Exercise E6

A stock portolio is followed over K periods of time with R_k the return in period k Dividend is excluded. The standard model for such assets is $\log(1+R_k) = \xi + \sigma\varepsilon_k$ with the drift ξ and the volatility σ fixed parameters and with $\varepsilon_1, \dots, \varepsilon_K$ independent random variables with mean 0 and standard deviation 1.

a) Show that the return of the investment after K periods is

$$R_{0:K} = (1+R_1)\cdots(1+R_K) - 1.$$

b) Argue that the model for the log-return $\log(1+R_{0:K})$ has drift $K\xi$ and volatility $\sigma/\sqrt{K}$ and that it becomes Gaussian for large K.

c) If $\xi = 0.05$ and $\sigma = 0.25$ are parameters on an annual time scale what is the chance of the investment showing a loss after 25 years and that it has doubled in value?

d) If only a fraction (weight) w of the original investment is in the stock market and the rest put in a bank account with fixed rate of interest r, show that the return of the portfolio after K periods is

$$\mathcal{R}_{0:K} = w(1+R_{0:K}) + (1-w)(1+r)^K.$$

e) Redo c) when $w = 0.35$ and $r = 0.04$.

Exercise E7

Extend the model in Exercise E6 by making the volatilities time-dependent and random so that now $\log(1+R_k) = \xi + \sigma_k\varepsilon_k$ where σ_k is independent of ε_k and all prior values $\varepsilon_{k-1}, \varepsilon_{k-2}, \dots$. Consider the so-called ARCH model for which

$$\sigma_k^2 = (1-\theta)\zeta^2 + \theta\{\log(1+R_k) - \xi\}^2, \qquad k = 1, \dots K$$

where $\zeta > 0$ and $\theta > 0$ are parameters.

a) Argue that the squared volatilities σ_k^2 satisfy the recursion

$$\sigma_k^2 = \eta^2 + \theta(\sigma_{k-1}^2\varepsilon_k^2 - \eta^2)$$

b) Show that

$$E(\sigma_k^2) - \eta^2 = \theta(E\sigma_{k-1}^2 - \eta^2) \qquad \text{so that} \qquad E(\sigma_k^2) = \eta^2 + \theta^k(\sigma_0^2 - \eta_2).$$

c) Also show that

$$\mathrm{var}(\sigma_k^2) = 3\theta^2\mathrm{var}(\sigma_{k.1}^2) + 2\theta^2\zeta + 2\theta^{2k}(\sigma_0 - \zeta^2), \qquad k = 1, 2, \dots.$$

d) Why is the variance zero $k = 0$? Verify that

$$\mathrm{var}(\sigma_k^2) = \frac{2\theta^2\zeta^2}{1-3\theta^2}(1 - 3^k\theta^{2k}) + (\sigma_0 - \zeta^2)(3^k - 1)\theta^{2k}$$

by inserting this expression into the recursion in c) which now becomes satisfied.

e) Offer a sensible definiation of stationarity, and use the results above to argue that the ARCH model is stationary if $\theta < 1/\sqrt{3}$.

Exercise E8

a) Define the concept of a financial weight mathematically.

b) Formulate and prove the theorem that tells us how portfolio returns are connected to the returns of the individual assets.

Exercise E9

Let R_k and r_k be the return to equity and the floating rate of interest in period k which allows the value of equity S_k and the value B_k of a bank account grow and fluctuate from a start $S_0 = s_0$ and $B_0 = b_0$.

a) What does it mean that the investment is of the buy and hold type? Write down recursions that determine how the equity, cash and portfolio value evolves.

b) What does it mean that the portfolio is invested through a fixed weight strategy? Develope the mathematics that determines how the portfolio values evolve now.

Exercise E10

A more realistic model for a bank account than the fixed rate of interest in Exercise E6 is a floating rate r_k that varies randomly. A standard model is

$$r_k = \zeta e^{-\tau^2/(1-a^2)+X_k}, \qquad X_k = aX_{k-1} + \tau\delta_k$$

where $\delta_1, \delta_2, \ldots,$ are independent and standard normal distributed.

a) What does it mean that the model is reversion to mean or stationary (which amounts to the same thing) and for which values of a does this occur?

b) If the equity returns in Exercise E6 and the floating rates of interest here are independent, sketch a computer program that simulates the returns of a portfolio consisting of equity and a bank account under a floating rate strategy.

c) The same question for a fixed weight strategy.

d) If the there is a regulatory rule that specifies that the fraction invested in equity should at most be w_0, how is that implemented in the computer programming?