Pricing of minimum interest guarantees: Is the arbitrage free price fair?

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1 Outline

- Stating the problem
- The savings account
- Case study
- Discussion

2 Stating the problem

- What is the "value" to the policyholder of an embedded interest rate guarantee, when it is assumed that the guarantee is priced according to the arbitrage free principle?

- Probability distributions for the amount on a linked savings account at retirement - respectively with and without a minimum interest rate guarantee embedded.

3 The saving account

Contributions are made annually in advance.

4 Financial market

- A bond with current value B_0 has a value at time t:

$$B_t = B_0\, e^{\delta t} \tag{1}$$

- A stock with current value S_0 has a value at time t :

$$S_t = S_0\, e^{L_t} \tag{2}$$

where the log-return is $L_t \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right) t, \sigma\sqrt{t}\right)$.

$$E[S_t] = S_0\, e^{\mu t} \tag{3}$$

5 Notation

μ	expected rate of return on the stock
σ	volatility of the stock
δ	rate of return of the risk free asset
γ	minimum interest rate
α	proportion in the stock - rebalanced
C	discrete premium payments
T	time at retirement

6 Return

The value at time t of a unit invested at time $t-1$:

$$a_t \;=\; \alpha\, e^{G_t} + (1-\alpha)\, e^{\delta} \tag{4}$$

- $G_t = L_t - L_{t-1} \sim N(\mu - \frac{\sigma^2}{2}, \sigma)$.
- $\alpha \in (0,1)$ is the share/ weight invested in a given stock which develops according to (2)

7 The savings account without guarantee

$$
\begin{aligned}
F_0 &= 0 \\
F_t &= a_t \left(C + F_{t-1}\right), t = 1, 2, ...T
\end{aligned} \tag{5}
$$

8 The savings account with guarantee

$$F_t^g = \max\left\{e^{\gamma}, a_t\ (1-p)\right\}\ (C + F_{t-1}^g) \qquad (6)$$

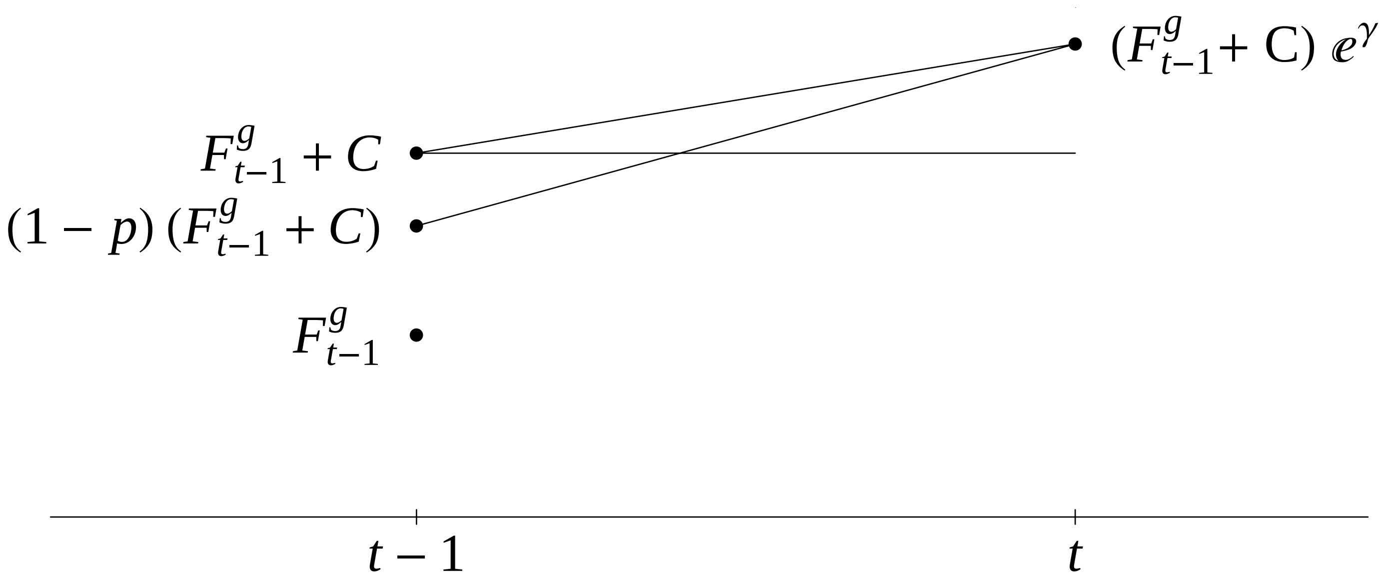
9 Guarantee premium p

The unit guarantee premium p is obtained as the solution of the equation

$$
\begin{aligned}
p &= e^{-\delta} E_Q[(e^{\gamma} - (1-p)\, a_t)^{+}] \\
&= K\, e^{-\delta}\Phi(-d_2) - S_0\, \Phi(-d_1), \quad Q \sim N\left(\delta - \frac{\sigma^2}{2}, \sigma\right) \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
d_2 &= \frac{\log(\frac{S_0}{K}) + (\delta - \frac{\sigma^2}{2})}{\sigma} \\
d_1 &= d_2 - \sigma \\
K &= e^{\gamma} - (1-p)\,(1-\alpha)\, e^{\delta} \\
S_0 &= (1-p)\, \alpha
\end{aligned}
$$

10 F_t^g – dynamics



11 Computation

- Analytical expressions for F_T- and F_T^g- distributions?

- Stochastic Monte Carlo simulation procedure:

$$G_t, \quad t \in \{1, 2, ..., T\} \longrightarrow a_t, \quad t \in \{1, 2, ..., T\} \longrightarrow F_T \text{ and } F_T^g \tag{8}$$

- Sufficiently large simulated samples will be distributed approximately according to the probability density function (pdf)

- A measurement of "over-performance resulting from guarantee":

$$\Psi_T = 100 \left(\frac{F_T^g}{F_T} - 1 \right) \tag{9}$$

12 Case study

$$
\begin{aligned}
\mu &= 10\ \%\ \text{per year} \\
\sigma &= 20\ \%\ \text{per year} \\
\delta &= 5\ \%\ \text{per year} \\
\gamma &= 3\ \%\ \text{per year} \\
\alpha &= 20\ \% \\
C &= 1 \\
T &= 20\ \text{years}
\end{aligned}
$$

13 p

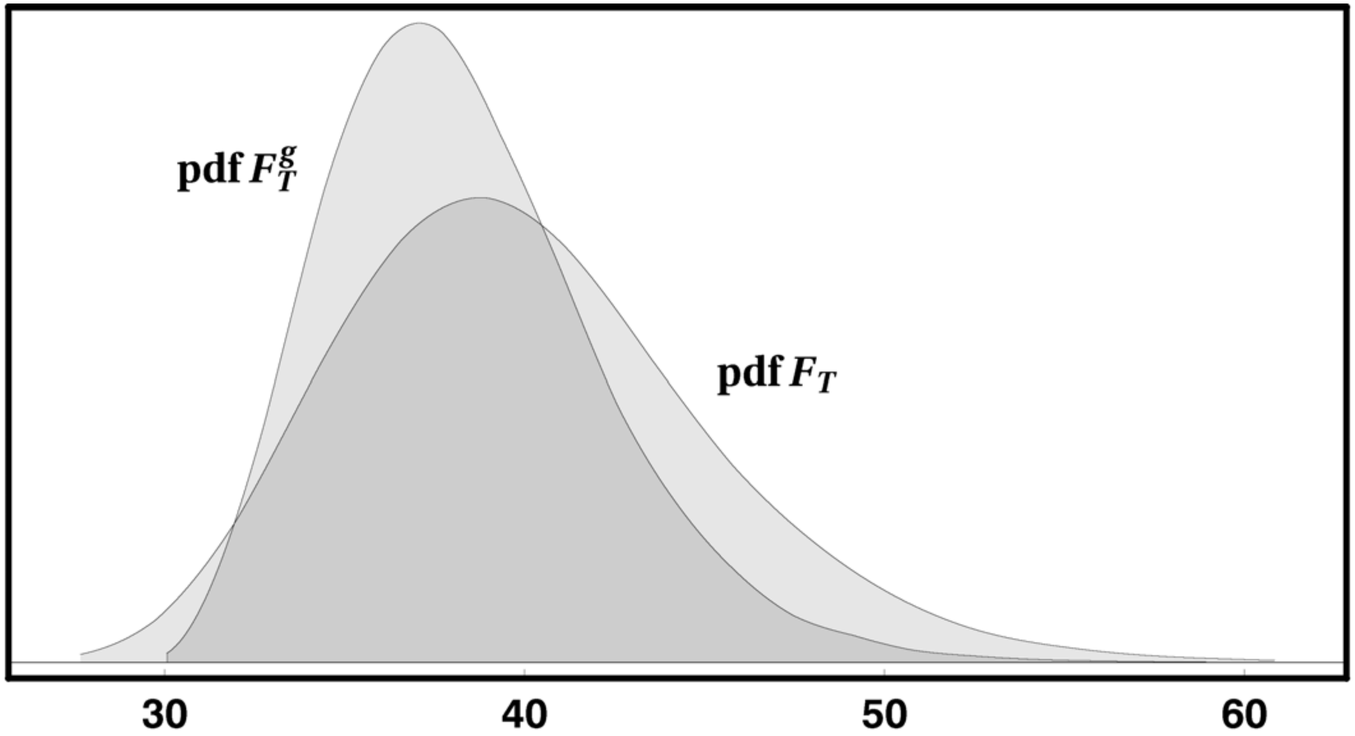
In this case the guarantee premium is

$$p = 0.0117$$

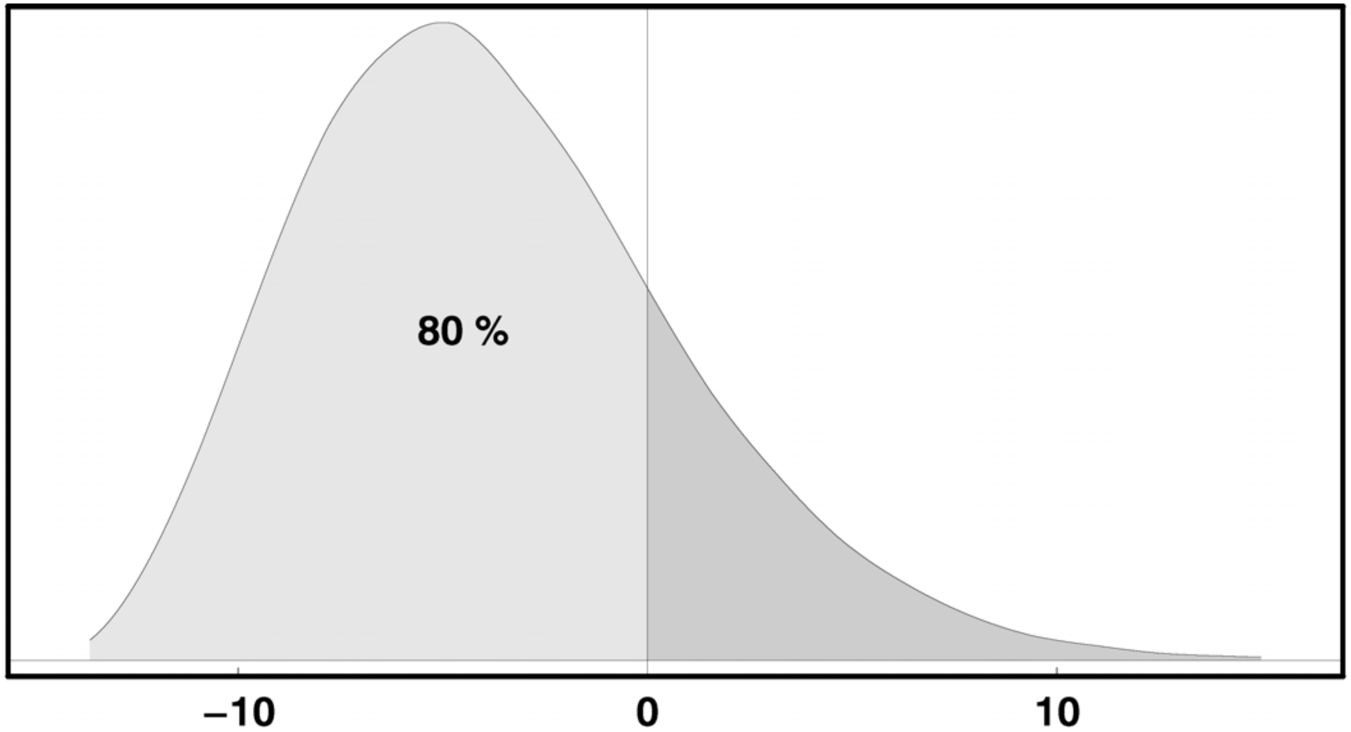
and the guarantee becomes effective if

$$a_t < \frac{e^\gamma}{1-p} = 1.0427$$

14 Approximate pdfs for F_T and F_T^g



15 Approximate pdf for Ψ_T



16 Risk measures

	min	$VaR(.05)$	$CVaR(.05)$
F_T	26.4	32.7	31.4
F_T^g	29.3	33.1	32.3

17 Sensitivity of $\Pr\{\Psi_T > 0\}$ to changes in the parameters μ and σ

		σ		
		.10	.20	.30
	.07	.26	.37	.46
μ	.10	.09	.20	.30
	.15	.01	.05	.12

18 Some conclusions

The safety the policyholder achieves from an interest rate guarantee is small compared to the reduced return resulting from the guarantee premium:

- Indeed in our illustrations. Generalizations?
- Intuition: Too expensive for the policyholder to "allow" the provider to do away with all risk
- Non-arbitrage vs. time diversification – reconcilable concepts?

With high probability similar safety can be achieved by having a slightly smaller proportion in the stock.

19 Some observations

Long standing tradition for interest rate guarantee in life and pension insurance:

- Pricing?

- Asset allocation – hedging?

Regulators seem to have a positive attitude towards interest rate guarantees – in the spirit of "consumer protection"

Is interest rate guarantee a user-friendly concept?

Will/should risk interest rate guarantees priced risk-neutral be in demand?

20 Appendix: Replicating portfolio

Assume we have a stock S_t. We want have the possibility to sell the stock at time T for the price K. We can use two investment strategies to achieve this:

- Buying at put option with strikeprice K. In this case we have the stock and a put option

- Buying the replicating portfolio. In this case we have a portfolio consisting of the stock and the replicating portfolio.

Option pricing and replicating portfolios are in essence two equivalent concepts.

21 The two investment strategies