Obligatory assignment for STK4500, Spring 2018 − Solution

Liability risk

Mortality risk

The mortality risk of the company is related to the term insurance portfolio. The term insurance contracts are all equal, start at the age of $l_0 = 30$ and last $K = 25$ years. The premium π_1 is then paid in advance at the start of the contract and the 24 following years, given that the individual is still alive. The expected contributions must then be

$$\pi_1 \sum_{k=0}^{K-1} d^k \; {}_kp_{l_0},$$

where $d = 1/(1+r)$ is the discount and ${}_kp_{l_0}$ the probability of surviving till at least age $l_0 + k$ given the current age l_0. These contributions are counted as negative in the expected present value of the cash flow. Further, a one-time payment s_1 is made in arrears if the individual dies within the end of the contract, i.e. within $K = 25$ years. Hence, the expected payment is

$$s_1 \sum_{k=1}^{K} d^k \; {}_kq_{l_0},$$

where ${}_kq_{l_0}$ is the probability of dying between the ages of $l_0 + k - 1$ and $l_0 + k$ given the current age l_0. Hence, the expected present value must be

$$\mathcal{PV}_0 = -\pi_1 \sum_{k=0}^{K-1} d^k \; {}_kp_{l_0} + s_1 \sum_{k=1}^{K} d^k \; {}_kq_{l_0},$$

such that the equivalence premium is

$$\pi_1 = s_1 \frac{\sum_{k=1}^{K} d^k \; {}_kq_{l_0}}{\sum_{k=0}^{K-1} d^k \; {}_kp_{l_0}}.$$

Plugging in the values, we obtain $\pi_1 = 0.00576$.

All individuals started their contract at the age of l_0. For a person of age $l_0 + m$ at $k = 0$, the liability related to this contract after k years must be

$$-\pi_1 \; {}_kp_{l_0+m} + s_1 \; {}_kq_{l_0+m},$$

except for $k = K$, when the last premium has already been paid and there only is a possibility for a one-time payment in case of death, and hence is

$$s_1 \; _Kq_{l_0}.$$

The number of individuals of age $l_0 + m$ at $k = 0$ is N_{1m}, hence the total for all individuals of this age is obtained by multiplying the above expressions by N_{1m}. Finally, to obtain the total for the portfolio, one must sum over all ages, and one obtains

$$L_{1k} = -\pi_1 \sum_{m=0}^{K-k} N_{1m} \; _kp_{l_0+m} + s_1 \sum_{m=0}^{K-k} N_{1m} \; _kq_{l_0+m}, \;\; k = 1, \ldots, K-1,$$

and

$$L_{1K} = s_1 N_{10} \; _Kq_{l_0}.$$

Finally, SCR^{mort} is obtained as

$$SCR^{mort} = \sum_{k=1}^{K} \frac{L_{1k}^{S}}{(1+r_k)^k} - \sum_{k=1}^{K} \frac{L_{1k}}{(1+r_k)^k},$$

where L_{1k}^{S} is computed like L_{1k}, but with the stressed mortalities

$$q_l^{S} = 1.15 q_l.$$

We obtain

$$SCR^{mort} = 1,005.$$

Longevity risk

The longevity risk of the company is related to the pension portfolio. The pension contracts are all equal, start at the age of $l_0 = 30$ and last till death (at most $l_e = 120$ years) and the age of retirement is $l_r = 67$ years. The premium π_2 is then paid in advance at the start of the contract and until the year before retirement, given that the individual is still alive. The expected contributions must then be

$$\pi_2 \sum_{k=0}^{l_r-l_0-1} d^k \; _kp_{l_0}.$$

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These contributions are counted as negative in the expected present value of the cash flow. Further, a yearly payment s_2 is made in advance from age 67 years, as long as the individual is alive. Hence, the expected payment is

$$s_2 \sum_{k=l_r-l_0}^{l_e-l_0} d^k \ _kp_{l_0},$$

Hence, the expected present value must be

$$\mathcal{PV}_0 = -\pi_2 \sum_{k=0}^{l_r-l_0-1} d^k \ _kp_{l_0} + s_2 \sum_{k=l_r-l_0}^{l_e-l_0} d^k \ _kp_{l_0}.$$

such that the equivalence premium is

$$\pi_2 = s_2 \frac{\sum_{k=l_r-l_0}^{l_e-l_0} d^k \ _kp_{l_0}}{\sum_{k=0}^{l_r-l_0-1} d^k \ _kp_{l_0}}.$$

Plugging in the values, we obtain $\pi_2 = 0.0622$.

All individuals started their contract at the age of l_0. For a person of age $l_0 + m$ at $k = 0$, the liability related to this contract after k years must be

$$-\pi_2 \ _kp_{l_0+m}$$

if $m < l_r - l_0$ and

$$s_2 \ _kp_{l_0+m}$$

if $m \geq l_r - l_0$. The number of individuals of age $l_0 + m$ at $k = 0$ is N_{2m}, hence the total for all individuals of this age is obtained by multiplying the above expressions by N_{2m}. Finally, to obtain the total for the portfolio, one must sum over all ages, and one obtains

$$L_{2k} = -\pi_2 \sum_{m=0}^{l_r-l_0-1-k} N_{2m} \ _kp_{l_0+m} + s_2 \sum_{m=l_r-l_0-k}^{\min(l_e-l_0-k,65)} N_{2m} \ _kp_{l_0+m}, \ k = 1, \ldots, l_r-l_0-1,$$

and

$$L_{2k} = s_2 \sum_{m=0}^{l_e-l_0-k} N_{2m} \ _kp_{l_0+m}, k = l_r - l_0, \ldots, l_e - l_0.$$

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Finally, SCR^{long} is obtained as

$$SCR^{long} = \sum_{k=1}^{K} \frac{L_{2k}^{S}}{(1+r_k)^k} - \sum_{k=1}^{K} \frac{L_{2k}}{(1+r_k)^k},$$

where L_{2k}^{S} is computed like L_{2k}, but with the stressed mortalities

$$q_l^{S} = 0.80 q_l.$$

We obtain

$$SCR^{long} = 5460.$$

We see that the longevity risk is much larger than the mortality risk. This is due to among others the difference in portfolio size, but more importantly to the fact that the pension liabilities stretch over a much longer period of time.

Finding SCR^{liab}

To obtain SCR^{liab}, we apply the standard aggregation formula:

$$SCR^{liab} = \sqrt{(SCR^{mort})^2 + (SCR^{long})^2 - 2 \cdot 0.25 SCR^{mort} SCR^{long}} = 5298.$$

Hence, the liability risk is dominated by the longevity risk, but a little lower, due to the lower value of the mortality risk and the negative correlation between the two risks.

Market risk

Interest rate risk

The interest rate risk considered here concerns decreasing interest rates. The corresponding SCR is found by $SCR^{int} = BE^S - BE$, where $BE = BE_1 + BE_2$ from the two portfolios under the liability risk, and BE^S is computed as BE, but with a shocked interest rate curve, given by

$$r_k^{S} = (1 - S_k) r_k,$$

S_k being the shock at time k. The normal and shocked interest rate curves are shown in the plot below. We obtain
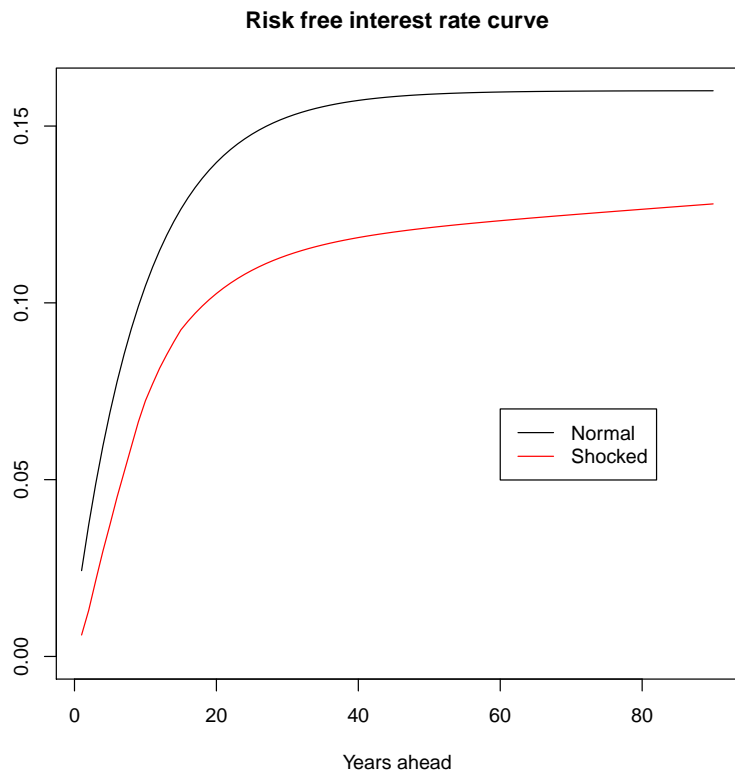
Figure 1: Normal and shocked risk free interest rate curves.

$$SCR^{int} = BE^S - BE$$

$$= \sum_{k=1}^{K} \frac{L_{1k}}{(1+r_k^S)^k} + \sum_{k=1}^{K} \frac{L_{2k}}{(1+r_k^S)^k} - \left(\sum_{k=1}^{K} \frac{L_{1k}}{(1+r_k)^k} + \sum_{k=1}^{K} \frac{L_{2k}}{(1+r_k)^k} \right)$$

$$= 37,993.$$

Equity risk

The market value of the equity investments is $0.3 \cdot 300,000 = 90,000$. These are spread on 20% in Type 1 and 80% in Type 2 stocks. Further, these investments are not protected by options. Hence, SCR^{equi} is given by

$$SCR^{equi} = 0.78 \cdot 0.2 \cdot 90,000 + 0.61 \cdot 0.8 \cdot 90,000 = 57,960.$$

Property risk

The market value of the property investments is $0.1 \cdot 300,000 = 30,000$. Then, SCR^{prop} is given by

$$SCR^{prop} = 0.75 \cdot 30,000 = 22,500.$$

Spread risk

The market value of the bond investments is $0.6 \cdot 300,000 = 180,000$. In turn this is divided $50-50$ between two bonds, one with AA rating, corresponding to the coefficient $b(1) = 0.011$, and duration $D_1 = 5.5$ and one with AAA rating, corresponding to the coefficient $b(0) = 0.009$, and duration $D_2 = 10.5$. This means that SCR^{spread} is given by

$$S_1 = 1 - 0.011 \cdot 5.5$$
$$S_2 = 1 - 0.009 \cdot 10.5$$
$$SCR^{spread} = (1 - S_1) \cdot 0.5 \cdot 180,000 + (1 - S_2) \cdot 0.5 \cdot 180,000 = 13,950.$$

Finding SCR^{mark}

To obtain SCR^{mark}, we apply the standard aggregation formula:

$$\begin{aligned}
SCR^{mark} = \Big(&\left(SCR^{int}\right)^2 + \left(SCR^{equi}\right)^2 + (SCR^{prop})^2 + \left(SCR^{spread}\right)^2 \\
&+ 2 \cdot 0.5 \cdot SCR^{int}SCR^{equi} + 2 \cdot 0.5 \cdot SCR^{int}SCR^{prop} \\
&+ 2 \cdot 0.5 \cdot SCR^{int}SCR^{spread} + 2 \cdot 0.75 \cdot SCR^{equi}SCR^{prop} \\
&+ 2 \cdot 0.75 \cdot SCR^{equi}SCR^{spread} + 2 \cdot 0.5 \cdot SCR^{prop}SCR^{spread}\Big)^{1/2} \\
=& 112,131.
\end{aligned}$$

We see that SCR^{mark} is much larger than SCR^{liab}, which reflects that the market risk is much larger than the liability risk. This is as one would expect for a life insurance company. We see that the equity risk, in particular, is quite high, and so is the property risk. The bond investments, which make up the largest part of the assets of the company, are not nearly as risky, again as expected.

Finding $SCR = BSCR$

Finally, the $BSCR$ is found using the standard formula:

$$\begin{aligned}
SCR =& BSCR \\
=& \sqrt{(SCR^{liab})^2 + (SCR^{mark})^2 + 2 \cdot 0.25 \cdot SCR^{liab}SCR^{mark}} \\
=& 113,572.
\end{aligned}$$

We see that the $BSCR$ consists mostly of market risk. It is a little higher than SCR^{mark} since the correlation between SCR^{mark} and SCR^{liab} is positive.

R-code

```
lifetab <- function(t0,t1,t2,l_0,l_e,Shock=0)
{
# Input: l_0 is age of policy holders when the contracts are drawn up,
# l_e maximum age, whereas t0, t1, t2 are Gomperz-Makeham parameters.
#
# Output: kp is the lifetable and q the mortalities. Indexing: kp[l,k]
# is the probability of surviving k-1 years given the initial age l_0+l-1.
```

```r
# Mortality q[l] is the probability of dying at age $l_0+l-1$.
#
ll <- l_0:(l_e+l_e)
q <- pmin(pmax((1-exp(-t0-(t1/t2)*(exp(t2)-1)*exp(t2*ll)))*(1+Shock),0),1)
kp <- matrix(0,l_e+1,l_e+1)
kp[,1] <- 1
for (l in 1:(l_e-1))
 {
  for (k in 2:(l_e+1-l))
   {
     kp[l,k] <- kp[l,k-1]*(1-q[l+k-2])
   }
 }

list(l_0=l_0,l_e=l_e,kp=kp,q=q)
}


ndist <- function(gamma,mu,K,J)
{
# The distribution of policies according to how long they have been in the
# company.
# Input: Parameters describing the distribution of the policies.
#
# Output: N number of policies at different age with indexing as
# described in function "lifetab".
#
p <- exp(-gamma*abs(0:K-mu))
N <- p*(J/sum(p))

N
}

riskfree <- function(a,b,theta,K=200,Shock=NULL)
{
# The risk-free rates of interest stored as rf.
#
rf <- a+b*(1-exp(-(1:K)*theta))
if(length(Shock) > 0)
```

```r
 {
   rf <- rf*pmax(1-Shock,0)
 }
rf
}

## Model parameters
t0 <- 0.00078
t1 <- 0.0000376
t2 <- 0.092759
l_0 <- 30
l_r <- 67
l_e <- 120
K <- 25
s_1 <- 2
s_2 <- 0.3
r <- 0.02
d <- 1/(1+r)
J_1 <- 150000
J_2 <- 200000
gamma_1 <- 0.1
mu_1 <- 20
K_1 <- K
gamma_2 <- 0.15
mu_2 <- 24
K_2 <- 65
a <- 0.01
b <- 0.15
theta <- 0.1
A <- 300000
D_1 <- 5.5
D_2 <- 10.5

## Life table
lt <- lifetab(t0,t1,t2,l_0,l_e,Shock=0)

## Life table with mortality shock
lt_mort <- lifetab(t0,t1,t2,l_0,l_e,Shock=0.15)
```

```
## Life table with longevity shock
lt_long <- lifetab(t0,t1,t2,l_0,l_e,Shock=-0.2)

## Age distribution in the two portfolios
N_1 <- ndist(gamma_1,mu_1,K_1,J_1)
N_2 <- ndist(gamma_2,mu_2,K_2,J_2)

## Risk free interest rate
rf <- riskfree(a,b,theta,K=150)

## Risk free interest rate with shock
x <- c(1:10,12,15,90)
y <- c(0.75,0.65,0.56,0.5,0.46,0.42,0.39,0.36,0.33,0.31,0.29,0.27,0.2)
interp <- approx(x,y,xout=1:150,rule=2)
r_Shock <- interp$y
rf_S <- riskfree(a,b,theta,K=150,Shock=r_Shock)

plot(1:90,rf[1:90],type="l",ylim=c(0,0.16),xlab="Years ahead",ylab="",
main="Risk free interest rate curve")
lines(1:90,rf_S[1:90],col=2)
legend(60,0.07,legend=c("Normal","Shocked"),lty=rep(1,2),col=1:2)


## Term insurance premium
a1 <- sum(d^(0:(K-1))*lt$kp[1,0:(K-1)+1])
a2 <- sum(d^(1:K)*lt$kp[1,1:K]*lt$q[1:K])
pi_1 <- s_1*a2/a1

## Pension premium
a1 <- sum(d^(0:(l_r-l_0-1))*lt$kp[1,0:(l_r-l_0-1)+1])
a2 <- sum(d^((l_r-l_0):(l_e-l_0))*lt$kp[1,(l_r-l_0):(l_e-l_0)+1])
pi_2 <- s_2*a2/a1

## Term insurance liablities
L_1 <- rep(0,K)
for (k in 1:(K-1))
  L_1[k] <- -pi_1*sum(N_1[0:(K-k)+1]*lt$kp[0:(K-k)+1,k+1])
```

```
for (k in 1:K)
  L_1[k] <- L_1[k]+s_1*sum(N_1[0:(K-k)+1]*lt$kp[0:(K-k)+1,k]*lt$q[k:K])

## Term insurance liablities with mortality risk
L_1_S <- rep(0,K)
for (k in 1:(K-1))
  L_1_S[k] <- -pi_1*sum(N_1[0:(K-k)+1]*lt_mort$kp[0:(K-k)+1,k+1])
for (k in 1:K)
  L_1_S[k] <- L_1_S[k]
              +s_1*sum(N_1[0:(K-k)+1]*lt_mort$kp[0:(K-k)+1,k]*lt_mort$q[k:K])

## Pension liabilities
L_2 <- rep(0,l_e-l_0)
for(k in 1:(l_r-l_0-1))
 L_2[k] <- sum(-pi_2*N_2[0:(l_r-l_0-1-k)+1]*lt$kp[0:(l_r-l_0-1-k)+1,k+1])
             +sum(s_2*N_2[(l_r-l_0-k):min(K_2,l_e-l_0-k)+1]
             *lt$kp[(l_r-l_0-k):min(K_2,l_e-l_0-k)+1,k+1])
for(k in (l_r-l_0):(l_e-l_0))
 L_2[k] <- sum(s_2*N_2[0:(l_e-l_0-k)+1]*lt$kp[0:(l_e-l_0-k)+1,k+1])

## Pension liabilities with longevity risk
L_2_S <- rep(0,l_e-l_0)
for(k in 1:(l_r-l_0-1))
 L_2_S[k] <- sum(-pi_2*N_2[0:(l_r-l_0-1-k)+1]
                  *lt_long$kp[0:(l_r-l_0-1-k)+1,k+1])
                  +sum(s_2*N_2[(l_r-l_0-k):min(K_2,l_e-l_0-k)+1]
                  *lt_long$kp[(l_r-l_0-k):min(K_2,l_e-l_0-k)+1,k+1])
for(k in (l_r-l_0):(l_e-l_0))
 L_2_S[k] <- sum(s_2*N_2[0:(l_e-l_0-k)+1]*lt_long$kp[0:(l_e-l_0-k)+1,k+1])

##### SCR mortality
BE <- sum(L_1/(1+rf[1:K])^(1:K))
BE_S <- sum(L_1_S/(1+rf[1:K])^(1:K))
SCR_mort <- BE_S-BE

##### SCR longevity
BE <- sum(L_2/(1+rf[1:(l_e-l_0)])^(1:(l_e-l_0)))
BE_S <- sum(L_2_S/(1+rf[1:(l_e-l_0)])^(1:(l_e-l_0)))
```

```
SCR_long <- BE_S-BE

##### SCR interest rate
BE <- sum(L_1/(1+rf[1:K])^(1:K))+sum(L_2/(1+rf[1:(l_e-l_0)])^(1:(l_e-l_0)))
BE_S <- sum(L_1/(1+rf_S[1:K])^(1:K))
           +sum(L_2/(1+rf_S[1:(l_e-l_0)])^(1:(l_e-l_0)))
SCR_int <- BE_S-BE

##### SCR equity
A_equi <- 0.3*A
A_equi_1 <- 0.2*A_equi
A_equi_2 <- 0.8*A_equi
SCR_equi <- (1-0.22)*A_equi_1+(1-0.39)*A_equi_2

##### SCR property
A_prop <- 0.1*A
SCR_prop <- (1-0.25)*A_prop

##### SCR spread
A_bond <- 0.6*A
A_bond_1 <- 0.5*A_bond
A_bond_2 <- 0.5*A_bond
S_1 <- 1-0.011*D_1
S_2 <- 1-0.009*D_2
SCR_spread <- (1-S_1)*A_bond_1+(1-S_2)*A_bond_2

##### SCR liabilities
SCR_liab <- sqrt(SCR_mort^2+SCR_long^2-2*0.25*SCR_mort*SCR_long)


##### SCR market
SCR_mark <- sqrt(SCR_int^2+SCR_equi^2+SCR_prop^2+SCR_spread^2
                 +2*0.5*SCR_int*SCR_equi+2*0.5*SCR_int*SCR_prop
                 +2*0.5*SCR_int*SCR_spread+2*0.75*SCR_equi*SCR_prop
                 +2*0.75*SCR_equi*SCR_spread+2*0.5*SCR_prop*SCR_spread)

##### BSCR
BSCR <- sqrt(SCR_liab^2+SCR_mark^2+2*0.25*SCR_mark*SCR_liab)
```