STK4500: LIFE INSURANCE AND FINANCE MANDATORY ASSIGNMENT SPRING 2023

This assignment consists of 3 exercises. The deadline is 20th of April, 2023 at 2:30 pm. To pass the assignment you need to have at least two out of three exercises correct. Good luck!

Exercise 1 (Learn your formulas). This exercise has to be **written by hand**. Provide a set of the most important formulas of the course so far. Divide the set into *continuous time* and *discrete time* formulas. We will start with the continuous time setting. Please, be precise, clear and tidy.

Let us introduce the following notation:

- $\mathcal{S}$ denotes the states of the insured.
- X_t denotes the state of the insured at time t. X is assumed to be Markov.
- For states $i, j \in \mathcal{S}$, let $p_{ij}(t,s) \triangleq \mathbb{P}[X_s = j | X_t = i]$ for $s, t \geq 0$, $t \leq s$ be the transition probabilities between times and states.
- Let $A(t)$ denote a general accumulated cash flow at time $t \geq 0$.
- Let r be an instantaneous rate of return (e.g. interest rate) and $v(t) = e^{-\int_0^t r(s)ds}$ the value of one monetary unit at time t (discount factor).

Continuous time setting:

Write down:

(a) The definition of transition rates μ_{ij} between states $i, j \in \mathcal{S}$.

(b) Kolmogorov's equations to find $p_{ij}(t,s)$ from μ_{ij}.

(c) Write down what is the value of the total cash flow A at $t = 0$, discounted accordingly with respect to $v(t)$.

(d) How would you change the value of the total cash flow above if considered at any time t?

(e) Write down the *retrospective* and *prospective value* of a general cash flow A.

(f) Define *policy functions* a_i and a_{ij} in the continuous time setting.

(g) Define the processes I_i^X and N_{ij}^X associated to X and write shortly what they describe.

(h) By means of the policy functions a_i and a_{ij} and the processes I_i^X and N_{ij}^X, give a formula for A that fully describes its evolution and describe it shortly.

(i) Recast the prospective value, hereby V_t^+ of an insurance cash flow in the form given in (h).

(j) Define *single premium* π_0.

(k) Define a cash flow A^π of yearly continuous payments of size π. Write down the associated policy function a_*^π and prospective value of $V_t^+(A^\pi)$.

(l) Write down the formula for the *expected prospective value*, given that the insured is in state i at time t, i.e. $X_t = i$.

(m) Given an insurance cash flow A, split A into the cash flow A^π like in (k) only dealing with the payment of premiums while in state $*$ and another cash flow A^B modelling only the benefits, i.e. $A = A^\pi + A^B$. Consider now their prospective values $V_t^+(A^\pi)$ and $V_t^+(A^B)$. What is then the prospective value of A?

(n) What is the expected prospective value of the cash flow A above?
(o) Explain what the *equivalence principle* is.
(p) How would you find the yearly premium π in (m)?
(q) Write down *Thiele's differential equation.*

Discrete time setting:

(a) Follow the same steps as above and write down the formulas for the discrete time case.

Exercise 2 (Friend Group Survival). Consider a group of $N \in \mathbb{Z}$, $N \geq 1$ friends all with the same age and individual mortality, hereby denoted as μ. Let $Z = \{Z_t\}_{t\geq 0}$ be the continuous time (regular) Markov chain which counts the number of living friends in this group, by time t. The state space of Z is then clearly $\mathcal{S} = \{0, 1, \ldots, N\}$. Define

$$p_{m\,n}(t,s) \triangleq \mathbb{P}[Z_s = n | Z_t = m]$$

and let

$$p(t,s) = e^{-\int_t^s \mu(u)du}$$

be the survival probability of an individual. Lastly we will assume that the lifespans of the friends are independent.

(a) Prove that for every $t \geq 0$ we have

$$\mu_{m\,n}(t) = 0,$$

for every $m, n \in \mathcal{S}$, $|m - n| \geq 2$ or $n = m + 1$, and that

$$\mu_{m\,m-1}(t) = m\mu(t),$$

for every $m \in \mathcal{S} \setminus \{0\}$.

(b) Argue that for every $t, s \geq 0$, $s \geq t$,

$$p_{m\,n}(t,s) = 0,$$

for every $m, n \in S$, $n \geq m + 1$ and that,

$$p_{m\,n}(t,s) = \binom{m}{n} p(t,s)^n (1 - p(t,s))^{m-n},$$

for every $m, n \in S$, $n \leq m$ and show that $s \mapsto p_{mn}(t,s)$ satisfies Kolmogorov's forward equation.

Exercise 3 (A Tontine of Friends in Continuous Time). A tontine is an old investment strategy named after Neapolitan banker Lorenzo de Tonti, who is popularly credited with inventing it in France in 1653.

The main idea of a tontine is that a group of people all pay a single lump sum and invest it into a shared fund. Every year the net profit of the fund is distributed amongst the investors as yearly dividends. Whenever any of the initial investors die, the yearly dividends will then be distributed between fewer and fewer people, thus increasing the payout for those who remain alive.

In this exercise we will assume that an insurance company oversees and manages a tontine insurance scheme with N friends under the same conditions and notations as in the previous exercise. We will focus on one of these participants, referred to as the *chosen one*. As such we model the state of everyone involved by the Markov process X with the state space

$$\mathcal{S} = \{0, 1, \ldots, N-1\} \times \{*, \dagger\}.$$

Here the state $(m, *)$ means that m participants, other than the chosen one, are alive and that that the chosen one is still living, while $(m, \dagger)$ means that m participants are alive, but our chosen one is dead.

(a) **Transition Probabilities.**

Argue in an almost analogous way as in Exercise 1, that

$$\begin{aligned}\mu_{(m,*)\,(m-1,\dagger)}(t) &= 0,\\ \mu_{(m,*)\,(m,\dagger)}(t) &= \mu(t),\\ \mu_{(m,*)\,(m-1,*)}(t) &= \mu_{(m,\dagger)\,(m-1,\dagger)}(t) = m\mu(t),\end{aligned}$$

for every $t \geq 0$ and $m \in \{0, 1, \ldots, N-1\}$. As a result, argue that for every $t, s \geq 0$, $s \geq t$,

$$p_{(m,j)\,(n,j)}(t, s) = 0,$$

for every $m, n \in \mathcal{S}$, $n \geq m+1$, $j \in \{*, \dagger\}$ and that,

$$p_{(m,*)\,(n,*)}(t, s) = \binom{m}{n} p(t, s)^{n+1}(1 - p(t, s))^{m-n},$$

and

$$p_{(m,*)\,(n,\dagger)}(t, s) = \binom{m}{n} p(t, s)^{n}(1 - p(t, s))^{m-n+1},$$

for every $m, n \in S$, $n \leq m$.

We assume that the contract starts at time $t = 0$ and that all participants pay a single premium π_0 at this time. The premiums are invested into a fund managed by the insurance company. Let $S = \{S(t), t \geq 0\}$ denote the value of a fund. We assume that the value of the fund evolves according to

$$S(t) = S(0)e^{\rho t}, \quad t \geq 0,$$

for some $\rho \in \mathbb{R}$, $\rho > 0$.

At retirement time $T_0 \geq 0$, we start paying out the returns from the fund to the living participants. Observe that the return on $[t, t+dt]$ is given by $S(t+dt) - S(t) = S(t)\left[e^{\rho dt} - 1\right]$ and if dt is infinitesimally small we have $e^{\rho dt} - 1 = \rho dt + O(dt^2)$. Hence, the instantaneous return at time t is given by $\rho S(t)$. In particular, our participants receive instantaneously $\rho S(T_0)$ after $t \geq T_0$ to be distributed among all surviving participants. We can also note that since all the profits of the fund are distributed immediately, the value of the fund stagnates at time T_0, that is

$$S(t) = S(T_0) = S(0)e^{\rho T_0} \text{ for all } t \geq T_0.$$

From now on, we assume a constant interest rate $r > 0$ that the insurance company uses to price its policies.

(b) **Policy Functions.** Take the perspective of the chosen one and therefore ignore all payments that are not going to them. Write down the policy functions for the chosen one's contract without taking into account premiums, yet. Note that since we are taking the perspective of the chosen one we have $a_{(m,\dagger)}(t) = 0$ for all m.

(c) **Cost of the insurance.** Show that the cost of this insurance, i.e., the present value at each time of the future liabilities is given by

$$V^+_{(m,*)}(t) = \rho S(0)e^{\rho T_0} \sum_{n=0}^{m} \int_{t\vee T_0}^{\infty} \binom{m}{n} \frac{p(t,s)^{n+1}(1-p(t,s))^{m-n}}{n+1} e^{-r(s-t)} ds$$

(d) **A simpler formula.** Show that the present value $V^+_{(m,*)}(t)$ can be written in the following simplified form,

$$V^+_{(m,*)}(t) = \frac{\rho S(0)e^{\rho T_0}}{m+1} \int_{t\vee T_0}^{\infty} \left(1-(1-p(t,s))^{m+1}\right) e^{-r(s-t)} ds$$

Note that for numerical implementations this form might be easier and faster to use.

(e) **Single and yearly premiums.** Compute the single and yearly premiums that our chosen one has to pay to enter this policy. The yearly premiums are only paid until retirement time T_0.

(f) **Thiele's Equation.** Derive an ordinary differential equation for the present value of the policy $V^+_{(m,*)}(t)$ for $t \geq 0$.

(g) **A numerical example.**

From now on and until the end of the assignment, let $r = 0.03, \rho = 0.07, S(0) = 100\,000, N = 10, T_0 = 40$. Furthermore, let $\mu(t)$ be given by the K2013 mortality rates from Finanstilsynet at time $t + 2023$ for a male aged 30 at the beginning of 2023.

Using these parameters compute the single and yearly premiums as described in exercise (e). Plot the present value $V^+_{(N-1,*)}(t)$, along with the reserves for the premiums and the total reserves for $t \in [0, 100]$.

(h) **Lifespan Simulation.**

An alternative way of computing the reserves is by the simulation of lifespans.

Simulate 10 000 outcomes and compute the (discounted) costs for each of them. Plot the histogram of these values. Compute the average of these payouts. What do you obtain?

Now, simulate 20 tontine outcomes and compute the cost average, as before. Repeat this procedure 10 000 times and plot the histogram of the sample means. What distribution do you see and why?