0^* \{1 + r_k^*(J)\}^J$ *%Buying bonds*

[4] Inflation contributes to risk too.

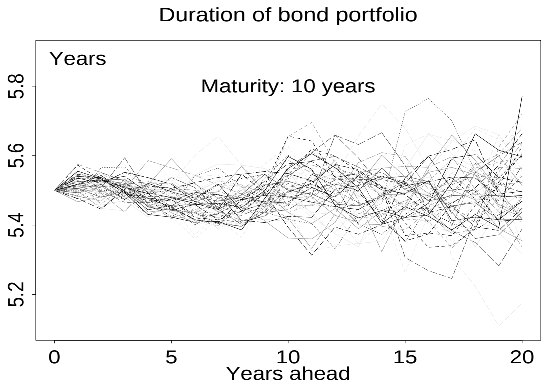

Figure 1.7: *Simulated development of duration for bond poertfolio described in text.*

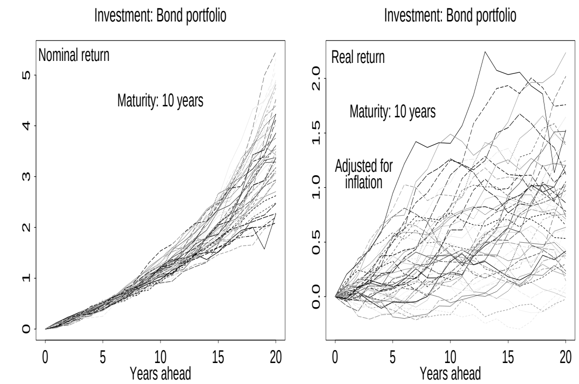

Figure 1.8: *Simulated k-step returns for the bond porfolio described in text,* **nominal** *to the left,* **real** *to the right.* **Varying vertical scale**

%*The term structure, Wilkie or implied*
4 For $j = 1, \ldots, J$, generate $r_k(j)^*$

%*Now valuation*
5 $\mathcal{V}_k^* \leftarrow 0$
6 For $j = 1, \ldots, J$ do
$$\mathcal{V}_k^* \leftarrow \mathcal{V}_k^* + B_j^*/\{1 + r_k(j)^*\}^j$$
7 End k − loop
8 Return $\mathcal{V}_1^*, \ldots, \mathcal{V}_K^*$.

We start (line 0) by storing the number of bonds bought, which are revised (lines 2 and 3) as the scheme develops. The cash released ($= B_0^*$) for bonds maturing is used to buy new bonds (line 3). Next (line 4) comes the term structure, which may be generated in many ways. We have below used the Wilkie model with the interpolation (1.55). An alternative approach could be implied, historical interest rate curves. Finally (lines 5-6) the valuation is carried out. The output is the values of the portfolio. We may also compute the one-step returns through

$$R_k^{b*} = \frac{\mathcal{V}_{k+1}}{\mathcal{V}_k^*}.$$

Adjustment for inflation is carried out in the way explained earlier.

The experiments were run with an annual time increment (i.e. $h = 1$) and a time to maturity of $L = 10$ years. Initially, equal value was placed on all the bonds. This means that during the first ten years the portfolio had a guaranteed cash-flow that was the same every year. The initial *duration* $\mathcal{D}_0$ of the portfolio is thus the average of the numbers $1, 2, \ldots 10$; hence $\mathcal{D}_0 = 5.5$. How the duration fluctuates over 20 years is shown in figure 13.7 under 50 different sceanrios; see exercise ?? for a discussion of implementation details. Clearly duration remains fairly stable.

How the portfolio value evolves is demonstrated in Figure 13.8 using the same 50 scenarios as in figure 13.7 (and also figure 13.6). Nominal returns are to the left and real ones to the right. If you compare with the results produced by the spot rate investments in figure 13.6, a more favourable picture has now emerged. Investment in equity is still in real terms superior over time.

Could we run *simpler* schemes and ignore the spread in the time to maturity? One way would be to run the recursion

$$\mathcal{V}_{k+1}^* = \{1 + r_k(\mathcal{D}_0)\}\mathcal{V}_k^*, \tag{1.39}$$

where we accumulate according to the interest rate defined by the initial durarion of the portfolio. Simulating according to (1.56) is a proxy for the more accurate (but also more complicated) algorithm 13.7. The two assessments have been compared in table 13.10. Error due to ignoring cash-flow spread sesm to large to be ignored.

	Percentiles				
	5%	25%	50%	75%	95%
Exact	3.6	4.1	4.6	5.2	5.7
Approximate	3.4	4.1	4.8	5.7	6.4

Table 1.10: *The distribution of* 20 *years returns (nominal) for bond portfolio by exact and approximate method.*

Examining different asset classes

In practice companies make use of many investment outlets, the principal tools being

- *cash* $\{\mathcal{V}_k^c\}$, earning the short rate $\{r_k\}$,
- *bonds* $\{\mathcal{V}_k^b\}$, earning bond returns $\{R_k^b\}$,
- *equity* $\{\mathcal{V}_k^e\}$, earning equity return $\{R_k^e\}$

Here bonds and equity would typically consist of many different assets, but those details are hidden in the following. The development over time of the combined holding is

$$\begin{aligned}\mathcal{V}_{k+1}^c &= (1+r_k)\mathcal{V}_k^c \\ \mathcal{V}_{k+1}^b &= (1+R_k^b)\mathcal{V}_k^b \\ \mathcal{V}_{k+1}^e &= (1+R_k^e)\mathcal{V}_k^e \\ \mathcal{V}_{k+1} &= \mathcal{V}_{k+1}^c + \mathcal{V}_{k+1}^b + \mathcal{V}_{k+1}^e,\end{aligned} \tag{1.40}$$

where the last line summarizes the total financial status.

How the portfolio evolves depends on strategy. In (1.56) the three accounts live their own life. That doesn't mean that they could be simulated separately from each other. Indeed, we have learned earlier that there is an important dependency in how they develop. What it does mean is that the relative values of the three assets are permitted to float freely. Such a strategy is called **buy-and-hold**.

In practice it does not work like that. First of all, there are persistent movements in and out of the cash account, and sometimes bonds and equity are liquidated to cover liabilites. The link to the liability side is taken up in section 13.6. What we have in mind at this point are shifts been assets classes for reasons of financial strategy according to the current view on what is profitable. This is known as **rebalancing** the portfolio. The mathematical formulation makes use of weights ω_k^c, ω_k^b and ω_k^e defining the relative share of assets held in the three instruments, i.e.

$$\omega_k^c = \frac{\mathcal{V}_k^c}{\mathcal{V}_k}, \quad \omega_k^b = \frac{\mathcal{V}_k^b}{\mathcal{V}_k}, \quad \omega_k^e = \frac{\mathcal{V}_k^e}{\mathcal{V}_k}. \tag{1.41}$$

Clearly

$$\omega_k^c + \omega_k^b + \omega_k^e = 1. \tag{1.42}$$

If we leave the system (1.56) to itself these weight will vary with time (as indicated by the mathematical notation).

In real life a company might have laid down rules as to what weights it will permit, often confining weights to certain intervals. The financial regulators might have a say in this too. In some countries the equity share must be below a certain upper limit. We shall on this point simplify and assume that weights are *fixed*, so that a given triple $(\omega^c, \omega^b, \omega^e)$ prevails at any point in time. How is that formulated mathematically?

If we ignore rebalancing costs (another simplification made), it is then inefficient to work with separate accounts for the three instruments, as in (1.58). It is better to combine assets classes through their *returns*. The recursion now takes the form

$$\begin{aligned}R_k &= \omega^c r_k + \omega^b R_k^b + \omega^e R_k^e \\ \mathcal{V}_{k+1} &= (1+R_k)\mathcal{V}_k,\end{aligned} \tag{1.43}$$

where the portfolio return R_k is a weighted sum of the returns from the three assets classes. Why that is so was explained in chapter 2.

Is the implementation of Monte Carlo schemes much affected by financial strategy? The answer is no. The backbone is in any case how scenarios $\{r_k^*, R_k^{b*}, R_k^{e*}\}$ of the returns are generated. Subsequently there is a simple matter to insert them into either of the recursions (1.58) and (1.59); see also exercise? for a discussion on how more complicated stratgies can be dealt with.

100000 simulations

	Percentiles				
Strategy	5%	25%	50%	75%	95%
Buy-and-hold	0.40	0.97	1.51	2.25	3.81
Fixed	0.48	1.05	1.57	2.24	3.51

Table 1.11: **Real** *return (20 years) on financial portfolio described in text*

A small comparision of the floating weight and fixed weight strategy is carried out in Table 13.11. Initial weights were

- $\omega_0^e = 0.5$ (equity)
- $\omega_0^b = 0.4$ (bonds)
- $\omega_0^c = 0.1$ (cash).

All assets followed the Wilkie model starting as described in ??. The bond portfolio was the rolling one introduced above; see, e.g., algorithm 13.7. In table 13.11 the real return is reported. The price index was again taken from the Wilkie model and simulated jointly with the financial variables, as outlined in section 13.4.

A principal difference of the two strategies is the higher variability of the buy-and-hold. That is plausible as we are then wholeheartedly a part of all upwards and downwards trends of the stock market without doing anything to correct. The other strategy amounts to constant rebalancing (carried out without cost), and means that shares are sold at high (when equity weight have been dragged upwards) and bought at low (when the equity weight has fallen below target). That is an entirely sensible way to conduct financial investments and its return is higher than that of its competitor up to the 75% percentile.

Using equity options

The buy- and-hold-strategy was more risky, but had also higher upside than when fixed asset weights were used. Could we have the best of both worlds by allowing weights to float and protect against the downside by buying options? Equity derivatives rarely last more than a year, but that is enough. We could renew annually. The issue addressed is whether that leads to a less volatile development than in table 13.11, and, above all, how we proceed to find out such things.

We shall study this through a simplified scheme. During year $k+1$ the value of the equity in our portfolio goes from $\mathcal{V}_k^e$ at the start to $\mathcal{V}_{k+1}^e$ at the end after having bought new stock with the dividend received. Suppose options on $\mathcal{V}_{k+1}^e$ can be bought in the market. Since an existing risky position is protected, it must be a put. At time t_k we enter a contract giving the right (but not the obligation) to sell our holding of equity for an agreed sum A_k. In that way we insure against the return on equity falling below a certain level.

The option will be payed for by liquidating a part of the equity owned. There is a slight technical obstacle here. The option premium payed up-front is

$$\pi_k = \pi_k(\mathcal{V}_k^e),$$

depending on the value of the equity. But when the option premium has been subracted, our holding is $\mathcal{V}_k^e - \pi_k$, rather than $\mathcal{V}_k^e$. That reduces the premium, and it becomes *smaller* than the original π_k. We shall have to adjust π_k so that it is exactly equal to the option on $\mathcal{V}_k^e - \pi_k$.

That leads to an equation, see, e.g., exercise ?. There is a neat solution when the option is at-the money. Consider a European contract, issued at t_k and expiring at t_{k+1}. At-the-money means that the exercise price A_k at t_{k+1} coincides with the value of the underlying asset at t_k. It will be shown below that the net return on equity now becomes

$$R_k^{eo} = \frac{\max(0, R_k^e) - \pi_k(1)}{1 + \pi_k(1)} \tag{1.44}$$

where

$$d_{1k} = \frac{r_k + \sigma_k^2/2}{\sigma_k}, \quad d_{2k} = d_{1k} - \sigma_k \tag{1.45}$$

$$\pi_k(1) = e^{-r_k}\Phi(-d_{2k}) - \Phi(-d_{1k}). \tag{1.46}$$

Model: Wilkie. Simulations: 100000

Use of options*	Percentiles 5%	25%	50%	75%	95%
	Nominal returns				
Yes	1.8	3.5	5.4	8.4	14.9
No	1.8	4.2	7.0	11.2	21.7
	Real returns				
Yes	0.2	0.9	1.5	2.5	4.5
No	0.3	1.2	2.1	3.5	6.4

*European put, at-the-money

Table 1.12: *The distribution of the* 20 *year returns for the* **portfolio described in the text**.

Model: Random walk. Simulations: 100000

Use of options*	Percentiles 5%	25%	50%	75%	95%
	Parameters: $\xi = 10\%$, $\sigma = 20\%$				
Yes	0.8	2.2	3.9	6.8	14.7
No	0.7	3.0	6.4	12.9	31.0
	Parameters: $\xi = 10\%$, $\sigma = 40\%$				
Yes	-0.1	1.4	4.3	11.2	42.8
No	-0.6	1.2	6.4	23.4	137.5

*European put, at-the-money

Table 1.13: *The distribution of the* 20 *year nominal returns for investments in* **equity** *only.*

Here σ_k is the volatility of $\mathcal{V}_k^e$ and r_k is the short rate of interest. The expression for $\pi_k(1)$ is the Black & Scholes put formula when $T = 1$ and $A = \mathcal{V}_k^e = 1$; see, e.g., (??).

The portfolio of asset classes still evolves according to (1.56), except for the earlier return R_k^e on stocks now being replaced by R_k^{eo}. What is the effect of introducing the option? It is easily seen that

$$\begin{aligned} R_k^{eo} &< R_k^e, \quad \text{if } R_k^e > 0 \\ R_k^{eo} &> R_k^e, \quad \text{if } R_k^e < 0 \\ R_k^{eo} &\geq -\frac{\pi_k(1)}{1+\pi_k(1)}. \end{aligned}$$

The downside is protected. Our return can't fall below a certain minimum level, and if $R_k^e < 0$, we lose *less* by having bought the option. The counterpart is lower returns than we could have had when $R_k^e > 0$. Whether the total effect is advantageous in the long run will be examined through simulations.

The expression (1.44) for the net return now follows, since

$$R_{k+1}^{eo} = \frac{\mathcal{V}_{k+1}^e}{\mathcal{V}_k^e} - 1.$$

Using at-the-money options for equity requires the return R_k^e in the recursion (1.56) to be replaced by R_k^{eo}. Apart from this the simulations are generated by the same algorithm as before. We saw earlier (section 13.4) that the Wilkie model for equity resembled a random walk on log-scale with annual drift and volatility $\xi = 10\%$ and $\sigma = 20\%$ respectively. The latter value was empoyed in (1.45) when calculating the option premium .

The results in table 13.12 suggest that downside protection through options may be a costly strategy. After twenty years the returns are higher *without* them from the 5% percentile and upwards. Using options only fare better at the extreme low tail. How general is such a conclusion? Does it depend on the underlying model? A second round of experiments are reported in table 13.13. This time investments are equity only, and their movements followed the random walk on log-scale, exactly the model on which the Black & Scholes argument is based. With annual drift and volatility 10% and 20%, parameters corresponding to those inherent in the Wilkie model, the earlier picture is largely confirmed in the upper half in table 13.13.

However, suppose the volatility is doubled to 40%. The investement then has a healthy expected return of

$$\exp(0.2 + 0.4^2/2) - 1 \approx 20\%,$$

but the risk is great too! Now (table 13.13, lower half) there could be more sense of salvaging some of the the high upper returns for security at the lower tail. Few people are likely to be very comfortable with a twenty-year loss of 60% with probability as high as 5%.

Mathematical derivation of (1.44)

We start by appealing to the Black & Scholes put option formulas in section 12.?. Since the exercise price A_k and the original value of the stock coincides, these formulas reduce to

$$\pi_k(\mathcal{V}_k^e) = \mathcal{V}_k^e \pi_k(1)$$

where expressions for d_{1k}, d_{2k} and $\pi_k(1)$ were given in (1.45)and (1.46). The true option premium $\pi_k(\mathcal{V}_k^e)$ is calculated from $\mathcal{V}_k^e - \pi_k(\mathcal{V}_k^e)$, and must be

$$\{\mathcal{V}_k^e - \pi_k(\mathcal{V}_k^e)\}\pi_k(1).$$

This yields the equation

$$\pi_k(\mathcal{V}_k^e) = \{\mathcal{V}_k^e - \pi_k(\mathcal{V}_k^e)\}\pi_k(1)$$

with the solution

$$\pi_k(\mathcal{V}_k^e) = \frac{\pi_k(1)}{1+\pi_k(1)}.\mathcal{V}_k^e,$$

and

$$\mathcal{V}_k^e - \pi_k(\mathcal{V}_k^e) = \frac{\mathcal{V}_k^e}{1+\pi_k(1)}$$

is the value of the equity *after* the option premium has been subtracted. This grows according to the factor

$$\max(1+R_k^e, 1),$$

since a *negative* return leads to the option holder selling out at the former price. Thus

$$\mathcal{V}_{k+1}^e = \max(1+R_k^e, 1)\frac{\mathcal{V}_k^e}{1+\pi_k(1)}.$$

or

$$\mathcal{V}_{k+1}^e = \frac{1+\max(0, R_k^e)}{1+\pi_k(1)}\mathcal{V}_k^e.$$

The expression (1.44) for the net return now follows, since

$$R_{k+1}^{eo} = \frac{\mathcal{V}_{k+1}^e}{\mathcal{V}_k^e} - 1.$$

Using interest rate floors.

How about downside risk protection through interest rate floors? An insurance company guranteeing its customers annual returns of say 3% 4% might consider such a financial instrument. The option premium is payed up-front, and if the amount of capital, after that expense has been subtracted out, is sufficient to cover the liabilites under 3 or 4% discounting, the company can not fail to meets its obligations. But such a strategy does cost money, possibly quite a lot. The question addressed is how we find such things out.

Interest rate derivatives were discussed in section 12.6, and the premium for a floor was of the form

$$\pi^f(\mathcal{V}_0) = \mathcal{V}_0 \pi^f(1),$$

where the so-called principal $\mathcal{V}_0$ is the amount insured, and $\pi^f(1)$ is the option fee when $\mathcal{V}_0 = 1$. As with equities the signing of a floor taps the original capital $\mathcal{V}_0$ so that we no longer need the coverage of the financial insurance to be quite so high. The equation determining this is

$$\pi^f(\mathcal{V}_0) = \{\mathcal{V}_0 - \pi^f(\mathcal{V}_0)\}\pi^f(1),$$

with the solution

$$\pi^f(\mathcal{V}_0) = \frac{\pi^f(1)}{1+\pi^f(1)}\mathcal{V}_0.$$

This yields

$$\mathcal{V}_0^f = \mathcal{V}_0 - \pi^f(\mathcal{V}_0) = \frac{\mathcal{V}_0}{1+\pi^f(1)} \qquad (1.47)$$

as the amount of capital *after* the premium for the floor have been subtracted. As compensation we receive at the end of each period k the amount

$$\max(r^f - r_k, 0)\mathcal{V}_0^f$$

as reimbursement for the spot rate r_k having fallen below the agreed floor rate r_f. If this amount is added the cash account, the latter develops according to

$$\mathcal{V}_{k+1}^c = (1+r_k)\mathcal{V}_k^c + \max(r^f - r_k, 0)\mathcal{V}_0^f, \qquad (1.48)$$

Model: Wilkie. Simulations: 100000

	Percentiles				
Use of floor	5%	25%	50%	75%	95%
	Nominal returns: **10** *years*				
$r_f = 7\%$	0.67	0.72	0.79	0.91	1.06
$r_f = 4\%$	0.55	0.69	0.83	1.01	1.37
No	0.56	0.73	0.87	1.06	1.43
	Nominal returns: **20** *years*				
$r_f = 7\%$	1.25	1.49	1.76	2.17	3.21
$r_f = 4\%$	1.22	1.66	2.10	2.71	4.08
No	1.50	2.04	2.56	3.26	4.81

Table 1.14: *The distribution of the* 10 *and* 20 *year nominal returns for a pure cash account.*

starting from

$$\mathcal{V}_0^c = \mathcal{V}_0^f. \tag{1.49}$$

The recursion shows how a cash account develops when protected by an interest floor regime. It is straightforward to insert Monte Carlo realisations of the spot rate for simulated scenarios. We may then compare with the growth rate from the *original* amount of capital $\mathcal{V}_0$ *without* interest rate derivatives being in use; se, e.g., (1.56). It is also easy to integrate several asset classes, as in ??.

In the numerical experiments reported in table 13.14 all assets are held in cash. Returns for floor rates 7% and 4% are compared with those obtained without any financial protection at all. The time horizon was ten or twenty years, and the options were bought at the start of the scheme at $t_0 = 0$. The model used for the spor rate of interest was Wilkie's.

The general picture is the same as with equities. Using interest rate floors is an expensive strategy. With the option premium a high amount of capital is drawn out up-front, and the subsequent development under a floor regime does not quite catch up. The ten-year contracts perform better than the twenty-year ones. An alternative strategy of buying contracts of shorter duration is discussed in exercise ?.

Judgment and errors

How to proceed if we believe risk processes ahead to be different from those of the past? It isn't a good idea to discard historical modelling alltogether, as we do not have much else to fall back on. What seems reasonable and practical would be to modify Wilkie's model on certain key points to bring it closer to our perceptions of the future.

For example, consider inflation and rates of interest, which are, as this book is being written (2004) on a much lower level than the average in Wilkie's model. It we judge this situation to continue, how could that view be integrated? One way is to change some of the key parameters. The most natural ones are ξ^i, ξ^y, ξ^d, ξ^f and ξ^r of the system in section 13.4. Those are the most important parameters, chiefly responsible for *expected* growth (without quite being mean parameters themselves).

They are also the most difficult to determine from historical time series; standard errors of their estimates in table 13.7 are huge[5]. Take the rate of inflation. The estimate for ξ^i is 4.8% with standard error (se) equal to 1.2%. This is enormously inaccurate if you recall that random error of one se is what we expect and that discrepancies up to twice that number are plausible. It means that even *if the forces driving inflation remain they were*, the true ξ^i with which the future should be projected could easily be as low as 3.6% or lower or 6% or higher[6]

What this mean in practical terms are shown in table 13.15 where the buy-and-hold experiment reported in table 13.11 have been recalculated with $\xi^i = 3.6\%$ and $\xi^i = 6.0\%$. The two scenarios deviate! Example: the ratio of the two 5% downsides is about 2.5. With such discrepancies there are limits to the usefulness of the historical estimate. But even if we 'take

[5]The parameter $\hat{\xi}^y$ is an exception.

[6]These numbers emerge when subtracting or adding one se to the estimate, for example $3.6\% = 4.8\% - 1.2\%$.

Model: Wilkie. *100000 simulations*

	Percentiles				
ξ^i	5%	25%	50%	75%	95%
3.6%	0.59	1.27	1.93	2.83	4.77
6.0%	0.23	0.72	1.18	1.79	3.05

Table 1.15: **Real** *return (20 years) on financial portfolio in Table 13.11 under buy-and-hold varying the rate of inflation.*

Model: Modified Wilkie. *100000 simulations*

	Percentiles				
Strategy	5%	25%	50%	75%	95%
Buy-and-hold	0.23	0.47	0.67	0.90	1.30
Fixed	0.25	0.48	0.66	0.88	1.23

Table 1.16: **Real** *return (20 years) on financial portfolio in Table 13.11 under modified Wilkie model.*

a position' on future inflation, it *not* necessary to let it stay fixed, since the other parameters (i.e. a^i and σ^i) are more accurately estimated and errors in that part of the model do not have such a severe adverse effect in any case. A similar line of argument could have been presented for most of the other parameters ξ^d,ξ^r and so forth, and substantiates the suggestion in Wilkie (1995) that these parameters might be specified to reflect a 'view'.

An alternative way is the following. Financial risk is for the model in section 13.4 tied to the the rate of inflation (I_k), the spot rate of interest (r_k), the long rate ($r_k(T)$) and the return on equity (R_k^e). If we are going 'to take a position', we could also select factors γ^i, γ^c, γ^r and γ^e and change the simulations from

$$I_k^*, \quad r_k^*, \quad r_k(T)^*, \quad R_k^{e*}.$$

to

$$\gamma^i I_k^*, \quad \gamma^c r_k^*, \quad \gamma^r r_k(T)^*, \quad \gamma^e R_k^{e*}.$$

This reduces (or upgrades) our expectations of financial earnings witout changing the basic structure of the model.

For example, the correlations between asset returns remain what they were and neither are the dynamic properties altered. Expectations and standard deviations do change, but their *ratio* does not. This is a pragmatic way of being faithful to the past, yet allowing for the future to deviate systematically. It is easy to accomodate the change in the simulations algorithms. Simply multiply by the appropriate factor.

Results of the same experiment as that reported in table 13.11 are shown in table 13.16 when

$$\gamma^i = \gamma^c = \gamma^r = \gamma^e = 0.5.$$

We are then assuming that inflation, interest rates and equity returns to be halved compared to what they were in the twentieth century. Does this cause changes in real terms? Yes indeed! If you compare tables 13.11 and 13.16 you will discover that the real upside and real downside are both severly altered by the new risk model. What remains stable is the relative performance of the two strategies.

1.6 Assets and liabilites I

Posing the problem

The liabilities of a life insurance company are, if not exactly given, at least reasonably forseeable. Causes for uncertainty were analysed in sections 13.2 and 13.3. They *could* have huge impact, but in section 13.4 it seeemed to emerge that financial risk is even more important. The way the insurance business operates creates large initial surpluses that have to be invested. It must then be a major aim not only to bring in profits to the shareholders in the short run, but also to allow liabilities to be covered in the long one[7]. Balancing between *high* returns on investments on one hand and *solvency* with respect financial obligations on the other is no easy task. What should be our strategy? Should the way we invest be influenced by existing liabilities or is that irrelevant?

Operationally it is simplest to separate the

[7]This is, of course, also in the shareholder interest, and the society as a whole accepts nothing less.

two. One department deals with insurance risk; another one manages the assets. It used to be like that. The investment arm of the companies once ran their business as if the liabilities didn't exist. However, ever since the pioneering efforts of Redington (1952) it has been realised that it might be beneficial to coordinate the handling of assets and liabilites.

This derives from two basic factors. One is the *distribution of liabilites over time*. From the dicussion in the preceding section, surely it could be sensible to invest more in high-yielding, but risky equities if a lion's share of the payments to the policy holders are put off in time, say a decade or two. The idea is to trust the higher *expected* return from the stock market to overcome its much higher uncertainty. If we can rely on past experience, the financial earnings would then be higher over time. The prerequisite is that the company can afford to wait for the equity market to realize its potential. In practice it isn't that simple since government regulation does not allow companies to be underfinanced with the respect to its obligations at any point in time. We shall see how that complication can be formulated mathematically in section 13.6; the solution is through simulation.

The second factor is whether liabilities are index-linked or not. If they are, we must be definitely concerned with inflation and might choose investment outlets that safeguards better against such risk. The longe-range portrayals of financial uncertainty of the preceding section will help us in evaluating this.

Simulation has become an indispensable tool to deal with these issues, but first we present the original line of argument known as immunization. This idea is very simple and tells a lot of what asset-liability matching is about. The drawback of the simplicity is restrictions on applicability; we can't use in all situations we would want to.

Figure 1.9: *The distribution of liabilities over time for the portfolio in section 11.3*

Immunisation: What we want

In 1952 the British actuary Frank Redington suggested (see, e.g., Redington, 1952) that insurance risk could be protected against interest rate movements through a strategy that was coined **immunisation**. To present the idea consider the portfolio introduced in section 14.2. Using the same mathematical notation as there the net cash-flow from company to customers is in period k

$$L_k = \sum_{l=l_0}^{l_e} ({}_k p_l \, J_l) \, {}_k\zeta_l \tag{1.50}$$

where ${}_k p_l \, J_l$ is the number of persons of age l that are alive at the entry of year k and ${}_k\zeta_l$ the payments, positive if the policy holders draw benefit, negative if they are contributors.

The liabilites L_k are plotted against k in Figure 14.6 for the portfolio in section 10.3. There is steady growth to a maximum occurring after 35 years. In practice administrative overhead and other costs come on top, but let us for simplicity accept figure 13.6 as a picture of how liabilities of the existing portfolio distribute over time.

Suppose we deem the interest rate r_0 suitable to discount the payment stream $\{L_k\}$. Its present value is then

$$\mathrm{PV}_0^l = \sum_{k=0}^{\infty} \frac{L_k}{(1+r_0)^k}. \tag{1.51}$$

Let us utilise the initial surplus on the contracts to buy bonds that in the absence of defaults give us a fixed, future cash-flow A_k. The present value of that is

$$\mathrm{PV}_0^a = \sum_{k=0}^{\infty} \frac{A_k}{(1+r_0)^k}, \tag{1.52}$$

using the same discount rate as in (1.51). We must ensure that

$$\mathrm{PV}_0^a \geq \mathrm{PV}_0^l.$$

Otherwise we would be insolvent.

Over long interest rate is going to fluctuate; that is a major problem for our asset management. Let $\mathrm{PV}^l(r)$ and $\mathrm{PV}^a(r)$ be the same present values when a general rate of interest r is used in (1.51) and (1.52) instead of r_0. What we might want, if we could, is to select the asset flow $\{A_k\}$ so that

$$\mathrm{PV}^a(r) \geq \mathrm{PV}^l(r) \tag{1.53}$$

for *all* r, not only the rate r_0 we have today. In theory, we are then solvent whatever happens![8]

Immunisation and convexity

The Redington argument leads to the situation in (1.53) if r does not deviate too much from r_0. It rests on the two cash-flows in (1.51) and (1.52) being *independent of the interest rate used for valuation.* For the assets this is satisified with bonds we have already purchased (as was assumed), but not if we are dealing with equities or have put the money on the short rate of interest. We are going to need the the *duration* of the two cash-flows $\{L_k\}$ and $\{A_k\}$ when using r_0 as the discount rate; see chapter 12. The way that concept was defined leads to

$$\mathcal{D}_0^l = \sum_{k=0}^{\infty} k q_k^l, \quad \mathcal{D}_0^a = \sum_{k=0}^{\infty} k q_k^a \tag{1.54}$$

where

$$q_k^l = \frac{L_k(1+r_0)^{-k}}{\mathrm{PV}_0^l},$$

$$q_k^a = \frac{A_k(1+r_0)^{-k}}{\mathrm{PV}_0^a},$$

[8]Many reasons for this being a *laboratory* statement have been given earlier.

and $\mathcal{D}_0^l$ and $\mathcal{D}_0^a$ are the durations of the liability and the asset cash-flow respectively. The 0 index reminds us that they are calculated under r_0. Similar notation is used for the *quadratic* coefficients

$$\mathcal{C}_0^l = \sum_{k=0}^{\infty} k^2 q_k^l, \quad \mathcal{C}_0^a = \sum_{k=0}^{\infty} k^2 q_k^a. \tag{1.55}$$

so-called coefficients of **convexity**. While the duration represents the time average of a cash-flow, these convey their *variance*[9].

It can now be proved rigorously (see the end of the section) that if

$$\gamma \mathcal{D}_0^a = \mathcal{D}_0^l, \qquad \gamma = \frac{\mathrm{PV}_0^a}{\mathrm{PV}_0^l}, \tag{1.56}$$

then

$$\mathrm{PV}^a(r) - \mathrm{PV}^l(r) \doteq \mathrm{PV}_0^a - \mathrm{PV}_0^l \tag{1.57}$$
$$+ \frac{1}{2}\frac{PV_0^l}{1+r_0^2}(\gamma\mathcal{C}_0^a - \mathcal{C}_0^l)(r-r_0)^2.$$

This is the first two terms of a series expansion. The next one would be proportional to $(r-r_0)^3$. The question is: What does it tell us about how the asset cash-flow should be organised?

Interpretation

Consider first the condition (1.56), which is given an algebraic reformulation in exercise ?. If we are dealing with portfolios that are not too small, the coefficient γ, as the ratio of the two present values, would not deviate too much from 1[10]. Thus to utilize the Redington result, we plan bond investments so that

$$\mathcal{D}_0^a \doteq \mathcal{D}_0^l;$$

i.e. the durations of the two cash-flows are about equally long.

Secondly, we would want

$$\gamma \mathcal{C}_0^a \doteq \mathcal{C}_0^a > \mathcal{C}_0^l. \tag{1.58}$$

[9]The ordinary variance would be $\mathcal{C}_0^l - (\mathcal{D}_0^l)^2$.

[10]PV_0^l may represents billions of euros or US \$, our assets are not *that* much higher.

Why? Because under this condition the quadratic term in (1.57) becomes *positive* and ensures that

$$\mathrm{PV}^a(r) - \mathrm{PV}^l(r) \geq \mathrm{PV}^a_0 - \mathrm{PV}^l_0$$

for all r. The interest rate r_0 *we are planning under has become a worst case scenario.* All movement of r away from it is benficial! If we are solvent under r_0, we are even more solvent under other interest rates.

The approximate mathematics is likely to be sufficient, but, as with all results of this nature: In practice it may be less clear how to implement it. For example, are the the bonds we require available? What about portfolio changes due to so-called **lapses**[11] and new recrutement? Yet the result gives us a recipy for bond investments in insurance. Ensure that they have about the same duration as the liabilities and then make *their cash-flow more spread out* than the one it is to match. We can achieve that by investing in short and very long bonds, perhaps going short in medium ones.

Why does that lead to (1.58)? Because the coefficients $\mathcal{C}^a_0$ and $\mathcal{C}^l_0$ essentially define a a cash flow time *variance*. That must be so when their time *expectation* , i.e. their duration was about equal; see exercise ?.

Mathematical derivation

To derive the precise form of immunisation we have to subject the two present values $\mathrm{PV}^l(r)$ and $\mathrm{PV}^a(r)$ to a Taylor expanson around a given interest rate r_0. For the former this yields

$$\mathrm{PV}^l(r) \doteq \mathrm{PV}^l_0 + \frac{\partial \mathrm{PV}^l}{\partial r}(r - r_0) + \frac{1}{2}\frac{\partial^2 \mathrm{PV}^l}{\partial r^2}(r - r_0)^2,$$

where the two partial derivatives are evaluated under $r = r_0$. For PV^a_0 we have the same; simply insert $l = a$ everwhere. Our interest is in their difference, i.e.

$$\mathrm{PV}^a(r) - \mathrm{PV}^l(r) \doteq \mathrm{PV}^a_0 - \mathrm{PV}^l_0 + b_1(r - r_0) + \frac{1}{2}b_2(r - r_0)^2. \quad (1.59)$$

where

$$b_1 = \frac{\partial \mathrm{PV}^a}{\partial r} - \frac{\partial \mathrm{PV}^l}{\partial r}$$
$$b_2 = \frac{\partial^2 \mathrm{PV}^a}{\partial r^2} - \frac{\partial^2 \mathrm{PV}^l}{\partial r^2}.$$

The first order partial derivative was calculated in exercise ?. That gave us

$$\frac{\partial \mathrm{PV}^l}{\partial r} = -\frac{\mathrm{PV}^l_0}{1 + r_0}\mathcal{D}^l_0$$

The linear term in the expansion (1.59) vanishes if $b_1 = 0$, or equivalently, if

$$\frac{\mathrm{PV}^l_0}{1 + r_0}\mathcal{D}^l_0 = \frac{\mathrm{PV}^a_0}{1 + r_0}\mathcal{D}^a_0$$

This leads to the condition (1.56).

For the second derivatives we have (exercise ?)

$$\frac{\partial^2 \mathrm{PV}^l}{\partial r^2} = \frac{\mathrm{PV}^l_0}{(1 + r_0)^2}(\mathcal{C}^l_0 + \mathcal{D}^l_0).$$

This yields for b_2

$$b_2 = \frac{\mathrm{PV}^a_0}{(1 + r_0)^2}(\mathcal{C}^a_0 + \mathcal{D}^a_0) - \frac{\mathrm{PV}^l_0}{(1 + r_0)^2}(\mathcal{C}^l_0 + \mathcal{D}^l_0).$$

Here the terms involving $\mathcal{D}^a_0$ and $\mathcal{D}^l_0$ cancel so that

$$b_2 = \frac{\mathrm{PV}^a_0}{(1 + r_0)^2}\mathcal{C}^a_0 - \frac{\mathrm{PV}^l_0}{(1 + r_0)^2}\mathcal{C}^l_0,$$

gives us the expression (1.57) after a brief spell of algebra.

[11] Customers leaving the portfolio

1.7 Assets and liabilities II

Posing the problem
With the powerful computational tools at our disposal it is possible to analyse asset and liability cash-flows without the restrictive condition underlying immunisation. *Technically* there is no limit on the complexity of the systems that can be handled, but another matter is what is useful. *Transparency* of analyses and results is of paramount importance. We simply have to control and oversee what we put into the simulation models in order to gauge, digest and communicate results properly.

In a textbook where techniques are to be learnt, simplicity is vital in any case. We shall below split investments into no more than three asset classes. One of them must be cash, since a bank account is needed to reimburse clients and collect their premia. We shall stick to the convention of the preceding section of using $\{L_k\}$ to denote *net* liabilites; i.e. payment streams that come from the insurance part of the business. This seems to be the most convenient way to formulate an asset-liability scheme that flexibly allows all variations with respect to index-linkage and guarantees to be fitted in.

The aim of this section is to present a fairly general dynamic system of investments and liabilites in a form suitable for simulation in a computer. A chief point is to make the design sufficiently general to accomodate details that arise in practice. One of the things you must learn is how these variations are formulated mathematically so that they can be fed into the key quantities of the dynamic systems. Of course we then draw on the models presented in earlier chapters of this book.

The approach amounts to a *hierarchical* way of working or 'top-down', as it is sometimes called in engineering. It means, above all, that simulation algorithms for more delimited problems are inserted into the coordinating schemes presented below. Inflation is a case in point. Both assets and liabilities may depend on inflation, and we shall be able to reflect that in the analysis by employing the Wilkie inflation model, presented in section 13.3, possibly in a modified version if that seems the more realistic.

Liabilities: Mathematical formulation
It is convenient to write the liabilites as

$$L_k = {}_kJ(c)\zeta(c,k). \tag{1.60}$$

Here ${}_kJ(c)$ the number of policy holders in state c at time t_k, as before. The payment function $\zeta(c,k)$ is now allowed to depend on time k.

Solvency: Mathematical formulation
The solvency of the company at time t_k is not only the question of its combined assets $\mathcal{Y}_k$ at tht time, but also on its future liabilities $\mathcal{R}_k$ and the amount it will receive when bonds are repayed at expiry.

This is for an insurance company *not* the same as solvency, since there may be a huge pile of future obligations that are not included. We come back to that in a moment. Also note that the second line does not include expiry of bonds at which point the original loan is repaid. We are not going to discuss that complication.

1.8 Bibliographical notes

1.9 Exercises