Single period modelling of financial assets

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1 Outline

- A possible - and common - approach to stochastic modelling of risk for financial assets

- Basic properties of the model

- Markets with more than one financial asset

- Funds

2 Market value of financial assets as Geometric Brownian Motion

$S_t =$ value at time t for a certain financial asset.

We believe that there is a probabilistic law governing how S_t evolves over time, by dynamics as follows:

$$dS_t = \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dW_t$$

3 W_t is a Wiener process:

For any $0 < t_1 < t_2 \leq t_3 < t_4$:

- $W_0 = 0$
- $\left(W_{t_2} - W_{t_1}\right) \sim N\left(0, \sqrt{t_2 - t_1}\right)$
- $\left(W_{t_2} - W_{t_1}\right)$ and $\left(W_{t_4} - W_{t_3}\right)$ are stochastically independent.

4 Terminology:

- μ: *drift* ("instantaneous expected growth rate")
- σ: is the *volatility* (also referred to as the *diffusion*)

5 Comments/observations on model:

- Value evolves as determined by "systematic force" disturbed by purely random noise with zero expectation

- Purely forward-looking dynamics:
  - No memory
  - No "mean reversion"
  - "Market will correct itself" inconsistent with the model
  - "Timing opportunities" inconsistent with the model

- Standard assumption that "risk is rewarded":
  - Two financial assets a and b characterized by (μ^a, σ^a) and $\left(\mu^b, \sigma^b\right)$ respectively. Then $\mu^a > \mu^b \; iff \; \sigma^a > \sigma^b$.

6 Probability distribution for S_t?

Motivation: Pretend that W_t is an ordinary differentiable function. Then:

$$\begin{aligned}
dS_t &= \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dW_t \\
\text{''} &= \text{''} \quad \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot W_t' \cdot dt \\
&= \left(\mu + \sigma \cdot W_t'\right) \cdot S_t \cdot dt \\
&\Updownarrow \\
S_t' \text{ ''} &= \text{''} \quad \left(\mu + \sigma \cdot W_t'\right) \cdot S_t
\end{aligned}$$

This is recognized as the differential equation for ordinary geometric growth, which yields the following erroneous expression:

$$\begin{aligned}
S_t^{err} &= S_0 \cdot Exp\left[\int_0^t \left(\mu + \sigma \cdot W_s'\right) ds\right] \\
&= S_0 \cdot Exp\left[\mu \cdot t + \sigma \cdot W_t\right]
\end{aligned}$$

Nice - but regrettably flawed!! Demonstrated by the problem in the following.

Precisely correct that:

$$\begin{aligned} E(dS_t) &= dE(S_t) = E(\mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dW_t) \\ &= \mu \cdot dt \cdot E(S_t) + \sigma \cdot E(S_t) \cdot E(dW_t) \\ &= \mu \cdot dt \cdot E(S_t) \\ &\Updownarrow \\ \frac{d}{dt} E(S_t) &= \mu \cdot E(S_t) \end{aligned}$$

with the obvious solution:

$$E(S_t) = S_0 \cdot Exp[\mu \cdot t]$$

On the other hand, since $W_t = (W_t - W_0) \sim N\left(0, \sqrt{t}\right)$ it is straight forward that:

$$\begin{aligned} E\left(S_t^{err}\right) &= E\left\{S_0 \cdot Exp\left[\mu \cdot t + \sigma \cdot W_t\right]\right\} \\ &= S_0 \cdot Exp\left[\mu \cdot t\right] \cdot E\left\{Exp\left[\sigma \cdot W_t\right]\right\} \\ &= S_0 \cdot Exp\left[\mu \cdot t\right] \cdot Exp\left[\frac{\sigma^2}{2} \cdot t\right] \\ &= S_0 \cdot Exp\left[\left(\mu + \frac{\sigma^2}{2}\right) \cdot t\right] \end{aligned}$$

Basic problem that W_t cannot be treated as an ordinary differentiable function.

Wanted: alternative calculus to deal with infinitesimal behavior of W_t.

Good news: alternative calculus has been identified and is alive and well.

Even better news: alternative calculus yields correct expression:

$$S_t = S_0 \cdot Exp\left[\left(\mu - \frac{\sigma^2}{2}\right) \cdot t + \sigma \cdot W_t\right]$$

S_t is said to be *lognormally* distributed with parameters $\left(\mu - \frac{\sigma^2}{2}, \sigma\right)$.

Argument of reasonableness:

- S_t dynamics gives reason to suspect W_t-driven geometric behavior
- corrective term $Exp\left[-\frac{\sigma^2}{2} \cdot t\right]$ helps achieve $E\left(S_t\right) = S_0 \cdot Exp\left[\mu \cdot t\right]$

7 Two financial assets

Two financial assets a and b . $\left(S_t^a, S_t^b\right)$-dynamics governed by:

$$\begin{aligned} dS_t^a &= \mu^a \cdot S_t^a \cdot dt + \sigma^a \cdot S_t^a \cdot dW_t^a \\ dS_t^b &= \mu^b \cdot S_t^b \cdot dt + \sigma^b \cdot S_t^b \cdot dW_t^b \end{aligned}$$

where W_t^a and W_t^b are *dependent* Wiener processes with correlation coefficient ρ.

Simultaneous probability distribution for $\left(S_t^a, S_t^b\right)$:

$$
\begin{aligned}
S_t^a &= S_0^a \cdot Exp\left[\left(\mu^a - \frac{(\sigma^a)^2}{2}\right) \cdot t + \sigma^a \cdot W_t^a\right] \\
S_t^b &= S_0^b \cdot Exp\left[\left(\mu^b - \frac{\left(\sigma^b\right)^2}{2}\right) \cdot t + \sigma^b \cdot W_t^b\right]
\end{aligned}
$$

where:

$$
\left(W_t^a, W_t^b\right) \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \cdot t\right)
$$

8 Funds

n financial assets. Values at time t: $\left\{S_t^i\right\}_{i=1}^n$ is "multi GBM" with drifts $\left\{\mu^i\right\}_{i=1}^n$ and some covariance matrix $\mathbf{\Sigma}$.

Fund: a mixed portfolio, $\left\{\alpha_t^i\right\}_{i=1}^n$, where α_t^i is the "number of units" invested in asset $i\,(= 1, ..., n)$ at time $t > 0$, and in practice (although not necessarily in theory): $\alpha_t^i \geq 0, \forall i, t$.

At time 0 allocation of units between assets subject to the constraint:

$$S_0^F = \sum_{i=1}^n \alpha_0^i \cdot S_0^i$$

where S_0^F is the fund's initial total amount available for investments. For subsequent reallocations:

$$\sum_{i=1}^{n} \alpha_{t^-}^i \cdot S_t^i = \sum_{i=1}^{n} \alpha_{t^+}^i \cdot S_t$$

Fund's value at time t:

$$S_t^F = \sum_{i=1}^{n} \alpha_t^i \cdot S_t^i$$

9 Funds: Limited potential for diversification

Where asset allocation is maintained constant over time, $\alpha_t^i \equiv \alpha_0^i; \forall i, t$:

$$var\left(S_t^F\right) = var\left[\sum_{i=1}^{n} \alpha_0^i \cdot S_t^i\right] = \sum_{i=1}^{n} \left(\alpha_0^i\right)^2 \cdot var\left(S_t^i\right) + \sum_{i \neq j} \alpha_0^i \cdot \alpha_0^j \cdot Cov\left(S_t^i, S_t^j\right).$$

In practice, funds will be structured so that $Cov\left(S_t^i, S_t^j\right) > 0$ is the main rule, so the aggregate effect of covariances does not vanish.

For illustrational purposes, consider the following simplistic case:

$\alpha_0^i = \frac{1}{n}$, $var\left(S_t^i\right) = \tau_t^2$, and $Cov\left(S_t^i, S_t^j\right) = \rho \cdot \tau_t^2 > 0; \forall t, i, j^*$.

*In general, this model specification is "legal" provided $\rho > -\frac{1}{n-1}$.

Then $var\left(S_t^F\right)$ simplifies to:

$$\begin{aligned} var\left(S_t^F\right) &= \sum_{i=1}^{n} \frac{1}{n^2} \cdot \tau_t^2 + \sum_{i \neq j} \frac{1}{n} \cdot \frac{1}{n} \cdot \rho \cdot \tau_t^2 \\ &= n \cdot \frac{1}{n^2} \cdot \tau_t^2 + n \cdot (n-1) \cdot \frac{1}{n^2} \cdot \rho \cdot \tau_t^2 \\ &= \frac{1}{n} \cdot \tau_t^2 \cdot [1 + (n-1) \cdot \rho] \\ &= \tau_t^2 \cdot \left[\frac{1}{n} + \left(1 - \frac{1}{n}\right) \cdot \rho\right] \end{aligned}$$

The portfolio risk does not converge to 0 as portfolio size increases, due to positive covariance.

In the illustration: $var\left(S_t^F\right) \rightarrow \tau_t^2 \cdot \rho$ as portfolio size increases, which is risk *reduction* as opposed to risk *elimination.*

10 Funds: GBM behavior

Disappointing mathematical property:

Although each and every $\left\{S_t^i\right\}_{i=1}^n$ are GBM, $S_t^F = \sum_{i=1}^n \alpha_t^i \cdot S_t^i$ is not GBM!

As an approximation it may be convenient - and often acceptable - to assume that the fund itself obeys GBM behavior:

$$dS_t^F = \mu^F \cdot S_t^F \cdot dt + \sigma^F \cdot S_t^F \cdot dW_t^F$$

Must be interpreted as a postulated property, as opposed to derived from property of the individual assets which go into the fund.